**1.** Prove: If some vector in a list of vectors in a vector space $V$ is a linear combination of the other vectors, then the list is linearly dependent.

**Answer.** Suppose that $v_k = \lambda_1 v_1 + \lambda_2 v_2 + \cdots \lambda_n v_n$. Subtracting $v_k$ from both sides gives $0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + (-1)v_k + \cdots \lambda_n v_n$ proving that $v_1, \ldots, v_n$ is dependent.

**2.** Does $(1,2,3,-5),(4,5,8,3),(9,6,7,-1)$ span $\mathbb{R}^4$? Explain.

**Answer.** No. Since $\mathbb{R}^4$ is four dimensional, no set with fewer than four vectors can span $\mathbb{R}^4$.

**3.** Is the list $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ linearly independent in $\mathbb{R}^3$? Explain.

**Answer.** No. Since $\mathbb{R}^3$ is three dimensional, no set with greater than three vectors can be independent.

**4.** Prove that $F^\infty$ is infinite-dimensional.

**Answer.** In any finite dimensional vector space, if a list of vectors is longer than the dimension, the list is dependent. Since the following list is independent in $F^\infty$: $(1,0,0,0\ldots)$, $(0,1,0,0,\ldots)$, $\ldots$, $(0,0,\ldots,0,1,0,\ldots)$—and I can make this list as long as I like—the dimension of $F^\infty$ cannot be finite.

**5.** Suppose that $p_1, p_2, p_3, p_4, p_5$ is a list polynomials in $\mathcal{P}_4(\mathbb{R})$ that all vanish at $x = 3$. Prove that $p_1, p_2, p_3, p_4, p_5$ is linearly dependent.

**Answer.** The space $U$ of polynomials that vanish at $x = 3$ is a proper subspace of the 5 dimensional space of all polynomials in $\mathcal{P}_4(\mathbb{R})$. Therefore, $U$ has dimension at most 4. So, any list of 5 or more polynomials in this space must be dependent.

**6.** Suppose that $v_1, v_2, v_3$ is a basis for a vector space $V$. Prove or disprove $v_1+v_2, v_1-v_2, v_3$ is also a basis for $V$.

**Answer.** If $v_1, v_2, v_3$ is a basis for $V$, then $V$ is three-dimensional, so it suffices to check whether the list $v_1+v_2, v_1-v_2, v_3$ spans $V$. Note that $v_1 = \frac{1}{2}(v_1+v_2) - \frac{1}{2}(v_1-v_2)$ and $v_2 = \frac{1}{2}(v_1+v_2) + \frac{1}{2}(v_1-v_2)$. So, $v_1, v_2, v_3$ are in the span of $v_1+v_2, v_1-v_2, v_3$, and we conclude the list $v_1+v_2, v_1-v_2, v_3$ spans $V$.

**7.** Let $U = \{(a,b,c) \in \mathbb{R}^3 : a+b+c = 0\}$.

(a) Find a basis for $U$.

**Answer.** As a proper subspace of $\mathbb{R}^3$, the dimension of $U$ is at most 2. The vectors $(1,0,-1),(0,1,-1)$ are an independent list in $U$, hence are a basis.

(b) Extend your basis to a basis of $\mathbb{R}^3$.

**Answer.** Adding any vector not in $U$ works. For example, $(1,0,-1),(0,1,-1),(1,0,0)$

(c) Find a subspace $W$ of $\mathbb{R}^3$ so that $\mathbb{R}^3 = U \oplus W$.

**Answer.** Let $W$ be the span of $(1,0,0)$.

**8.** Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5)\}$.

(a) Find a basis for $U$.

**Answer.** Notice that since $U$ is a proper subspace of of the five dimensional space $\mathcal{P}_4(\mathbb{R})$, we know the dimension of $U$ is at most 4. Here's an independent list of four polynomials in $U$:

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$$

and so it is a basis for $U$. To see that the list is independent, note that no polynomial in this list can be a linear combination of the previous polynomials since the degree of each polynomial is strictly greater than the degrees of the polynomials that preceed it.

(b) Extend your basis to a basis of $\mathcal{P}_4(\mathbb{R})$

**Answer.** It suffices to add any polynomial not in $U$. For example $x$ works. So

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2, x$$

is a basis for $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{R})$ so that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

**Answer.** Let $W$ be the span of $x$.

**9.** Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$.

(a) Find a basis for $U$.

**Answer.** As a proper subspace of the five dimensional space $\mathcal{P}_4(\mathbb{R})$, the dimension of $U$ is at most four. The list $x, 3x^2 - 1, x^3, 5x^4 - 1$ is a list of four independent polynomials in $U$, hence is a basis. To see that the list is independent, note that no polynomial in this list can be a linear combination of the previous polynomials since the degree of each polynomial is strictly greater than the degrees of the polynomials that preceed it.

(b) Extend your basis to a basis of $\mathcal{P}_4(\mathbb{R})$

**Answer.** It suffices to add any polynomial not in $U$. For example, the polynomial 1 works: $x, 3x^2 - 1, x^3, 5x^4 - 1, 1$.

(c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{R})$ so that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

**Answer.** Let $W$ be the span of 1 — that's the space of constant polynomials.

**10.** Prove that any two three dimensional subspaces of $\mathbb{R}^5$ must have a nonzero vector in their intersection.

**Answer.** Let $U_1, U_2$ be two three dimensional subspaces of $\mathbb{R}^5$. We know

$$\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) = \dim(U_1 + U_2)$$

Therefore, $3 + 3 - \dim(U_1 \cap U_2) \leq 5 \Rightarrow \dim(U_1 \cap U_2) > 0$ and therefore must contain a nonzero vector.