Compositions

Question: In how many ways can we write a positive integer n as a sum of positive integers?

If order doesn't matter:

A **partition**: $n = p_1 + p_2 + \cdots + p_\ell$ for positive integers $p_1, p_2 \ldots, p_\ell$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_\ell$.

If order does matter:

A **composition**: $n = i_1 + i_2 + \cdots + i_\ell$ for positive integers $i_1, i_2 \ldots, i_\ell$ with no restrictions.

$$\begin{array}{cl}
4 & \begin{cases} 4 \end{cases} \\
3+1 & \begin{cases} 3+1 \\ 1+3 \end{cases} \\
2+2 & \begin{cases} 2+2 \end{cases} \\
2+1+1 & \begin{cases} 2+1+1 \\ 1+2+1 \\ 1+1+2 \end{cases} \\
1+1+1+1 & \begin{cases} 1+1+1+1 \end{cases}
\end{array}$$

There are 2^{n-1} compositions of n.

Compositions of Generating Functions

Question: Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$.
What can we learn about the composition $H(x) = F\big(G(x)\big)$?

Investigate $F(x) = 1/(1-x)$.

$$H(x) = F(G(x)) = \frac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + G(x)^3 + \cdots .$$

- This is an infinite sum of (likely infinite) power series. Is this OK?
- The constant term h_0 of $H(x)$ only makes sense if ______ .
- This implies that x^n divides $G(x)^n$.
  Hence, there are at most $n-1$ summands which contain x^{n-1}.
  We conclude that the infinite sum makes sense.

For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

Compositions. of. Generating Functions.

Interpreting $$\frac{1}{1-G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots :$$

Recall: The generating function $G(x)^n$ counts sequences of length n of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k](G(x)^n)$ counts those n-sequences that have total size equal to k.

Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

Alternatively: Interpret $[x^k]\frac{1}{1-G(x)}$ thinking of k as this *total size*. First, find **all ways** to break down k into integers $i_1 + \cdots + i_\ell = k$. Then create **all sequences** of objects of type G in which object j has size i_j.

Think: A composition of generating functions equals
a composition. of. generating. functions.

An Example, Compositions

Example. How many compositions of k are there?

Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

The objects in the sequence are positive integers; we need the g.f. that counts how many positive integers there are with "size i".

What does size correspond to?

How many have value i? Exactly one: the number i.

So the generating function for our objects is
$G(x) = 0 + 1x^1 + 1x^2 + 1x^3 + 1x^4 + \cdots =$ ____________________.

We conclude that the generating function for compositions is
$H(x) = \frac{1}{1-G(x)} =$

So the number of compositions of n is

A Composition Example

Example. How many ways are there to take a line of k soldiers, divide the line into non-empty platoons, and from each platoon choose one soldier in that platoon to be a leader?

Solution. A soldier assignment corresponds to a sequence of platoons of size $(i_1, \ldots, i_\ell)$.

Given i soldiers in a platoon, in how many ways can we assign the platoon a leader? __________

Therefore $G(x) =$

And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$

Domino Tilings

Example. How many square-domino tilings are there of a $1 \times n$ board?

Solution. A tiling corresponds to a sequence $(i_1, \ldots, i_\ell)$,
where i_j ____________________.
So $G(x) =$ __________________, and therefore $H(x) =$ ________________.

Another way to see this:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$G(x)^0 =$													
$G(x)^1 =$													
$G(x)^2 =$													
$G(x)^3 =$													
$G(x)^4 =$													
$G(x)^5 =$													
$G(x)^6 =$													
$1/(1 - G(x)) =$													