Name:

1	b
2	e
3	a
4	c
5	T
6	F
7	T
8	T
9	T
10	T

Mutliple Choice

1. Which of the following is *not* a field?

(a) The numbers $\{0, 1\}$ with $+$ and $\times$ defined modulo 2

(b) The natural numbers $\mathbb{N}$

(c) The real numbers $\mathbb{R}$

(d) The complex numbers $\mathbb{C}$

Answer. (b) The natural numbers $\mathbb{N}$ are not a field. For one reason, there's no additive identity. For another, $3 \in \mathbb{N}$ has no multiplicative inverse.

2. The multiplicative inverse of $1 + i$ is

(a) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

(b) $1 - i$

(c) $\dfrac{i}{2}$

(d) $\sqrt{2} - \sqrt{2}i$

(e) $\dfrac{1}{2} - \dfrac{1}{2}i$

Answer. (e). To check, multiply $(1+i)\left(\dfrac{1}{2} - \dfrac{1}{2}i\right) = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right)i = 1$.

3. Which number satisfies $z^8 = 1$?

(a) $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

(b) $z = 1 - i$

(c) $z = \dfrac{i}{2}$

(d) $z = \sqrt{2} - \sqrt{2}i$

(e) $z = \dfrac{1}{2} - \dfrac{1}{2}i$

Answer. (a) You can check this using polar coordinates since (a) is positioned one unit away from the origin at an angle of $\pi/4$. Also straightforward to check directly: $\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^2 = i$ so $\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^4 = i^2 = -1$ and so $\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^8 = (-1)^2 = 1$.

4. Consider the vector space $\mathbb{R}^4$. Which one of the following subsets is a subspace?

(a) $\{(a, b, c, d) \in \mathbb{R}^4 : a = 1\}$

(b) $\{(a, b, c, d) \in \mathbb{R}^4 : a + b + c + d = 1\}$

(c) $\{(a, b, c, d) \in \mathbb{R}^4 : a + 2b = 3c + 4d\}$

(d) $\{(a, b, c, d) \in \mathbb{R}^4 : a + b + c \geq d\}$

(e) $\{(a, b, c, d) \in \mathbb{R}^4 : ab = cd\}$

(f) $\{a, b, c, d,) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\}$

Answer. (c). One can check that (c) is correct. To see that the other choices are incorrect, notice that the zero vector isn't in the sets in (a), (b), (f). For (d), the set isn't closed under scalar multiplication: $(1, 1, 1, 1)$ is in the set, but $(-1, -1, -1, -1)$ isn't. For (e), the set isn't closed under addition: $(2, 6, 3, 4)$ and $(1, 1, 1, 1)$ but the sum $(3, 7, 4, 5)$ isn't.

True or False

5. The set $\left\{f : [0, 1] \to \mathbb{R} : f \text{ is integrable and } \displaystyle\int_0^1 f = 0\right\}$ is a subspace of $\mathbb{R}^{[0,1]}$.

Answer. True. The zero function is in the set since $\int_0^1 0 = 0$. It's closed under addition since if $\int_0^1 f = 0$ and $\int_0^1 g = 0$ then $\int_0^1 f + g = \int_0^1 f + \int_0^1 g = 0 + 0 = 0$ and it's closed under scalar mutliplication since if $\int_0^1 f = 0$ then for any $a \in \mathbb{R}$, we have $\int_0^1 af = a \int_0^1 f = (a)(0) = 0$.

6. The set $\big\{f : (-1, 1) \to \mathbb{R} : f \text{ is twice differentiable and } f''(0) = 2\big\}$ is a subspace of $\mathbb{R}^{(-1,1)}$.

Answer. False. The zero function does not satisfy $f''(0) = 2$. Also, this set is not closed under addition, nor under scalar multiplication.

7. The set $\big\{f : \mathbb{R} \to \mathbb{R} : f'(-1) = f(1)\big\}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Answer. True. The zero function satisfies $f'(1) = 0 = f(1)$. If f and g satisfy $f'(-1) = f(1)$ then $(f + g)'(-1) = f'(-1) + g'(-1) = f(1) + g(1) = (f + g)(1)$ so this set is closed under addition. And if f satisfies $f'(-1) = f(1)$ and $\lambda \in \mathbb{R}$ then $(\lambda f)'(-1) = \lambda = f'(-1) = \lambda f(1) = (\lambda f)(1)$ so the set is closed under scalar multiplication.

8. The set of all sequences of real numbers with limit 0 is a subspace of $\mathbb{R}^\infty$.

Answer. True. The zero sequence converges to 0. If $\{a_n\}$ and $\{b_n\}$ converge to zero then so do $\{a_n + b_n\}$ and $\{\lambda a_n\}$.

9. If $X = \{(a, 2a, 3a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$ and $Y = \{(a, a, a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$ then $(1, 3, 5) \in X + Y$.

Answer. True. $(1, 3, 5) = (2, 4, 6) + (-1, -1, -1)$

10. If $X = \{(0, 0, a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$, $Y = \{(a, a, a) \in \mathbb{R}^3 : a \in \mathbb{R}\}$ and $Z = \{(a, a, b) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$, then $X \oplus Y = Z$.

Answer. True. To verify it, we check that $Z = X + Y$ and $X \cap Y = \{0\}$.

To see that every vector in Z can be expressed as a sum of a vector in X and a vector in Y. Suppose $(a, a, b) \in Z$. Then we have $(a, a, a) \in Y$ and $(0, 0, b - a) \in X$ and $(a, a, b) = (0, 0, b - a) + (a, a, a)$.

To see that $X \cap Y = \{(0, 0, 0)\}$ note that if $(x, y, z) \in X$, then $x = y = 0$. If $(x, y, z) \in Y$ then $x = y = z$. So, if (x, y, z) is a vector in both X and W then $x = y = 0$ and $x = y = z$, which together mean that $(x, y, z) = (0, 0, 0)$.