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- Characterize what solutions look like.
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In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is ________.

Question: How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

The sum principle

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- Length 2:
- Length 3:
- Length 4:
- Length 5,6:
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This illustrates the **sum principle**:

Suppose the objects to be counted can be broken into k disjoint and exhaustive cases. If there are n_j objects in case j, then there are $n_1 + n_2 + \cdots + n_k$ objects in all.

Counting pitfalls

When counting, there are two common pitfalls:

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  - **Ask:** Does the same object appear in multiple ways?

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Common example: A deck of cards.

There are four suits: Diamond ♢, Heart ♡, Club ♣, Spade ♠.
Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

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Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

Example. Suppose you are dealt two diamonds between 2 and 10.
In how many ways can the product be even?

Overcounting

Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a Heart ♡?

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Remember to ask: Do cases need to be counted in different ways?

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Example. How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555 Non-examples: 1231, 9898.

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Q1: How many 4-lists taken from [9] have **at least one** pair of adjacent elements equal?

—Compare this to—

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What can we say about:

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Example. When playing five-card poker, what is the probability that you are dealt a full house?

[*Three cards of one type and two cards of another type.*] 5 5 5 K K

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  $\frac{3744}{2598960} \approx 0.14\%$

Pascal's triangle

Pascal's identity gives us the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
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$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
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2	1		1					
3	1			1				
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Seq's in Pascal's triangle:

- $1, 2, 3, 4, 5, \ldots$ $\binom{n}{1}$ ($a_n = n$)
- $1, 3, 6, 10, 15, \ldots$ $\binom{n}{2}$ triangular
- $1, 4, 10, 20, 35, \ldots$ $\binom{n}{3}$ tetrahedral
- $1, 2, 6, 20, 70, \ldots$ $\binom{2n}{n}$ centr. binom.

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Online Encyclopedia of Integer Sequences:
http://oeis.org/

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

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Rewrite in summation notation!
Determine the generic term $[\binom{n}{k}x \quad y \quad]$ and the bounds on k

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From the n factors $(x+y)$, you must choose a "y" exactly k times. Therefore, $\binom{n}{k}$ ways. We recover the desired equation. □