Part I: Multiple Choice [1 point each]

1. One of the following lists is linearly independent. Which one?

(a) $(1, 2, 3),\ (4, 5, 6),\ (7, 8, 9),\ (10, 11, 12)$

(b) $1 + x + x^2 + 4x^3,\ 2 + 2x + 2x^2 + 8x^3$

(c) $(1, 2, 3, 4),\ (0, 0, 0, 1),\ (1, 2, 3, 5)$

(d) $1 + x^6,\ 1 - x^5,\ 1 + x^2 - x^3,\ 1 + x$

(e) $1 + x + x^2,\ 4 - x^2,\ x + x^2,\ 6 - x$

Answer. (d) is independent. Since each polynomial in (d) has a term with a degree that none of the others have, the only linear combination of them that will produce the zero polynomial is the trivial linear combination. To see that the other lists are dependent, notice (a) and (e) consist of four vectors from three dimensional spaces; the second vector in (b) is a multiple of the first; and in (c) the third vector is the sum of the first two.

2. Let $T : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^4$ be the linear map defined by $T(p) = (p(0), p(1), p(2), p(3))$. Which is a basis for $\text{null}(T)$?

(a) $x(x - 1)(x - 2)(x - 3)$

(b) $2x^4 - 12x^3 + 22x^2 - 12x,\ 2x^5 - 12x^4 + 22x^3 - 12x^2$

(c) $x^3 - 6x^2 + 11x - 6$

(d) $x^3 - 3x^2 + 2x, x^3 - 4x^2 + 3x,\ x^3 - 6x^2 + 11x - 6$

(e) $x^3 - 3x^2 + 2x,\ x^3 - 4x^2 + 3x$

Answer. (a) is correct. Recall that $p(\lambda) = 0$ if and only if $(x - \lambda)$ divides $p(x)$. So, in order for $(p(0), p(1), p(2), p(3)) = (0, 0, 0, 0)$, the polynomial $p(x)$ must be a multiple of $x(x-1)(x-2)(x-3)$. So, the scalar multiples of $x(x - 1)(x - 2)(x - 3)$ are the only polynomials of degree 4 or less that vanish at $x = 0, 1, 2, 3$.

Note (b) is wrong since the second polynomial isn't even in the domain of T, and none of the other polynomials listed in (c), (d), or (e) are in the nullspace of T.

3. Which statement about the matrix $A = \begin{pmatrix} 3 & -1 & 5 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is true?

(a) There exists a basis of $\mathbb{R}^4$ consisting of eigenvectors for A.

(b) The nullspace of A is trivial.

(c) The matrix A is invertible.

(d) -1 is an eigenvalue for A

(e) $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for A

Answer. (a) is true. Since A is upper triangular, the eigenvalues for A are the diagonal entries $3, 0, 2, 4$. Any corresponding eigenvectors v_1, v_2, v_3, v_4 will be independent since they correspond to distinct eigenvalues. Therefore they are a basis for $\mathbb{R}^4$.

(b) is false since 0 is an eigenvalue for A, so there is a nonzero vector v with $Av = 0v = 0$. Since the nullspace of A is nontrivial, A is not invertible so (c) is false. Since A can have at most 4 distinct eigenvalues, -1 cannot be one so (d) is false. A quick computation reveals that (e) is false since the product of A with the given vector has a nonzero first entry.

4. Which of the following vectors are *not* eigenvectors for the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$?

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (e) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Answer. (e) is not an eigenvector, look:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

note that the vector in (a) is an eigenvector with eigenvalue 2, the vectors in (b), (c), and (d) are eigenvectors with eigenvalues -1.

5. Which statement about the operator $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ defined by $D(p) = p'$ is false?

(a) D is surjective

(b) 0 is an eigenvalue for D

(c) D is injective

(d) $\mathcal{P}_2(\mathbb{R})$ is an invariant subspace for D

(e) $\text{null}(D) \subseteq \text{range}(D)$

Answer. (c) is false since, for example, $D(x^2 + 1) = D(x^2 + 3)$. Since D is not injective, the nullspace of D is nontrivial, so 0 is an eigenvalue for D and (b) is true. (a) is true since every polynomial is the derivative of another polynomial, which you can find by taking an antiderivative. (d) is true since the derivative of a polynomial of degree less than or equal to two, is again a polynomial of degree less than or equal to two. Finally (e) is true since we've already noted that D is surjective, so every set of polynomials is a subset of range(D). More precisely, null(D) consists of the constant polynomials, which are themselves the derivatives of degree one polynomials.

