Answers

1	B
2	C
3	E
4	A
5	C
6	A
7	F
8	B
9	F
10	T
11	T
12	T
13	T
14	T

Part I: Multiple Choice. 1 point each

1. (b) Note that $T(1+x^2) = 2+2x^2 = 2(1+x^2)$, $T(2-x^2) = -2+x^2 = -1(2-x^2)$ and $T(2-2x+x^2) = 2-2x+x^2$. Therefore, the matrix for T using the given basis is

$$\mathscr{M}(T) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The $(2,2)$ entry is -1.

2. (c) is False. From the previous problem, we see that the polynomials in the given basis are eigenvectors for T with eigenvalues $2,-1,1$, in particular $1+x^2$ corresponds to the eigenvalue 2. Thus (a), (d), and (e) are true. Moreover, none of the eigenvalues are zero so T is invertible and we see that (b) is true.

3. (e) is the answer. We solve the problem in two steps. The first step is to find the eigenvalues for the given matrix. λ is an eigenvalue for $\begin{pmatrix} -14 & 5 \\ -24 & 9 \end{pmatrix}$ if and only if $\begin{pmatrix} -14-\lambda & 5 \\ -24 & 9-\lambda \end{pmatrix}$ is not invertible if and only if $(-14-\lambda)(9-\lambda)-(-24)(5) = 0$ if and only if $\lambda^2+5\lambda-6 = 0$ if and only if $\lambda = -6$ or $\lambda = 1$.

Once we know that $-6, 1$ are the eigenvalues, the next step is:

$$\begin{pmatrix} -14 & 5 \\ -24 & 9 \end{pmatrix} \begin{pmatrix} 5 \\ x \end{pmatrix} = 1 \begin{pmatrix} 5 \\ x \end{pmatrix} \Leftrightarrow \begin{pmatrix} 5x-70 \\ 9x-120 \end{pmatrix} = \begin{pmatrix} 5 \\ x \end{pmatrix} \Rightarrow x = 15$$

and

$$\begin{pmatrix} -14 & 5 \\ -24 & 9 \end{pmatrix} \begin{pmatrix} 5 \\ x \end{pmatrix} = -6 \begin{pmatrix} 5 \\ x \end{pmatrix} \Leftrightarrow \begin{pmatrix} 5x-70 \\ 9x-120 \end{pmatrix} = \begin{pmatrix} -30 \\ -6x \end{pmatrix} \Rightarrow x = 8.$$

4. (b)

$$\begin{pmatrix} -2 & 3 \\ 4 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}$$

and we see the $(1,1)$ entry is $-\frac{5}{2}$.

5. (c) is the answer. Statements (a) and (b) are equivalent. Statement (c) implies (a) and (b), but it's not equivalent since it's possible to have a basis of eigenvectors with the same eigenvalue (eg: the identity map on $\mathbb{R}^3$.)

6. (a) If $\text{null}(T-\lambda\,\text{id}) = \{0\}$ then there are no nonzero vectors v with $(T-\lambda\,\text{id})v = 0$ which means that λ is *not* an eigenvalue for T.

7. (f) is the answer. It's possible to have a basis of V consisting of eigenvectors for a non-invertible operator $T : V \to V$. For example, $(1,0),(0,1)$ is a basis of eigenvectors for the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (x,0)$, but T is not invertible.

To see why the others are equivalent, note that if $\dim(V) = n$ and $T : V \to V$, then $\dim(\text{range}(T)) + \dim(\text{null}(T)) = n$. Thus, T is injective $\Leftrightarrow \text{null}(T) = \{0\} \Leftrightarrow \dim(\text{null}(T)) = 0 \Leftrightarrow \dim(\text{range}(T)) = n \Leftrightarrow \text{range}(T) = V \Leftrightarrow T$ is surjective. Together, injective and surjective mean bijective which is statement (d) and bijective is also equivalent to invertible which is statement (b). Finally, to see that (e) is equivalent, note that if $Tv_1,\ldots,Tv_n$ is a basis of V then it spans, hence T is surjective hence and conversely if T is surjective and $v_1,\ldots,v_n$ is a basis for V, then $Tv_1,\ldots,Tv_n$ must span V and a spanning set of the right size must be a basis.

8. (b) is false since S maps both $(1,1,1,\ldots)$ and $(2,1,1,\ldots)$ to the same sequence. The other statements are true. Note that $S(1,1,1\ldots) = (1,1,\ldots)$ so $(1,1,1\ldots)$ is an eigenvector with eigenvalue 1 so (d) and (e) are true. Also since S is not injective, there's a nonzero vector in the nullspace, which means that 0 is an eigenvalue for S so (c) is true. To see that (a) is true, note that if $(a_1,a_2,\ldots)$ is any sequence in $\mathbb{R}^\infty$, $S(1,a_1,a_2,\ldots) = (a_1,a_2,\ldots)$, so S is surjective. This problem contrasts with the previous two problems. For operators on finite dimensional spaces, surjective and injective are equivalent, but not so for operators on infinite dimensional spaces.

9. False. Note that $\{(x,2x,3x,4x,5x) : x \in \mathbb{R}\} = \text{span}((1,2,3,4,5))$ is one-dimensional. If $T : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear map, the dimension of $\text{range}(T) \leq 3$ hence the dimension of $\text{null}(T) \geq 2$.

10. True. To see this, suppose $v \in \text{null}(S)$. We need to check if Tv is again in $\text{null}(S)$. So, apply S to Tv to get $(Tv) = T(Sv) = T(0) = 0$, as needed for Tv to be in $\text{null}(S)$. The first equal sign is because S and T commute, the second is because $v \in \text{null}(S)$ and the third is because linear maps always map 0 to 0.

11. True. To prove that $\mathbb{R}^\mathbb{R} = U \oplus V$, let $f \in \mathbb{R}^\mathbb{R}$ be any function. Write f as $f = g + h$ where $g(x) = f(x) + f(-x)$ and $h(x) = f(x) - f(-x)$. Note that $g \in U$ and $h \in W$. This shows $\mathbb{R}^\mathbb{R} = U + W$. To see that the sum is direct, notice that if $f \in U \cap W$ then $f(-x) = f(x) = -f(x) \Rightarrow f(x) = 0$ so $U \cap W = \{0\}$.

12. True. First, since $\mathscr{P}_3(\mathbb{R})$ is four dimensional, and I is not the zero map, the nullspace of I is at most three dimensional. A quick check shows that all three polynomials are in the nullspace of I (both x and x^3 are odd and it's easy to compute $\int_{-1}^{1} 3x^2 - 1 = 0$). These three polynomials are independent since they have different degree. Hence, the three polynomials given must be a basis.

13. True. Let $p(x) = 4 - 8x + 5x^2 + 7x^3 - 4x^4 - 2x^5 + x^6$. Note that $p'(x) = 6x^5 - 10x^4 - 16x^3 + 21x^2 + 10x - 8$ and both $p(2) = 0$ and $p'(2) = 0$. Hence $(x-2)^2$ must divide $p(x)$ and $p'(x)$.

14. True. Note that $T(1,2,-2) = (2,3,5)$ and $T(2,3,5) = (-1,-2,2) = -(1,2,-2)$. So, yes, $\text{span}((1,2,-2),(2,3,5))$ is an invariant subspace for T.