Name:

1	T
2	F
3	T
4	F
5	T
6	T
7	F
8	T
9	T
10	T
11	T
12	T
13	T
14	T

15.

Part I

Use the functions e, f and g as defined below for problems 1–6

$$\begin{array}{ccc} \{1,2,3\} \xrightarrow{e} \{1,2,3\} & \{1,2,3\} \xrightarrow{f} \{1,2,3\} & \{1,2,3\} \xrightarrow{g} \{1,2,3\} \\ 1 \longmapsto 1 & 1 \longmapsto 2 & 1 \longmapsto 3 \\ 2 \longmapsto 2 & 2 \longmapsto 1 & 2 \longmapsto 1 \\ 3 \longmapsto 3 & 3 \longmapsto 3 & 3 \longmapsto 2 \end{array}$$

1. $2 \xmapsto{fg} 2$.

Answer. True. $fg(2) = f(g(2)) = f(1) = 2$.

2. $fg = e$.

Answer. False. $fg(1) = f(g(1)) = f(3) = 3$ and $e(1) = 1$.

3. f is bijective.

Answer. True. A nice way to see that is to notice that $f^2 = e$, hence f is invertible.

4. $gf = fg$.

Answer. False. $gf(1) = g(f(1)) = g(2) = 1$ and $fg(1) = f(g(1)) = f(3) = 3$.

5. $ggf = fg$.

Answer. True. One can check

$$\begin{array}{cc} \{1,2,3\} \xrightarrow{fg} \{1,2,3\} & \{1,2,3\} \xrightarrow{ggf} \{1,2,3\} \\ 1 \longmapsto 3 & 1 \longmapsto 3 \\ 2 \longmapsto 2 & 2 \longmapsto 2 \\ 3 \longmapsto 1 & 3 \longmapsto 1 \end{array}$$

6. ggf is an inverse for fg.

Answer. True. There are a few ways to see this quickly. For one, notice that $ff = e$ and $ggg = e$, so $ggffg = ggeg = ggg = e$ and $fgggf = fef = ff = e$.

Part II: More True/False

7. The relation $\sim$ defined on $\mathbb{Z}$ by $m \sim n \Leftrightarrow \gcd(m,n) = 1$ is an equivalence relation.

Answer. False. This relation is neither reflexive nor transitive. To see that it's not reflexive, notice that $\gcd(2,2) = 2 \neq 1$. To see that it's not transitive, observe that $\gcd(35,11) = 1$ and $\gcd(11,21) = 1$ but $\gcd(35,21) = 7 \neq 1$. As a side note, one should *not* us the notation $\sim$ for a relation that is not an equivalence relation.

8. The composition of two monomorphisms is a monomorphism.

Answer. True. Suppose $f : X \to Y$ and $g : Y \to Z$ are monic. To see that gf is monic, let $h : A \to X$ and $k : A \to X$ be two functions and assume that $(gf)h = (gf)k$. By the associativity of composition, we have $g(fh) = g(fk)$. Because g is monic, we know $fh = fk$. Because f is monic, we know $h = k$, as needed to conclude gf is monic.

9. A function $f : X \to Y$ between two sets X and Y is surjective if and only if there exists a function $h : Y \to X$ with $fh = \mathrm{id}_Y$.

Answer. True. To prove it, first suppose that f is surjective. Define a function $h : Y \to X$ as follows. For each $y \in Y$, we know (because f is surjective) that there exists at least one element $x \in X$ with $f(x) = y$. Choose one such x and set $h(y) = x$. Then $fh(y) = f(x) = y$ as needed to show $fh = \mathrm{id}_Y$.

Now, suppose that $fh = \mathrm{id}_Y$. To prove that f is surjective, let $y \in Y$. Since $y = fh(y)$, we know that for $x = h(y)$, we have $f(x) = y$.

10. Let X and Y be sets and suppose $f : X \to Y$. If for all functions $g : Y \to Z$ and $h : Y \to Z$, $gf = hf \Rightarrow g = h$, then f is surjective.

Answer. True. To prove it, let $Z = \{\text{red}, \text{blue}\}$ and define functions $g, h : Y \to Z$ by

$$g(y) = \text{red for all } y \in Y$$

$$h(y) = \begin{cases} \text{red} & \text{if } y \text{ is in the image of } f \\ \text{blue} & \text{if } y \text{ is not in the image of } f \end{cases}$$

Notice that $gf = hf$. So, by the property that we've assumed f has, we conclude that $g = h$. The only way that g and h can be equal is if y is in the image of f for every $y \in Y$. This proves f is surjective.

