# Combinatorics of Core Partitions

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qc.edu/~chanusa/research

# Partitions

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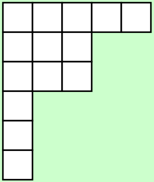
The **hook length** of a box = # boxes below + # boxes to right + box

$\lambda$ is an ***a*-core** if no boxes have hook length $a$.



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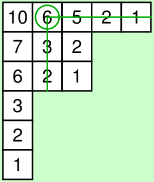


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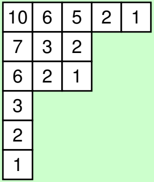


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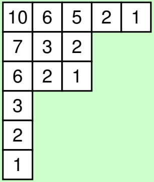


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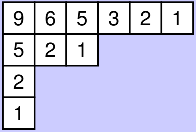


*Simultaneous*
$(4, 7)$-core partition

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| 5 | 2 | 1 | | | |
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- (Anderson, 2002): # $(a, b)$-core partitions equals $\frac{1}{a+b}\binom{a+b}{a}$.

# Partitions and Abacus Diagrams

An **abacus diagram** is a function $\mathcal{A} : \mathbb{Z} \to \{\bullet, \text{␣}\}$.

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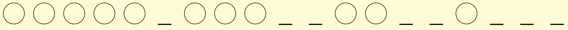

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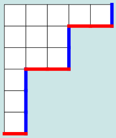

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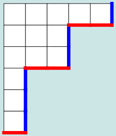

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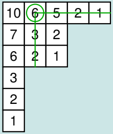


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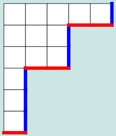

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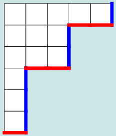
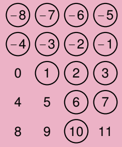
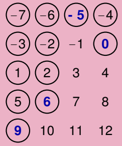



**Normalized**




**Balanced**



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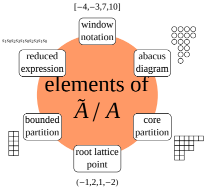
- **Group Theory:** By Lascoux 2001, $a$-cores $\longleftrightarrow$ coset reps in $\widetilde{S}_a/S_a$
  Group actions on combinatorial objects!!!!



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| | |
|---|---|
| 123 | 123 |
| 213 | 132 |
| 231 | 312 |
| 321 | 321 |

These **generators** interact:

- Consecutive generators don't commute: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
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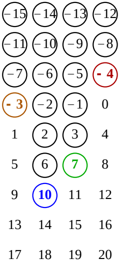
**Affine *n*-Permutations** $\pi \in \widetilde{S}_n$

- Generators: $\{s_0, s_1, \ldots, s_{n-1}\}$
- Can think of as permutations of $\mathbb{Z}$.
- Window notation: $[-4, -3, 7, 10]$

# Action of generators on abacus diagrams

(James and Kerber, 1981) Given an affine permutation $[w_1, \ldots, w_n]$,

- Create a balanced abacus on $n$ runners where each runner has a lowest bead at $w_i$.

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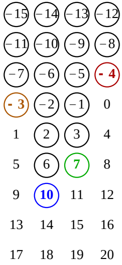


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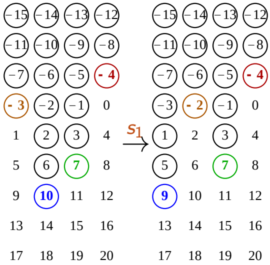


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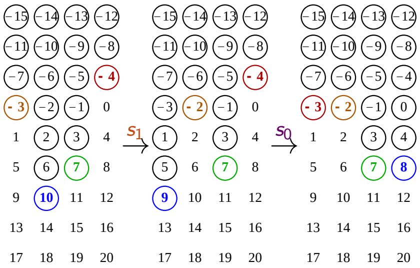


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# Action of generators on core partition

- Label the boxes of $\lambda$ with residues.
- $s_i$ acts by adding or removing boxes with residue $i$.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 0 | 1 |
| 3 | 0 | 1 | 2 | 3 | 0 |
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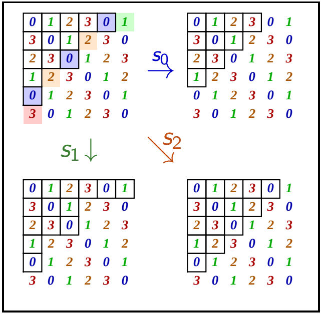

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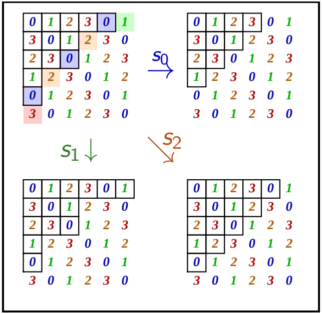
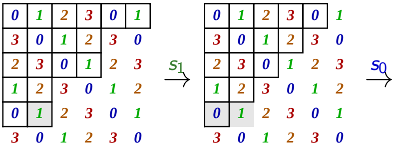


