Introduction to Bijections

Key tool: A useful method of proving that two sets A and B are of the same size is by way of a *bijection*.

A **bijection** is a function or rule that pairs up elements of A and B.

Example. The set A of subsets of $\{s_1, s_2, s_3\}$ are in bijection with the set B of binary words of length 3.

$$\begin{array}{rcccccccccc}
\text{Set A:} & \{ & \emptyset, & \{s_1\}, & \{s_2\}, & \{s_1,s_2\}, & \{s_3\}, & \{s_1,s_3\}, & \{s_2,s_3\}, & \{s_1,s_2,s_3\} & \} \\
\textbf{Bijection:} & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\text{Set B:} & \{ & 000, & 100, & 010, & 110, & 001, & 101, & 011, & 111 & \}
\end{array}$$

Rule: Given $a \in A$, (a is a subset), define $b \in B$ (b is a word): If $s_i \in a$, then letter i in b is 1. If $s_i \notin a$, then letter i in b is 0.

Difficulties:

- Finding the function or rule (requires rearranging, ordering)
- Proving the function or rule (show it **IS** a bijection).

What is a Function?

Reminder: A **function** f from A to B (write $f : A \to B$) is a rule where for each element $a \in A$, $f(a)$ is defined as an element $b \in B$ (write $f : a \mapsto b$).

- A is called the **domain**. (We write $A = \text{dom}(f)$)
- B is called the **codomain**. (We write $B = \text{cod}(f)$)
- The **range** of f is the set of values that f takes on:

$$\text{rng}(f) = \{b \in B : f(a) = b \text{ for at least one } a \in A\}$$

Example. Let A be the set of 3-subsets of $[n]$ and let B be the set of 3-lists of $[n]$. Then define $f : A \to B$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in A$ (with $i_1 \leq i_2 \leq i_3$) to the word $i_1 i_2 i_3 \in B$.

Question: Is $\text{rng}(f) = B$?

What is a Bijection?

Definition: A function $f : A \to B$ is **one-to-one** (an **injection**) when

For each $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Equivalently,

For each $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

"When the inputs are different, the outputs are different." (picture)

Definition: A function $f : A \to B$ is **onto** (a **surjection**) when

For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

"Every output gets hit."

Definition: A function $f : A \to B$ is a **bijection** if it is both one-to-one and onto.

The function from the previous page is ________________________.

What is an example of a function that is onto and not one-to-one?

Proving a Bijection

Example. Use a bijection to prove that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$.

Proof. Let A be the set of k-subsets of $[n]$
and let B be the set of $(n-k)$-subsets of $[n]$.

A bijection between A and B will prove $\binom{n}{k} = |A| = |B| = \binom{n}{n-k}$.

Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

$$\left\{ \begin{matrix} \{1,2\}, \ \{1,3\} \\ \{1,4\}, \ \{1,5\} \\ \{2,3\}, \ \{2,4\} \\ \{2,5\}, \ \{3,4\} \\ \{3,5\}, \ \{4,5\} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \{1,2,3\}, \ \{1,2,4\} \\ \{1,2,5\}, \ \{1,3,4\} \\ \{1,3,5\}, \ \{1,4,5\} \\ \{2,3,4\}, \ \{2,3,5\} \\ \{2,4,5\}, \ \{3,4,5\} \end{matrix} \right\}$$

Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) =$ ______.

Proving a Bijection

Step 2: Prove f is well defined.

The function f is well defined. If S is any k-subset of $[n]$, then S^c is a subset of $[n]$ with $n-k$ members. Therefore $f : A \to B$.

Step 3: Prove f is a bijection.

Strategy. Prove that f is both one-to-one and onto.

f is 1-to-1: Suppose that S_1 and S_2 are two k-subsets of $[n]$ such that $f(S_1) = f(S_2)$. That is, $S_1^c = S_2^c$. This means that for all $i \in [n]$, then $i \notin S_1$ if and only if $i \notin S_2$. Therefore $S_1 = S_2$ and f is 1-to-1.

f is onto: Suppose that $T \in B$ is an $(n-k)$-subset of $[n]$. We must find a set $S \in A$ satisfying $f(S) = T$. Choose $S =$ ______. Then $S \in A$ (why?), and $f(S) = S^c = T$, so f is onto.

We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

Alternative methods to prove bijections

Prove that a rule f is a bijection by finding f's **inverse**:

- Determine a rule for a candidate inverse function g.
- Show that f is a well defined function from A to B.
- Show that g is a well defined function from B to A.
- Show for all $a \in A$, $g(f(a)) = a$
  and for all $b \in B$, $f(g(b)) = b$

When g is the inverse of f, both f and g are bijections.

Using the inverse function

Example. There exists as many even-sized subsets of $[n]$ as odd-sized subsets of $[n]$.

$$\begin{array}{rl} \text{even:} & \{\ \emptyset,\ \{s_1,s_2\},\{s_1,s_3\},\ \{s_2,s_3\}\ \} \\ \text{odd:} & \{\{s_1\},\ \{s_2\},\ \ \{s_3\},\ \{s_1,s_2,s_3\}\} \end{array}$$

Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

- f is a well defined function from A to B (why?).
- f is also a well defined function from B to A (why?).
- f^2 is the identity function.

Therefore, f is a bijection, proving the statement, as desired.

Eyebrow-Raising Consequence: $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$