1	E
2	C
3	D
4	A
5	T
6	F
7	F
8	T
9	F
10	T
11	F
12	F
13	T

14.

Part I: Multiple Choice. 1 point each

1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & -1 & -3 \\ 0 & -2 & 1 & 0 & -3 \\ -1 & -2 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 0 & -2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 3 & -2 \\ -3 & 0 & -2 \end{pmatrix}.$$

The $(2,3)$ entry of matrix AB is

(a) -6

(b) -3

(c) 0

(d) 3

(e) 6

Answer. The answer is 6, which you can find by multiplying the second row of A by the third column of B.

For the record $AB = \begin{pmatrix} 4 & -3 & 2 \\ 7 & 0 & 6 \\ -1 & -3 & 4 \\ -9 & 0 & -8 \end{pmatrix}$.

2. Using the basis $(x-1),(x-2)$ for $\mathscr{P}_1(\mathbb{R})$, the matrix $\mathscr{M}(x-5)$ is

(a) $\left(\frac{9}{2},-4\right)$

(b) $(3,2,1)$

(c) $(-3,4)$

(d) $(1,0,-1)$

(e) $(8,-3)$

Answer. The answer is $(-3,4)$ since $-3(x-1)+4(x-2) = x-5$.

3. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x,y,z) = (x-2y-2z, 2x-6y-7z, -4x+10y+10z).$$

Using the standard basis for the domain and the standard basis for the codomain, the first row of the matrix for T is

(a) $(1,0,0)$

(b) $(1,2,-4)$

(c) $(5,4,-2)$

(d) $(1,-2,-2)$

(e) $(5,0,1)$

Answer. To be complete, we find the entire matrix for T. To do that, apply T to each of the standard basis of $\mathbb{R}^3$

$$\begin{aligned} T(1,0,0) &= (1,2,-4) = 1(1,0,0) + 2(0,1,0) - 4(0,0,1) \\ T(0,1,0) &= (-2,-6,10) = -2(1,0,0) + -6(0,1,0) + 10(0,0,1) \\ T(0,0,1) &= (-2,-7,10) = -2(1,0,0) + -7(0,1,0) + 10(0,0,1) \end{aligned}$$

and use the coeffcients of the result for the columns of the matrix: $\begin{pmatrix} 1 & -2 & -2 \\ 2 & -6 & -7 \\ -4 & 10 & 10 \end{pmatrix}$.

4. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x,y,z) = (x-2y-2z, 2x-6y-7z, -4x+10y+10z).$$

Using the basis $(5,4,-2), (0,1,-1), \left(1, \frac{3}{2}, -1\right)$ for the domain and the standard basis for the codomain, the first row of the matrix for T is

(a) $(1,0,0)$

(b) $(1,2,-4)$

(c) $(5,4,-2)$

(d) $(1,-2,-2)$

(e) $(5,0,1)$

Answer. To be complete, we find the entire matrix for T. To do that, apply T to each of the vectors in the given basis

$$\begin{aligned} T(5,4,-2) &= (1,0,0) = 1(1,0,0)+0(0,1,0)+0(0,0,1) \\ T(0,1,-1) &= (0,1,0) = -0(1,0,0)+1(0,1,0)+0(0,0,1) \\ T(1,3/2,-1) &= (0,0,1) = 0(1,0,0)+0(0,1,0)+1(0,0,1) \end{aligned}$$

and use the coeffcients of the result for the columns of the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Part II: True/False. 1 point each

5. If $S : \mathbb{R}^3 \to \mathbb{R}$ be given by $S(x,y,z) = x+y+z$ then $\text{null}(S) = \text{span}((5,-3,-2),(0,3,-3))$.

Answer. True. Since S is a nonzero map to $\mathbb{R}$, the range is 1 dimensional. Therefore, the nullspace is two dimensional. The two vectors $(5,-3,-2),(0,3,-3)$ are two linearly independent vectors in the two dimensional nullspace, hence must be a basis, hence must span the nullspace.

6. The linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y,z,x+y-z)$ is surjective.

Answer. False. Note that $T(1,-1,0) = (0,0,0)$, so $\text{null}(T)$ has dimension at least 1, so $\text{range}(T)$ has dimension at most 2.

