Short Answer [1 pt each] Problems 1 through 5 concern computations in the symmetric group S_5. For these problems, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$.

1. $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}$.

2. $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$.

3. If $\sigma^2 x = \tau$ then $x = \sigma^{-2}\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$.

4. $|\sigma| = 6$.

5. $|S_5| = 120$.

True/False [1 pt each] For problems 6–14, decide whether each statement is true or false. Put **T** or **F** on the answer sheet.

6. The binary operation $\star$ defined on $\mathbb{Z}$ by $a \star b = 3ab$ is associative.

Answer. True. $(a \star b) \star c = 3ab \star c = 3(3ab)c$ and $a \star (b \star c) = 3a(b \star c) = 3a(3bc) = 9abc$.

7. Suppose G is a group and $x, y \in G$ satisfy $x^2 = y^3 = e$ and $yx = xy^2$. Then $(yx)^{-1} = xy^2$.

Answer. True. To check, multiply $xy^2 * yx = xy^3x = xex = x^2 = e$.

8. If G is a group and $g^2 = e$ for every $g \in G$, then G is abelian.

Answer. True. Suppose G is a group with $g^2 = e$ for every $g \in G$. Now consider two elements $x, y \in G$. We know $x^2 = y^2 = (xy)^2 = e$, so $x^{-1} = x$, $y^{-1} = y$, and $(xy)^{-1} = xy$. We also have $(xy)^{-1} = y^{-1}x^{-1} = yx$. Since xy and yx both equal $(xy)^{-1}$, they must be equal.

9. Every element of S_3 has order 1, 2, or 3.

Answer. True. You can list the six elements and see this. Here's another way to see it. Every element of S_3 must have an order that divides 6, but no element can have order 6 since then S_3 would be cyclic, hence Abelian, which it is not.

10. $|\,(\mathbb{Z}/12\mathbb{Z})^*\,| = 5$.

Answer. False. $(\mathbb{Z}/12\mathbb{Z})^*$ has consists of the elements in $\mathbb{Z}/12\mathbb{Z}$ whose greatest common divisor with 12 is one. So, $(\mathbb{Z}/12\mathbb{Z})^* = \{1, 5, 7, 11\}$.

11. $|[3]_7| = 6$ in $(\mathbb{Z}/7\mathbb{Z})^*$.

Answer. True. We check. We'll drop the $[\quad]_7$ from our notation.

$$3^2 = 2, \quad 3^3 = 3 * 2 = 6, \quad 3^4 = 3 * 6 = 4, \quad 3^5 = 3 * 4 = 5, \quad 3^6 = 3 * 5 = 1.$$

12. If $\phi : G \to H$ is a group homomorphism, then $|\phi(g)| = |g|$ for all $g \in G$.

Answer. False. Consider the group homomorphism $\phi : \mathbb{Z}/6/Z \to \mathbb{Z}/3\mathbb{Z}$ defined by $\phi([n]_6) = [n]_3$. Here, $|\phi([1]_6)| = |[1]_3| = 3$ and $|[1]_6| = 6$.

13. S_3 and $\mathbb{Z}/6\mathbb{Z}$ are isomorphic.

Answer. False. S_3 is not abelian and $\mathbb{Z}$ is abelian.

14. $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ are isomorphic.

Answer. True. The map $[n]_6 \mapsto ([n]_2, [n]_3)$ is an isomorphism.