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The cardinality of a set is a combinatorial statistic on $\mathcal{S}$.

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*less information* *more information*

$\longleftarrow\qquad\longrightarrow$

**counting** **statistics** **complete enumeration**

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**counting**

8

**statistics**

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 3 | 3 | 1 |

**complete enumeration**

$\emptyset \quad \{1\} \quad \{2\} \quad \{3\}$

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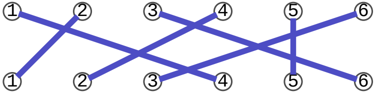
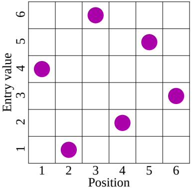



(only two crossings at a time)

Matrix-like diagram:



# Descent statistic

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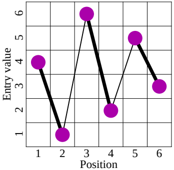


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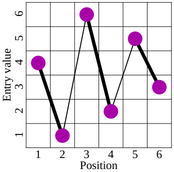


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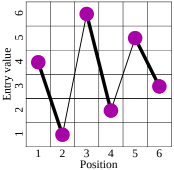


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|---|---|---|---|---|---|
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| 2 | 1 | 1 | | | |
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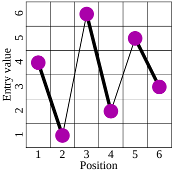


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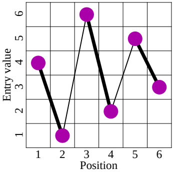


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These are the **Eulerian numbers**.

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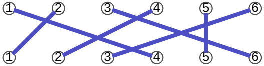


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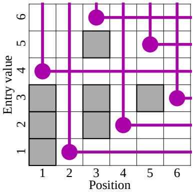

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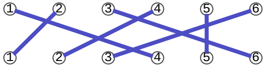


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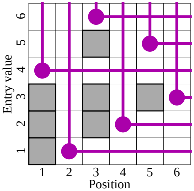


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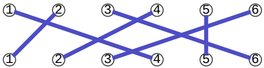


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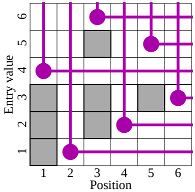


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The inversion number is a good way to count how “far away” a permutation is from the identity.

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Define maj($\pi$), the **major index** of $\pi$, to be sum of the descents of $\pi$.

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A statistic that has the same distribution as inv is called **Mahonian**.

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*Definition:* A $q$-**analog** of a number $c$ is an expression $f(q)$ such that $\lim_{q\to 1} f(q) = c$.

Example. $\dfrac{1-q^n}{1-q} = \left(1 + q + q^2 + \cdots + q^{n-2} + q^{n-1}\right)$ is a $q$-analog of $n$ because $\lim_{q\to 1} \frac{1-q^n}{1-q} = n$.

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|---|---|---|
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| 2 | $1q^0 + 1q^1$ | $= (1+q)$ |
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| 3 | $1q^0 + 2q^1 + 2q^2 + 1q^3$ | $= (1+q+q^2)(1+q)$ |
| 4 | $1q^0 + 3q^1 + 5q^2 + 6q^3 + 5q^4 + 3q^5 + 1q^6$ | $=$ |

*Conjecture:* $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q \cdots [1]_q =: [n]_q!$, the $q$-**factorial**.

*Claim:* This equation makes sense when $q = 1$.

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*Proof.* There exists a bijection

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$$\begin{aligned} \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n} \\ &= \left( \sum_{a_1=0}^{n-1} q^{a_1} \right) \left( \sum_{a_2=0}^{n-2} q^{a_2} \right) \cdots \left( \sum_{a_n=0}^{0} q^{a_n} \right) \\ &= \quad [n]_q \quad [n-1]_q \quad \cdots \quad [1]_q \quad = \quad [n]_q! \end{aligned}$$

# Notes

We said that inv and maj are equidistributed. Two possible proofs:

- Find a bijection $f : S_n \to S_n$ such that $\mathrm{maj}(\pi) = \mathrm{inv}(f(\pi))$.
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Define $\mathrm{inv}(\pi) = |\{i < j \colon \pi(i) > \pi(j)\}|$.

Example. $\pi = 1\,1\,2\,2\,1\,2\,1\,1\,2\,2$ is a permutation of $\{1^5, 2^5\}$.
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This can also be used to give a $q$-analog of the Catalan numbers.

# There's always more to learn!!!

**References :**


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