7. The linear map $T : \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p(x)) = (p(1),p(2),p(3))$ is surjective.

Answer. False. Since the domain is two dimensional, the range can have dimension at most two.

8. Let $T : \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}^3$ be the linear map defined by

$$T(p(x)) = (p(1),p(2),p(3)).$$

Using the basis $(x-1),(x-2)$ for $\mathscr{P}_1(\mathbb{R})$ and the standard basis is used for $\mathbb{R}^3$, the matrix equation $\mathscr{M}(T)\mathscr{M}(x-5) = \mathscr{M}(T(x-5))$ is the equation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}$$

Answer. True. We know from problem 3 that $\mathscr{M}(x-5) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ To see that the matrix $\mathscr{M}(T)$ is correct, apply T to each of the given basis vectors of $\mathscr{P}_1(\mathbb{R})$

$$\begin{aligned} T(x-1) &= (0,1,2) = 0(1,0,0)+1(0,1,0)+2(0,0,1) \\ T(x-2) &= (-1,0,1) = -1(1,0,0)+0(0,1,0)+1(0,0,1) \end{aligned}$$

Finally, note that the matrix multiplcation is correct and that the righthand side is the vector for $T(x-5) = (-4,-3,-2) = -4(1,0,0)+-3(0,1,0)-2(0,0,1)$ in the standard basis of $\mathbb{R}^3$.

9. The follow homogeneous system of equations has infinitely many solutions:

$$\begin{aligned} x - 2y - 2z &= 0 \\ 2x - 6y - 7z &= 0 \\ -4x + 10y + 10z &= 0 \end{aligned}$$

Answer. False. Note that the given statement is equivalent to the statement that there are infinitely many vectors in the nullspace of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x,y,z) = (x - 2y - 2z, 2x - 6y - 7z, -4x + 10y + 10z).$$

Since $(1,0,0),(0,1,0),(0,0,1)$ are all in the range of T, the range of T must be at least three, hence three, dimensional. Therefore (T) is zero dimensional. Therefore, there is only one solution to the given system—the trivial solution $(x,y,z) = (0,0,0)$.

10. The follow inhomogeneous system of equations has exactly one solution:

$$\begin{aligned} x - 2y - 2z &= \frac{1}{5} \\ 2x - 6y - 7z &= -17 \\ -4x + 10y + 10z &= 9 \end{aligned}$$

Answer. True. Since the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x,y,z) = (x - 2y - 2z, 2x - 6y - 7z, -4x + 10y + 10z)$$

is surjective, there is a solution to the given system. Since the linear map T is injective, there is only one solution.

11. There is a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with

$$\text{null}(T) = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Answer. False. Note that the given space is two dimensional (it is spanned by $(3,1,0,0,0),(0,0,1,1,1)$). If there existed a linear map with the given subspace as a nullspace, then it would have a three dimensional range, impossible for a map with the two dimensional codomain $\mathbb{R}^2$.

12. Suppose $T : V \to W$. If $v_1,\ldots,v_n$ is independent in V, then $Tv_1,\ldots,Tv_n$ is independent in W.

Answer. False. For example, if $T : \mathbb{R}^3 \to \mathbb{R}^2$ is zero map and v_1, v_2 is the independent list $(1,0,0),(0,1,0)$, Tv_1, Tv_2 is the dependent list $(0,0,0),(0,0,0)$.

13. Suppose $T : V \to W$. If $v_1,\ldots,v_n$ spans V, then $Tv_1,\ldots,Tv_n$ spans the range(T).

Answer. True. Every vector in range(T) has the form Tv for some $v \in V$. If $v_1,\ldots,v_n$ spans V, then there exist scalars $a_1,\ldots,a_n$ so that $v = a_1v_1 + \cdots + a_nv_n$. Then $Tv = a_1Tv_1 + \cdots a_nTv_n$ and we see Tv is in the span of $Tv_1,\ldots,Tv_n$.

Part III: Short Answer. 3 points

14. Choose one of the True / False problems and write a complete justification of your answer.