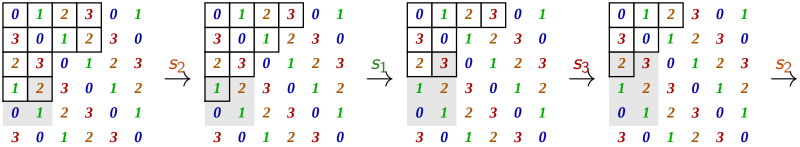

Idea: We can use this to figure out a *word* for $\lambda$.

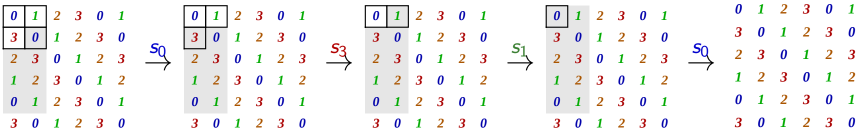
# Finding the word corresponding to a core partition.

*Example:* The word in $S_4$ corresponding to $\lambda = (6,4,4,2,2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$



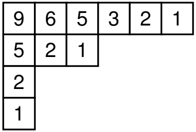



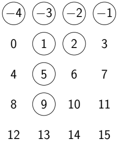



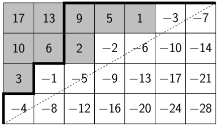
# Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{ \begin{array}{c} \text{simultaneous} \\ (a,b)\text{-cores} \end{array} \right\} \overset{\textit{James}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-flush} \\ \text{balanced abaci} \end{array} \right\} \overset{\textit{And.}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-Dyck paths} \\ (0,0) \to (b,a) \\ \text{above } y = \frac{a}{b}x \end{array} \right\}$$



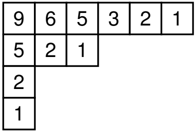



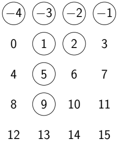



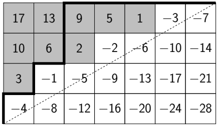
# Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{ \begin{array}{c} \text{simultaneous} \\ (a,b)\text{-cores} \end{array} \right\} \overset{James}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-flush} \\ \text{balanced abaci} \end{array} \right\} \overset{And.}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-Dyck paths} \\ (0,0) \to (b,a) \\ \text{above } y = \frac{a}{b}x \end{array} \right\}$$




Proof that the number of $(a,b)$-Dyck paths is $\frac{1}{a+b}\binom{a+b}{a}$: (Bizley '55)

- Path rotation gives an equivalence relation on the set of all lattice paths from $(0,0) \to (b,a)$.
- There are $\binom{a+b}{a}$ such paths and the equivalence classes have $a+b$ elements each.

# Familiar numbers

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| # of $(t, t+1)$-cores: | | | | | | | |

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|---|---|---|---|---|---|---|---|
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Specialize Anderson's result:

$$\#\ (t, t+1)\text{-cores}$$
$$\frac{1}{2t+1}\binom{2t+1}{t}$$

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**Question:** Is there a simple statistic on simultaneous core partitions that gives us a $q$-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{stat}(\lambda)} = \frac{1}{[t+1]_q}\begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

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**Answer: Yes.** We will create an analog of the **major statistic**.

## The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of $\pi$:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

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For a $(t, t+1)$-core $\lambda$, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#$ 1st col. boxes with hook length $\equiv i \bmod t$.

Define $$\text{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

See: maj defined as a sum over descents in a sequence.

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**Theorem.** (AHJ '13)

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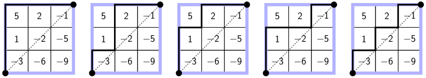
Why? Major index on Dyck paths!

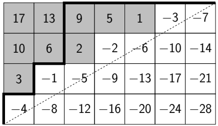



Add positions of valleys: $\frac{1}{[4]_q}\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = q^0 + q^2 + q^3 + q^4 + q^{2+4}$

# The Zeta Map (via cores)

**Follow this recipe:**

1. Start with any $(a, b)$-Dyck path $P$.



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**Follow this recipe:**

1. Start with any $(a, b)$-Dyck path $P$.
2. Find the corresponding $(a, b)$-core $\kappa$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 17 | 13 | 9 | 5 | 1 | −3 | −7 |
| 10 | 6 | 2 | −2 | −6 | −10 | −14 |
| 3 | −1 | −5 | −9 | −13 | −17 | −21 |
| −4 | −8 | −12 | −16 | −20 | −24 | −28 |

| | | | | | |
|---|---|---|---|---|---|
| 9 | 6 | 5 | 3 | 2 | 1 |
| 5 | 2 | 1 | | | |
| 2 | | | | | |
| 1 | | | | | |

