Name

1	A
2	A
3	C
4	B
5	D
6	D
7	A
8	A
9	B
10	T
11	T
12	T

13.

Part I: Multiple Choice. 1 point each

1. If A is a $50,000 \times 30$ matrix and B is a $30 \times 50,000$ matrix, then the product AB is a matrix with

(a) 2.5 billion entries

(b) 3 million entries

(c) 1.5 million entries

(d) 900 entries

Answer. (a) The matrix AB is a $50,000$ by $50,000$ matrix, so that's $50,000 \times 50,000$ entries.

2. The matrix $C = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$ is invertible. The $(2,1)$ entry of C^{-1} is

(a) $\frac{4}{11}$ (b) $\frac{1}{4}$ (c) 2 (d) 4 (e) -1

Answer. (a). Recall, if $ad - bc \neq 0$, then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. So, $C^{-1} = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{pmatrix}$

3. Let $T \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathbb{R}^3)$. If you choose bases for $\mathcal{P}_5(\mathbb{R})$ and $\mathbb{R}^3$ and express T as a matrix $\mathcal{M}(T)$, then how many rows will the matrix $\mathcal{M}(T)$ have?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) 6

Answer. (c). The number of rows will be the dimension of the codomain, which in this case is $\dim(\mathbb{R}^3) = 3$.

4. Suppose that $S \in \mathcal{L}(\mathbb{R}^3, \mathcal{P}_2(\mathbb{R}))$ and that $S(1,0,0) = x^2 + 2$, $S(0,1,0) = 2x^2 + 4$, and $S(0,0,1) = 3x^2 + 6$. Then the dimension of range(S) is

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Answer. (b). The range of S is spanned by $x^2 + 2, 2x^2 + 4, 3x^3 + 6$, a basis for which is $x^2 + 2$.

5. The map $T : \mathcal{P}_5(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p) = \left(p(0), \int_0^1 p(x)\,dx, p(1)\right)$ is surjective. Which is a basis for null(T)?

(a) $3x^5 - 5x^4 + 2x^3$

(b) $3x^5 - 5x^4 + 2x^3, \quad 5x^4 - 20x^3 + 15x^2$

(c) $3x^5 - 5x^4 + 2x^3, \quad 5x^4 - 8x^3 + 3x^2$

(d) $3x^5 - 5x^4 + 2x^3, \quad 5x^4 - 8x^3 + 3x^2, \quad 3x^5 - 2x^3 - 3x^2 + 2x$

(e) $3x^5 - 5x^4 + 2x^3, \quad 5x^4 - 8x^3 + 3x^2, \quad 3x^5 - 2x^3 - 3x^2 + 2x, \quad 4x^3 - 6x^2 + 2x$

Answer. (d). Since T is surjective, $\dim(\text{range}(T)) = 3$. Since $\dim \mathcal{P}_5 = 6$, it must be that $\dim(\text{null}(T)) = 3$. So, $\text{null}(T)$ is spanned by three polynomials. That's enough to know that if the correct answer is listed, it must be (d).

6. Let $T : \mathcal{P}_5(\mathbb{R}) \to \mathbb{R}^3$ be the map defined by $T(p) = \left(p(0), \int_0^1 p(x)\,dx, p(1)\right)$. Using the basis $1, x, x^2, x^3, x^4, x^5$ for $\mathcal{P}_5(\mathbb{R})$ and the standard basis for $\mathbb{R}^3$, the matrix for T is

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Answer. (d). To find the first column for matrix for T, we evaluate T on the first basis vector of $\mathcal{P}_5$ and find that $T(1) = (1, 1, 1)$. So the first column of T is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, which conclusively determines the answer.

7. Suppose that $S : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and suppose that $S(1, 0, -5) = (0, 0, 0)$. Four of the following statements about S are true and one is false. Which one is false?

(a) S is surjective.

(b) S is not injective.

(c) S is not surjective.

(d) There are numbers $a, b, c \in \mathbb{R}$ so that the equation $S(x, y, z) = (a, b, c)$ has no solution for $(x, y, z) \in \mathbb{R}^3$.

