**1.** The dimension of $\mathcal{P}_2(\mathbb{R})$ is

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5

**Answer.** (d) Since $\{1, x, x^2\}$ is a basis and has three elements, $\dim(\mathcal{P}_2(\mathbb{R}))=3$.

**2.** True or False: There exist real numbers $a, b$ so that $a(-7, 4) + b(5, -2) = (12, 11)$.

**Answer.** True. Since $(-7, 4), (5, -2)$ is an independent list of two vectors in $\mathbb{R}^2$, which is two dimensional, it is a basis. Hence, $\operatorname{span}((-7, 4), (5, -2)) = \mathbb{R}^2$. In particular, $(12, 11) \in \operatorname{span}((-7, 4), (5, -2))$.

**3.** Which one of the following lists of vectors is indpendent?

(a) $(2, 3, 0, 4), (4, 6, 0, 8)$

(b) $(1, 2, 3), (0, 0, 0)$

(c) $(1, 1, 1), (1, 0, 0), (2, 1, 1)$

(d) $(1, 1, 1), (1, 2, 3), (-3, 8, 1), (3, 10, 15)$

(e) $(0, 2, 0, 4), (0, 2, 0, 5), (1, 2, 3, 4)$

**Answer.** $(0, 2, 0, 4), (0, 2, 0, 5), (1, 2, 3, 4)$ is independent. To see this, observe that the first two vectors are not multiples of one another. Any linear combination of $(0, 2, 0, 4), (0, 2, 0, 5)$ will have 0 as the first entry, so $(1, 2, 3, 4)$ is not a linear combination of $(0, 2, 0, 4), (0, 2, 0, 5)$.

(a) is dependent since $(4, 6, 0, 8) = 2(2, 3, 0, 4)$. (b) is dependent since it contains the zero vector. (c) is dependent since $(2, 1, 1) = (1, 1, 1) + (1, 0, 0)$ and (d) is depdendent since it's a list of four vectors in a three dimensional space.

**4.** True or False: If $\dim(V) = 4$ and $v_1, v_2, v_3, v_4$ is a linearly independent list of four vectors from $V$ then $\operatorname{span}(v_1, v_2, v_3, v_4) = V$.

**Answer.** True. Four linearly independent vectors in a four dimensional space must be a basis, hence spans.

**5.** Which one of the following lists of polynomials is linearly dependent?

(a) $1, x, x^3$

(b) $1, x, x^2, x^3, x^4, x^5$

(c) $x^2 + 2x + 3, x^2 - x, 3x^2 + x + 1, 2x^2 + 1$

(d) $1 - x, 1 + x$

(e) $5x^2 + 1, x^2 + 1, x^2 + x + 4$

**Answer.** (c) is dependent since it's a list of four vectors from the three dimensional space $\mathcal{P}_2(\mathbb{R})$.

**6.** Which of the following is a basis for $\mathbb{R}^3$ ?

(a) $(1, 2, 0), (0, 0, 5), (1, 0, 3), (1, 2, 3)$

(b) $(1, 2, 0), (0, 0, 5), (1, 0, 3)$

(c) $(1, 2, 0), (0, 1, 5)$

(d) $(1, 0, 0), (0, 0, 1), (1, 0, 1)$

(e) $(1, 2, 3), (4, 5, 8), (9, 6, 7), (3, 2, 8)$

**Answer.** Since $\mathbb{R}^3$ is three dimensional, any basis has three elements. So, (a), (c), and (e) are wrong. Also, (d) is wrong since $(0, 5, 0)$ is not in the span of $(1, 0, 0), (0, 0, 1), (1, 0, 1)$ (any linear combination of $(1, 0, 0), (0, 0, 1), (1, 0, 1)$ has 0 for the second entry).

This leaves (b). It suffices to note that the three vectors $(1, 2, 0), (0, 0, 5), (1, 0, 3)$ are linearly independent. To see that, note that any linear combination of the first two will have the form $(a, 2a, b)$ for some real numbers $a, b$ and $(1, 0, 3)$ does not have that form.

