True/False [1 pt each] For each of the following statements, decide whether it is true or false. Put **T** or **F** on the answer sheet.

1. Let A = {red square, red octagon, blue triangle, yellow hexagon, blue hexagon}. Let $P(x, y)$ be the propositional function *"x has the same number of sides as y"* and let $Q(x, y)$ be the propositional function *"x and y have the same color."*

$$\forall x \in A\ \exists y \in A\ (\sim P(x, y) \wedge Q(x, y)) \vee (P(x, y) \wedge \sim Q(x, y))$$

Answer. True. For every element of A, there is another element that either has the same number of sides and a different color, or the same color and a different number of sides. You can go through the elements of A to check: red square has red octagon, red octagon has red square, blue triangle has blue hexagon, yellow hexagon has blue hexabon, and blue hexagon has blue triangle.

2. For any propositions P, Q, and R, the compound propositions $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$ are logically equivalent. (*Hint:* make a truth table.)

Answer. This is false, as a truth table will show. If P, Q, and R are all false, then $Q \Rightarrow R$ is true and so $P \Rightarrow (Q \Rightarrow R)$ has the form $F \Rightarrow F$ which is true. However, $(P \Rightarrow Q)$ is true, so the implication $(P \Rightarrow Q) \Rightarrow R$ has the form $T \Rightarrow F$ which is false.

3. There are $(7)(6)(5)(4) = 840$ different injective function $f : \{1, 2, 3, 4\} \longrightarrow \{A, B, C, D, E, F, G\}$.

Answer. True. We construct such an f by listing 4 distinct elements of A, B, C, D, E, F, G for the values of $f(1), f(2), f(3), f(4)$. Since there are 7 possible choices for $f(1)$, 6 for $f(2)$, 5 for $f(3)$, and 4 for $f(4)$, we have a total of $(7)(6)(5)(4) = 840$ possibilities.

4. If A, B, C are subsets of some universal set U, then $A - (B - C) = (A - B) - C$. (*Hint:* draw a Venn diagram or try an example.)

Answer. False. For an example, let $A = \{2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, and $C = \{0, 2, 4, 6\}$. Then $(B - C) = \{5, 7\}$ and $A - (B - C) = \{2, 3, 4\}$. On the other hand, $A - B = \{2, 3\}$ and $(A - B) - C = \{3\}$.

5. If $f : X \longrightarrow Y$ is a function and $A \subseteq X$ then $f^{-1}(f(A)) = A$.

Answer. False. Let $f : \{1, 2, 3, 4\} \longrightarrow \{r, g, b, y\}$ be given by

$$\begin{aligned} 1 &\mapsto r \\ 2 &\mapsto r \\ 3 &\mapsto b \\ 4 &\mapsto g \end{aligned}$$

Let $A = \{1, 3\}$. Then $f(A) = \{r, b\}$ and $f^{-1}(f(A)) = f^{-1}(\{r, b\}) = \{1, 2, 3\}$.

6. If $f : X \to Y$ is an injective function and $A \subseteq X$ then $f^{-1}(f(A)) = A$.

Answer. True.

For any function $f : X \to Y$ and any set $A \subseteq X$, we have $A \subset f^{-1}(f(A))$. To prove it, let $a \in A$. Then $f(a) \in f(A)$. Since $f(a) \in f(A)$ we have $a \in f^{-1}(f(A))$.

Now, suppose f is injective. To show that $f^{-1}(f(A)) \subseteq A$, let $a \in f^{-1}(f(A))$. This means that $f(a) \in f(A)$. Therefore, there exists an element $a' \in A$ with $f(a') = f(a)$. Since f is injective, $a' = a$ and we've shown $a \in A$.

7. For each $n \in \mathbb{N}$, $2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 2$.

Answer. True. We use induction. For the base step, notice that when $n = 1$, the statement $2 = 2^2 - 2$ is true.

For the inductive step, suppose that $2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 2$. Adding 2^{n+1} to both sides yields $2 + 4 + 8 + 16 + \cdots + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2(2^{n+1}) - 2 = 2^{n+2} - 2$, completing the inductive step.

Short answer [3 points]

8. Choose one of the true/false problems above and explain why it is true or false. Write your answer clearly and carefully. Neatness counts.