(e) The equation $S(x, y, z) = (0, 0, 0)$ has infinitely many solutions for $(x, y, z) \in \mathbb{R}^3$.

Answer. (a). Since S is a linear operator on a finite dimensional vector space, being surjective, injective, bijective, and invertible are all equivalent. Since (b) and (c) are equivalent, and (a) is the logical negation of (c), the answer must be (a).

A more positive argument is that since $S(1, 0, -5) = (0, 0, 0)$, we know $(1, 0, 5) \in \operatorname{null}(S)$, so S is not injective implying that S is not surjective. The statement in (d) just means S is not surjective. The statement in (e) is correct since every multiple of $(1, 0, -5)$ is a solution to $S(x, y, z) = 0$.

8. Let $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be the derivative operator defined by $D(p) = p'$. One of the following statements about D is true and the rest are false. Which is true?

(a) D is surjective

(b) D is injective

(c) D is invertible

(d) D is bijective

Answer. (a). The map D is not injective since $D(x^2+1) = 2x = D(x^2+3)$. Therefore D cannot be invertible or bijective. This leaves (a). To see that (a) is correct, let p be any polynomial. We can find a polynomail q so that $D(q) = p$ by integrating. Specifically, for $q(x) = \int_0^x p(t)dt$ we have $q'(x) = p(x)$.

9. Consider the derivative operator $D : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ defined by $D(p) = p'$. Using the basis

$$2x^2 + x + 1, \quad 4x + 1, \quad 4$$

for both the domain and codomain to express D as a matrix results in $\mathcal{M}(D) =$

(a) $\begin{pmatrix} 2 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Answer. (b) To find the first column of the matrix for D, we evaluate D on the first basis vector:

$$D(2x^2 + x + 1) = 4x + 1 = 0(2x^2 + x + 1) + 1(4x + 1) + 0(4)$$

and see that the first column of $\mathcal{M}(D)$ is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which determines the answer conclusively.

Part II: True/False. 1 point each

It will be helpful to know that

$$\begin{pmatrix} 1 & 6 & -7 \\ -2 & -5 & 4 \\ -1 & -4 & 4 \end{pmatrix}\begin{pmatrix} 4 & -4 & 11 \\ -4 & 3 & -10 \\ -3 & 2 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$T(x, y, z) = (4x - 4y + 11z, -4x + 3y - 10z, -3x + 2y - 7z).$$

The map T is an isomorphism.

Answer. True. Using standard basis for $\mathbb{R}^3$ gives the $\mathcal{M}(T) = \begin{pmatrix} 4 & -4 & 11 \\ -4 & 3 & -10 \\ -3 & 2 & -7 \end{pmatrix}$. Since this matrix is invertible, the operator T is invertible.

11. The system of equations

$$\begin{aligned} 4x - 4y + 11z &= \sqrt{5} \\ -4x + 3y - 10z &= \frac{13}{17} \\ -3x + 2y - 7z &= -11 \end{aligned}$$

has a unique solution for $x, y, z \in \mathbb{R}$.

Answer. True. Since T is invertible, it is surjective and so for any $(a, b, c) \in \mathbb{R}^3$, there is a point $(x, y, z) \in \mathbb{R}^3$ with $T(x, y, z) = (a, b, c)$. In particular, there is a point $(x, y, z) \in \mathbb{R}^3$ with $T(x, y, z) = \left(\sqrt{5}, \frac{13}{17}, -11\right)$.

12. The system of homogeneous equations

$$\begin{aligned} 4x - 4y + 11z &= 0 \\ -4x + 3y - 10z &= 0 \\ -3x + 2y - 7z &= 0 \end{aligned}$$

has no nontrivial solutions for $x, y, z \in \mathbb{R}$.

Answer. True. Since T is invertible, it is injective. Therefore, $\operatorname{null}(T) = \{(0, 0, 0\}$. This means that the only point $(x, y, z) \in \mathbb{R}^3$ with $T(x, y, z) = (0, 0, 0)$ is $(x, y, z) = (0, 0, 0)$.

Part III: Short Answer. 3 points

13. Choose one of the previous problems and write a complete justification of your answer.