# The Zeta Map (via cores)

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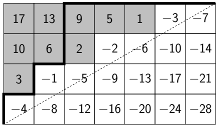


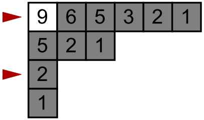

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4. Let $\lambda$ be the partition
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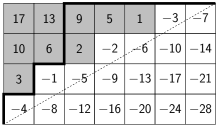


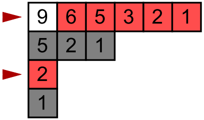


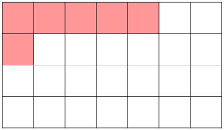

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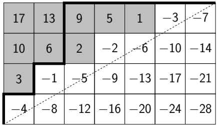


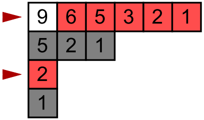


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5. Draw the $(a, b)$-Dyck path $Q$
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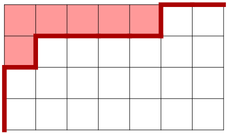

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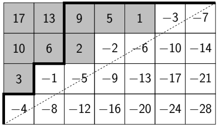


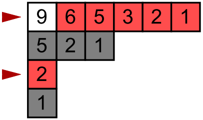


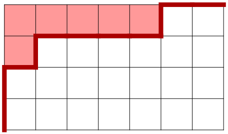

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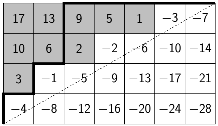


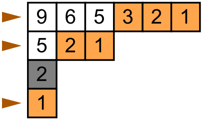


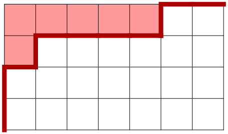

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4. Let $\lambda$ $(\mu)$ be the partition
   with those number of boxes.
5. Draw the $(a, b)$-Dyck path $Q$ $(R)$
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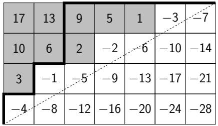


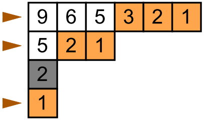


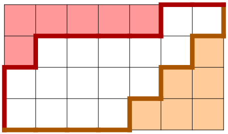

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**Follow this recipe:**

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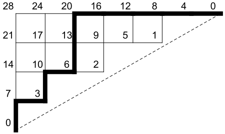
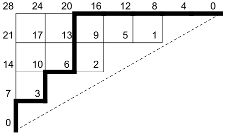


# The Zeta Map (via the sweep map)

**Follow this recipe:**

1. Start with any $(a, b)$-Dyck path $P$.
2. Assign to each lattice point its level.



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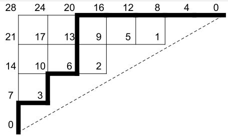


**Follow this recipe:**

1. Start with any $(a, b)$-Dyck path $P$.
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$$\overset{N}{0}, \overset{E}{7}, \overset{N}{3}, \overset{E}{10}, \overset{N}{6}, \overset{N}{13}, \overset{E}{20}, \overset{E}{16}, \overset{E}{12}, \overset{E}{8}, \overset{E}{4}$$

3. Write down the sequence of levels $(0, 0) \rightsquigarrow (b, a)$ with their associated $N$ or $E$.

# The Zeta Map (via the sweep map)

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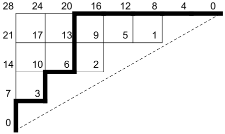


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4. Sort these from smallest to largest, permuting the $N$'s and $E$'s too.

# The Zeta Map (via the sweep map)

**Follow this recipe:**

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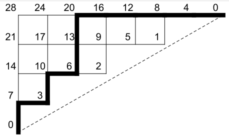
4. Sort these from smallest to largest, permuting the $N$'s and $E$'s too.
5. Read the steps as a new $(a, b)$-Dyck path $Q$.

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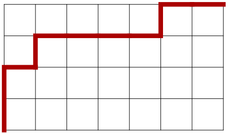
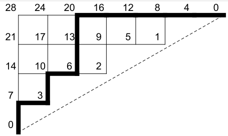


# The Zeta Map (via the sweep map)

**Follow this recipe:**




1. Start with any $(a, b)$-Dyck path $P$.
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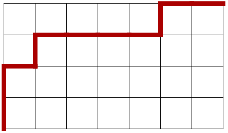
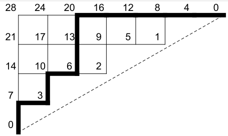
$N\ E\ N\ E\ N\ N\ E\ E\ E\ E\ E$
$0, 7, 3, 10, 6, 13, 20, 16, 12, 8, 4$

$N\ N\ E\ N\ E\ E\ E\ E\ N\ E\ E$
$0, 3, 4, 6, 7, 8, 10, 12, 13, 16, 20$



This **is also** the zeta map; $\zeta(P) = Q$!