Part II: True or False [1 point each]

6. The matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is invertible.

Answer. True. There are several ways to see this. Here's one. From the multiple choice problems, we know the vectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors for A. They are, in particular, in the range of the operator $\mathbb{R}^3 \to \mathbb{R}^3$ defined by $v \mapsto Av$. By inspection, we see that these vectors are linearly independent, hence they span $\mathbb{R}^3$. So the operator $v \mapsto Av$ is surjective. Surjective operators on a finite dimensional vector spaces are invertible, and so are the matrices for them found using any basis. The matrix A is the matrix for this operator in the standard basis for $\mathbb{R}^3$, so A is invertible.

7. The map $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ defined by $T(p) = (p(0), p(1), p(2), p(3))$ is invertible.

Answer. True. Recall that $p(\lambda) = 0$ if and only if $(x - \lambda)$ divides $p(x)$. So, in order for $(p(0), p(1), p(2), p(3)) = (0, 0, 0, 0)$, the polynomial $p(x)$ must be a multiple of $x(x-1)(x-2)(x-3)$. Since the only polynomial of degree less than or equal to three that is a multiple of $x(x - 1)(x - 2)(x - 3)$ is the zero polynomial, we see $\text{null}(T) = \{0\}$ and know T is injective. Since $\dim(\text{null}(T)) + \dim(\text{range}(T)) = \dim(\mathcal{P}_3(\mathbb{R})) = 4$, we see that $\dim(\text{range}(T)) = 4$, so T is surjective also. Together, this implies T is bijective, hence invertible.

8. For any numbers a, b, c, the system of equations $y + z = a$, $x + z = b$, $x + y = c$ has a unique solution $(x, y, z) \in \mathbb{R}^3$.

Answer. True. The system of equations given is equivalent to the matrix equation

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Since the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is invertible (see above), there is a unique solution, namely

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

9. If $T : V \to V$ is a linear operator on a vector space V then $V = \operatorname{null}(T) \oplus \operatorname{range}(T)$.

Answer. False. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (0, x)$. Notice that $T(1, 0) = (0, 1)$ so $(0, 1) \in \operatorname{range}(T)$. Also $T(0, 1) = (0, 0)$ so $(0, 1) \in \operatorname{null}(T)$. So $\operatorname{null}(T) \cap \operatorname{range}(T) \neq \{0\}$ and the sum cannot be direct.

For another example, consider the derivative operator on polynomials $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$. As was noted previously on this exam, $\operatorname{null}(D)$ consists of the constant polynomials and $\operatorname{range}(D) = \mathcal{P}(\mathbb{R})$, so $\operatorname{null}(D) \cap \operatorname{range}(D) = \{\text{constant polynomials}\} \neq \{0\}$.

10. If a matrix A satisfies $A^2 = I$ then $A = I$ or $A = -I$.

Answer. False. For example, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $A^2 = I$ and $A \neq I$ and $A \neq -I$. For another example, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ works.

That's the end of the answer, but let me leave you with something you might find interesting. If $A^2 = I$, then $A^2 = I \Rightarrow A^2 - I = 0 \Rightarrow (A - I)(A + I) = 0$. Since the composition of the maps $A - I$ and $A + I$ is the zero map, at least one of these maps must not be injective. So, there exist a nonzero vector v with $(A - I)v = 0$ or $(A + I)v = 0$, so $\lambda = 1$ or $\lambda = -1$ must be an eigenvalue for A. So, if $A^2 = I$, you can't conclude that $A = I$ or $A = -I$, but you *can* conclude that $\lambda = 1$ or $\lambda = -1$ (or both) is an eigenvalue for A.

Part III: Compute [2 points]. Show your work.

11. Find the eigenvalues of the matrix $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

Answer. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. A scalar λ is an eigenvalue for A if and only if $A - \lambda I$ is not invertible, which is true if and only if the determinant of $A - \lambda I$ is zero. We compute

$$\begin{aligned} \det A - \lambda I &= \det \left(\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{pmatrix} \\ &= (3 - \lambda)(4 - \lambda) - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (\lambda - 5)(\lambda - 2) \end{aligned}$$

So, $\det A - \lambda I = 0$ iff $\lambda = 2$ or $\lambda = 5$. Therefore, 2 and 5 are eigenvectors for the matrix A.

It's not hard, once you know the eigenvalues to find corresponding eigenvectors. Here are eigenvectors v_1 and v_2 that correspond to the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$ respectively:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Which is a good way to be certain of your computation.

Part IV: Short Answer [2 points]

12. Choose one of the true/false problems above and explain your answer. Write your explanation clearly and concisely.