Name: ______________________________

1	T
2	F
3	F
4	T
5	T
6	F
7	T
8	F
9	T
10	T
11	T
12	T
13	F
14	F
15	T

Part I: True/False [1 pt each]

For each of the following statements, decide whether it is true or false. Put **T** or **F** on the answer sheet.

1. $\dfrac{1}{3+4i} = \dfrac{3}{25} - \dfrac{4}{25}i.$

Answer. True. The expression $\dfrac{1}{3+4i}$ means "the number that when multiplied by $3+4i$ results in 1". So, you just check

$$\left(\frac{3}{25} - \frac{4}{25}i\right)(3+4i) = \frac{9}{25} + \frac{16}{25} + \left(\frac{12}{25} - \frac{12}{25}\right)i = 1.$$

2. There is only one number $\alpha \in \mathbb{C}$ so that $\alpha^3 = 1$.

Answer. False. $1^3 = 1$ and $\left(\dfrac{-1+\sqrt{3}\,i}{2}\right)^3 = 1$.

3. The set $\{f : [0,1] \to \mathbb{R} : f \text{ is one-to-one}\}$ is a subspace of $\mathbb{R}^{[0,1]}$.

Answer. False. The zero function isn't one-to-one.

4. The set $\{(a,b,c,d) \in \mathbb{R}^4 : c = 2a\}$ is a subspace of $\mathbb{R}^4$.

Answer. True. Note that $(0,0,0,0)$ is the given set. If (a,b,c,d) satisfies $c = 2a$ and (a',b',c',d') satisfies $c' = 2a'$ then the sum $(a+a', b+b', c+c', d+d')$ satisfies $c + c' = 2a + 2a' = 2(a+a')$, so the set is closed under addition. And if (a,b,c,d) satisfies $c = 2a$ then $\lambda(a,b,c,d) = (\lambda a, \lambda b, \lambda c, \lambda d)$ satisfies $\lambda c = 2\lambda a$ showing the set is closed under scalar multiplication.

5. The list $(1+i, 1+i, 1+i), (1+i, 0, 0)$ is independent in $\mathbb{C}^3$.

Answer. True. For a list of two vectors to be dependent, one has to be a multiple of the other, which isn't true here.

6. The vectors $(1,1,1,1,1), (0,1,0,1,0), \left(\frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}\right), (1,2,3,4,5)$ span $\mathbb{R}^5$.

Answer. False. The dimension of $\mathbb{R}^5$ is five. So no list of fewer than five vectors can span $\mathbb{R}^5$.

7. There exist real numbers a, b, c, d, e, f, not all zero, with

$$a(1,1,1,1,1) + b(0,1,0,1,0) + c\left(\frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}\right) + d(1,2,3,4,5) + e(1,2,9,1,1) + f(-1,-1,-2,-2,-3) = (0,0,0,0,0).$$

Answer. True. This is the statement that the followin list is dependent

$$(1,1,1,1,1), (0,1,0,1,0), \left(\frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}\right), (1,2,3,4,5), (1,2,9,1,1), (-1,-1,-2,-2,-3)$$

which must be true since every list with greater than five vectors in the five dimensional space $\mathbb{R}^5$ is dependent.

8. The list of polynomials $(x-1), (x-1)^2, (x-1)^3$ is a basis for $\mathcal{P}_3(\mathbb{R})$.

Answer. False. No list of three polynomials can span the four dimensional space $\mathcal{P}_3(\mathbb{R})$.

9. The list of polynomials $(x-1), (x-1)^2, (x-1)^3$ is a basis for the space $\{p \in \mathcal{P}_3(\mathbb{R}) : p(1) = 0\}$.

Answer. True. Since the degrees of these polynomials strictly increases, no polynomial in this list can be a linear combination of the previous ones. So, this list is independent. Since the space in question is a proper subspace of a four dimensional space, the dimension must be at most three. Since we have a list of three independent vectors in a space of dimension at most three, the dimension of that space must be three and the given list must be a basis.

Part II: more True/False [1pt each]

For problems 10–15, consider the following subspaces of $\mathbb{R}^3$

$$W = \{(0,0,a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$$
$$X = \{(a,a,a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$$
$$Y = \{(a,b,c) \in \mathbb{R}^3 : a+b+c = 0\}$$
$$Z = \{(a,a,b) \in \mathbb{R}^3 : a,b \in \mathbb{R}\}$$

10. $(1,1,-2) \in Y \cap Z$

Answer. True. To be in Y, the sum of the entries must be zero. To be in Z, the first two entries must be the same.

11. $W \subseteq Z$

Answer. True. Every vector in W has 0 for the first two entries. Since the first two entries are the same, those vectors are also in Z.

12. $W \cap X = \{(0,0,0)\}$

Answer.

True. If $(a,b,c) \in W$, then $a = b = 0$. If $(0,0,c)$ is in X, then $c = 0$.

13. $\dim(Y) = 1$

Answer. False. $(1,1,-2)$,$(1,0,-1)$ is a linearly independent list in Y, hence $\dim(Y) \geq 2$.

14. $\mathbb{R}^3 = Y \oplus Z$

Answer. False. Since $(1,1,-2) \in Y \cap Z$, we see $Y \cap Z \neq 0$ so the sum isn't direct.

15. $Z = W \oplus X$

Answer. True. We have $W \cap X = 0$. To see that $W + X = Z$, notice that $(a,a,b) = (a,a,a) + (0,0,b-a)$, which expresses an arbitrary element of Z as the sum of a vector in W and a vector in X.

Part III: Short answer [2 points]

16. Choose one of the true/false problems above and explain why it is true or false. Write your answer clearly and carefully. Neatness counts.