Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), ... (a^8) to (y^8).

Now a book!

triangulations of an $(n+2)$-gon	lattice paths from $(0,0)$ to (n,n) above $y=x$
sequences with n $+1$'s, n -1's with positive partial sums	multiplication schemes to multiply $n+1$ numbers

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n+2)$-gon

2. Lattice paths from $(0,0)$ to (n,n) staying above $y = x$

3. Sequences of length $2n$ with n $+1$'s and n -1's such that every partial sum is ≥ 0

4. Ways to multiply $n+1$ numbers together two at a time.

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Bijection 1: | triangulations of an $(n+2)$-gon | $\longleftrightarrow$ | multiplication schemes to multiply $n+1$ numbers |

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

Catalan Bijections

Bijection 2: | multiplication schemes to multiply $n+1$ #s | $\longleftrightarrow$ | seqs with n $+1$'s, n -1's with positive partial sums |
|---|---|---|

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Catalan Bijections

Bijection 3:

seqs with n $+1$'s, n -1's with positive partial sums	$\longleftrightarrow$	lattice paths $(0,0)$ to (n,n) above $y=x$

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

Example. triangulations of an $(n+2)$-gon

Here, x represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): x^0.
2. Every other triangulation has one side (x contribution) and is a sequence of two other triangulations $C(x)^2$.

Catalan Numbers

Example.

lattice paths $(0,0)$ to (n,n) above $y=x$

Here, x represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: x^0.
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (x contribution). "Between the V & H steps" and "after the H step" is a sequence of two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k \geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k$. (Next slide.)

Conclusion.
$$\begin{aligned}\tfrac{1}{2x}(1-\sqrt{1-4x}) &= -\tfrac{1}{2x}\sum_{k\geq 1} \tfrac{-2}{k}\binom{2(k-1)}{k-1} x^k \\ &= \sum_{k\geq 1} \tfrac{1}{k}\binom{2(k-1)}{k-1} x^{k-1} \\ &= \sum_{n\geq 0} \tfrac{1}{n+1}\binom{2n}{n} x^n\end{aligned}$$

Therefore, $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\begin{aligned}
\sqrt{1-4x} &= ((-4x)+1)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Expand } \tbinom{1/2}{k} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Denom. of } \tfrac{1}{2} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Factor } -2\text{'s} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Simplify; rewrite prod.} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k && \text{Write as factorials} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k
\end{aligned}$$