11. Let X and Y be sets and consider functions $f : X \to Y$, $g : Y \to X$, and $h : Y \to X$. If $hf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$ then $g = h$.

Answer. True. To prove it, suppose $hf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. Then $h = h(fg) = (hf)g = g$.

Part III

Use the following information for the problems 12, 13, and 14. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be the following sets

$\mathcal{A} = \{$A,B,C, . . . , Y, Z$\}$

$\mathcal{B} = \{$Isabella, Madison, Olivia, Liam, Joseph, Jayden, Esther, Elijah, Moshe, Emma, Mia$\}$

$\mathcal{C} = \{\text{yellow}, \text{purple}\}$.

and let $f : \mathcal{B} \to \mathcal{A}$ and $g : \mathcal{B} \to C$ be the functions defined by

$$f(x) = \text{the first letter of the name } x$$

$$g(x) = \begin{cases} \text{yellow} & \text{if name } x \text{ has an even number of letters} \\ \text{purple} & \text{if name } x \text{ has an odd number of letters} \end{cases}$$

12. The function defined by

$$C \xrightarrow{h} \mathcal{B}$$

$$\text{yellow} \longmapsto \text{Elijah}$$

$$\text{purple} \longmapsto \text{Madison}$$

is a right inverse for g

Answer. True. We compute and see that $gh = \mathrm{id}_C$:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[rr, bend left, "gh"] & \mathcal{B} \arrow[r, "g"] & C \end{tikzcd}$$

$$\text{yellow} \overset{h}{\longmapsto} \text{Elijah} \overset{g}{\longmapsto} \text{yellow}$$

$$\text{purple} \overset{h}{\longmapsto} \text{Madison} \overset{g}{\longmapsto} \text{purple}$$

13. Define an equivalence relation $\sim$ on $\mathcal{B}$ by $x \sim x' \Leftrightarrow f(x) = f(x')$. That is, two names are equivalent if and only if they begin with the same letter. There are Let $p : \mathcal{B} \twoheadrightarrow \mathcal{B}/\sim$ be the natural surjection mapping a name to its equivalence class.

The map g factors through p. That is, there exists a well defined map $h : \mathcal{B}/\sim \,\to C$ so that $hp = g$ as pictured below:

$$\begin{tikzcd} \mathcal{B} \arrow[d, two heads, "p"'] \arrow[dr, "g"] & \\ \mathcal{B}/\sim \arrow[r, dashed, "h"] & C \end{tikzcd}$$

Answer. True. Note that g is constant on these equivalence classes since Madison, Moshe, and Mia all have an odd number of letters, both Joseph and Jayden have an even number of letters, and Esther, Elijah, and Emma all have an even number of letters. So, we can define

$h : \mathcal{B}/{\sim}$ by

$$\begin{aligned} \mathcal{B}/{\sim} &\xrightarrow{\quad h \quad} C \\ \{Isabella\} &\longmapsto \text{yellow} \\ \{Madison, Moshe, Mia\} &\longmapsto \text{purple} \\ \{Olivia\} &\longmapsto \text{yellow} \\ \{Liam\} &\longmapsto \text{yellow} \\ \{Joseph, Jayden\} &\longmapsto \text{yellow} \\ \{Esther, Elijah, Emma\} &\longmapsto \text{yellow} \end{aligned}$$

14. Consider the natural projection $\pi_{\mathcal{B}} : \mathcal{B} \times C \to \mathcal{B}$. Define a map $s : \mathcal{B} \to \mathcal{B} \times C$ by $s(x) = (x, g(x))$. Then s is a section of $\pi_{\mathcal{B}}$. That is $\pi_{\mathcal{B}} s = \mathrm{id}_{\mathcal{B}}$. Here is a picture:

$$\begin{tikzcd} & \mathcal{B} \times C \arrow[dl, two heads, "\pi_{\mathcal{B}}"] \\ \mathcal{B} \arrow[ur, bend left, "s"] & \end{tikzcd}$$

Answer. True. We check that $\pi_{\mathcal{B}} \circ s = \mathrm{id}_{\mathcal{B}}$.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "s"] \arrow[rr, bend left, "\pi_{\mathcal{B}}\, s"] & \mathcal{B} \times C \arrow[r, "\pi_{\mathcal{B}}"] & \mathcal{B} \\ x \arrow[r, mapsto] & (x, g(x)) \arrow[r, mapsto] & x \end{tikzcd}$$

Part IV

Short answer [3 points]

15. Choose one of the True/False problems above and explain why it is true or false. Write your answer clearly and carefully. Neatness counts.