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Matching [1 pt each] How many? The answers (our of order) are:

15, 32, 37, 64, 240, 360, 671, 1296

1. How many length-4 lists from symbols A, B, C, D, E, F are possible if repetition is allowed?

Answer. $6^4 = 1296$.

2. How many length-4 lists from symbols A, B, C, D, E, F are possible if repetition is not allowed?

Answer. $6 * 5 * 4 * 3 = 360$.

3. How many length-4 lists from symbols A, B, C, D, E, F are possible if repetition is not allowed and the symbol E must appear somewhere in the list?

Answer. Think of these lists as coming in 4 types. The first type has an 'E' in the first position, the second type has an 'E' in the second position, and so on. For each type, we have choose three letters from the symbols A, B, C, D, F resulting in $5 * 4 * 3 = 60$ choices for each of the four types. That's $60 + 60 + 60 + 60 = 240$ lists in total.

4. How many length-4 lists from symbols A, B, C, D, E, F are possible if repetition is allowed and the symbol E must appear at least once in the list?

Answer. There are 1296 lists total. Let's subtract the number of lists that *do not* have an 'E'. There are $5^4 = 625$ of them. That leaves $1296 - 625 = 671$ lists that have an E somewhere in the list.

5. How many subsets of {A, B, C, D, E, F} are there?

Answer. $2^6 = 64$

6. How many subsets of {A, B, C, D, E, F} have four elements?

Answer. $\binom{6}{4} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = 15$.

7. How many subsets of {A, B, C, D, E, F} have an E in them?

Answer. Think of listing these sets by listing all the subsets of {A, B, C, D, F, G} and then adding an 'E' to each. The result is $2^5 = 32$ sets.

8. How many subsets of {A, B, C, D, E, F} either have an E in them or have four elements?

Answer. First, we count the four element subsets that also have an 'E' in them. We do that by counting the three element subsets of {A, B, C, D, F}, since addint an 'E' to these sets results in a four element subset of {A, B, C, D, E, F} with an 'E'. There are $\binom{5}{3} = 10$ of them.

Now, we use the inclusion-exclusion principle. There are 32 subsets with an 'E'. There are 15 subsets with four elements. There are 10 subsets that both have four elements and contain an 'E'. So, there are $32 + 15 - 10 = 37$ subsets that either have an E in them or have four elements.

Mathematical writing [3 pts each] Prove each of the following statements. Write your proofs clearly and carefully on the back of your answer sheet.

9. Suppose $n \in \mathbb{Z}$. Prove that if n^2 is odd, then n is odd.

Answer. We will prove the contrapositive. Suppose that n is not odd. This means n is even. This means that $n = 2k$ for some integer $k \in \mathbb{Z}$. Squaring yields $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Let l be the integer $l = 2k^2$. We've written $n^2 = 2l$ for an integer $l \in \mathbb{Z}$, so n^2 is even, hence n^2 is not odd.

10. Prove that if $a, b \in \mathbb{Z}$ then $a^2 - 4b - 3 \neq 0$.

Answer. Let's prove this by contradition. Suppose that $a, b \in \mathbb{Z}$ and that $a^2 - 4b - 3 = 0$. Then we have $a^2 = 4b + 3$. Since $4b$ is even and 3 is odd, the sum a^2 is odd. By the proposition above, a^2 being odd implies a is odd. Therefore $a = 2k + 1$ for some integer k. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1$. Setting this expression for a^2 equal to $4b + 3$ yields

$$4k^2 + 4k + 1 = 4b + 3 \Rightarrow 4k^2 + 4k - 4b = 2.$$

This is a contradiction since the left hand side is a multiple of 4 but the right hand side is not.