**7.** True or False: The vector space $\{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$ is infinite dimensional.

**Answer.** True. The given vector space has the polynomials $\mathcal{P}(\mathbb{R})$ as a subspace and $\mathcal{P}(\mathbb{R})$ is infinite dimensional.

**8.** True or False: If $a, b, c, d, e, f, g, h, i$ are real numbers satisfying

$$a + b + c = 0 \qquad d + e + f = 0 \qquad g + h + i = 0$$

then $(a, b, c), (d, e, f), (g, h, i)$ is a list of linearly dependent vectors in $\mathbb{R}^3$.

**Answer.** True. The set $\{(r, s, t) \in \mathbb{R}^3 : r + s + t = 0\}$ is a proper subspace of $\mathbb{R}^3$, hence has dimension at most two. Therefore, any list of three or more vectors from this space must be dependent.

**9.** Which one of the following sets of polynomials is *not* a subspace of $\mathcal{P}(\mathbb{R})$?

(a) {polynomials of degree 3}

(b) {polynomials $p(x)$ satisfying $p(1) = 0$ and $p'(1) = 0$}

(c) {even degree polynomials}

(d) {polynomials $p(x)$ with $\int_0^1 p(x)dx = 0$}

(e) {polynomials of degree $\leq 100$}

**Answer.** (a). {polynomials of degree 3} is not a subspace since, for example, it doesn't contain the zero vector. It's also not closed under addition since, for example, $-2x^3 - x^2 + x$ and $2x^3 + 3x + 1$ are both degree three polynomials but their sum $-x^2 + 4x + 1$ is not.

**10.** True or False: The list of polynomials $1, (x-5)^2, (x-5)^3$ is a basis for the subspace $U$ of $\mathcal{P}_3(\mathbb{R})$ defined by $U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}$.

**Answer.** True. Note that $U$ is a proper subspace of a four dimensional space, hence it's dimension is three or less. Since $1, (x-5)^2, (x-5)^3$ is a list consisting of three indpendent polynomials in $U$, it is a basis.

**11.** True or False: A list of vectors $v_1, \ldots, v_n$ is a basis for a vector space $V$ if and only if every vector $v \in V$ can be expressed as a unique linear combination of the vectors $v_1, \ldots, v_n$.

**Answer.** True. Every vector can be expressed as a linear combination if and only if the list spans and the expression is unique if and only if the list is indepdendent.

**12.** Let

$$U = \{p(x) \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5)\} \text{ and } W = \{p(x) \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5) = p(6)\}.$$

Which of the following statements is true:

(a) $U$ is a subspace of $W$

(b) $\dim(U) < \dim(W)$

(c) $U \oplus W = \mathcal{P}_4(\mathbb{R})$

(d) $\dim(W) = 3$

(e) $x^2 - 7x + 11 \in U \cap W$

**Answer.** Because any polynomial that has the same value at $2, 5$, and 6 has the same value at 2 and 5, we see that $W \subseteq U \subseteq \mathcal{P}(\mathbb{R})$. Since $x \in \mathcal{P}_4(\mathbb{R}) \setminus U$ and $(x-2)(x-5) \in U \setminus W$, the containments are proper, $\dim(W) < \dim(U) < \dim(\mathcal{P}(\mathbb{R})) = 5$, and we know $\dim(W) \leq 3$. Since $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ are three independent polynomials in $W$, we can conclude $\dim(W) = 3$ and we see that (d) is correct.

It also follows from the analysis above that (a) and (b) are wrong. Since $U \cap W = W$ it follows that $U \cap W \neq \emptyset$ so the sum $U + W$ is not direct.

Finally, a quick check shows that $x^2 - 7x + 11$ has different values at $x = 2$ and $x = 5$ and so is not an element of $W$.