# The Zeta Map (via the sweep map)

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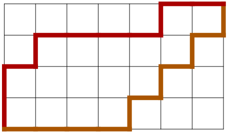
4. Sort these from smallest to largest, permuting the $N$'s and $E$'s too.
5. Read the steps as a new $(a, b)$-Dyck path $Q$. ( $R$ after rotating 180°.)

N E N E N N E E E E E
0, 7, 3, 10, 6, 13, 20, 16, 12, 8, 4

E E E E E N N E N E N
0, 4, 8, 12, 16, 20, 13, 6, 10, 3, 7

N N E N E E E E N E E
0, 3, 4, 6, 7, 8, 10, 12, 13, 16, 20

E E E E N E N E N E N
0, 3, 4, 6, 7, 8, 10, 12, 13, 16, 20



This **is also** the zeta map; $\zeta(P) = Q$! (Or the eta map $\eta(P) = R$! )

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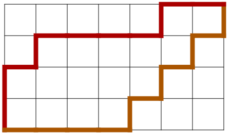

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 7 & 1 & 9 & 2 & 3 & 4 & 6 & 11 & 8 & 10 \end{array}$$

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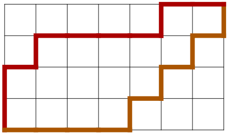

| | | | | | | | | | | |
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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
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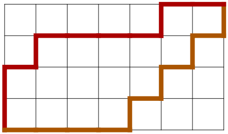

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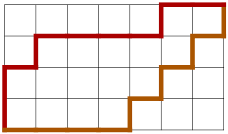
4. Recover $P$ from $\gamma$ by converting ascents to $N$ and descents to $E$.



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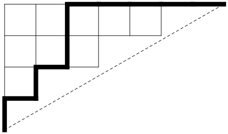

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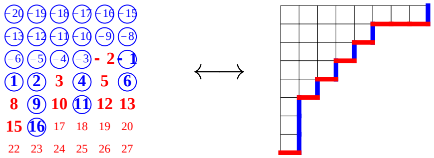

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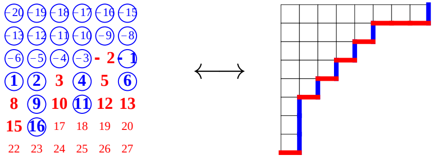

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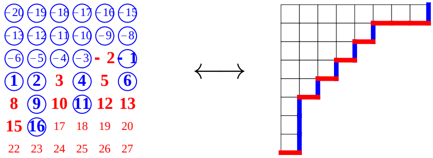

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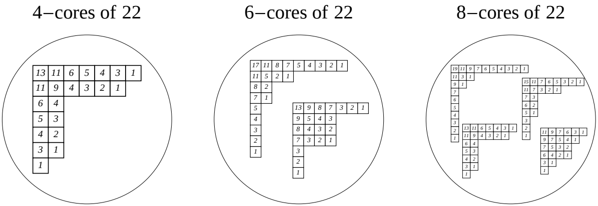

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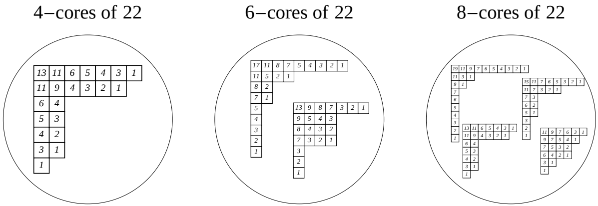

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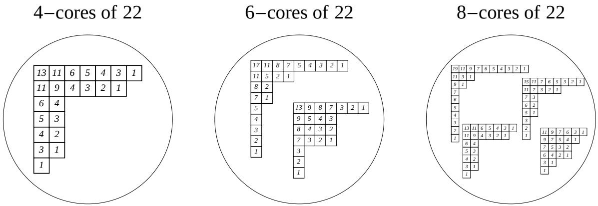

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- *Progress:* Mystère et boule de gomme!

★ Happy to have students who would like to do research! ★

# Course Evaluation

Please comment on:

- Prof. Chris's effectiveness as a teacher.
- Prof. Chris's contribution to your learning.
- The course material: What you enjoyed and/or found challenging.
- Is there anything you would change about the course?
- How did the reality of the course compare to your expectations?
- Is there anything else Prof. Chris should know?

Place completed evaluations in the provided folder.