Name:

1	T
2	F
3	T
4	F
5	T
6	F
7	T
8	F
9	T

True or False [1 pt each]

1. In a group G if $ab = ac$ then $b = c$.

Answer. True. If $ab = ac$, multiplying by a^{-1} on the left yields $b = c$.

2. For functions $f : B \to C$, $g : A \to B$ and $h : A \to B$, if $fg = fh$ then $g = h$.

Answer. False. Let $g : \{1, 2\} \to \mathbb{Z}$ be defined by $g(1) = 1$, $g(2) = -1$ and let $h : \{1, 2\} \to \mathbb{Z}$ be defined by $h(1) = 1$ and $h(2) = 1$. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n^2$. Then $fg = fh$ and $g \neq h$.

3. If $A \neq \emptyset$ and $f : A \to B$ is an injective function, then there exists a function $g : B \to A$ with $gf = \mathrm{id}_A$.

Answer. True. For any b in the range of f, define $g(b)$ to be the unique element $a \in A$ with $f(a) = b$. For every b not in the range of f, just choose any element $a_0 \in A$ and defined $g(b) = a_0$. This defines a function g with $gf = \mathrm{id}_A$.

4. If $\phi : G \to G'$ is an injective group homomorphism, then there exists a group homomorphism $\psi : G' \to G$ with $\psi\phi = \mathrm{id}_G$.

Answer. False. The map $\phi : \mathbb{Z}/3\mathbb{Z} \to S_3$ defined by $1 \mapsto (123)$ is injective. But there cannot be a nontrivial group homomorphism $\psi : S_3 \to \mathbb{Z}/3\mathbb{Z}$ since such a map would be surjective, hence $S_3/\ker(\psi) \equiv \mathbb{Z}/3\mathbb{Z}$. This would require $|\ker \psi| = 2$, but S_3 has no normal subgroups of order 2.

5. The function $f : \mathbb{Z} \to GL_2(\mathbb{R})$ defined by $f(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is a group homomorphism.

Answer. True. We check: $f(a)f(b) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = f(a + b)$.

6. The subgroup $S_3 \subset S_4$ is a normal subgroup of S_4.

Answer. False. We compute the left and right cosets $(24)S_3$ and $S_3(24)$ and see that they are different:

$$\begin{aligned}
(24)S_3 &= \{(24)e, (24)(12), (24)(13), (24)(23), (24)(123), (24)(132)\} \\
&= \{(24), (241), (24)(13), (243), (1243), (1324)\} \\
S_3(24) &= \{e(24), (12)(24), (13)(24), (23)(24), (123)(24), (132)(24)\} \\
&= \{(24), (142), (13)(24), (234), (1423), (1342)\}
\end{aligned}$$

7. If H and K are subgroups of a group G and $|H| = 8$ and $|K| = 15$, then $H \cap K = \{e\}$.

Answer. True. $H \cap K$ will be a subgroup of both H and K, so $|H \cap K|$ will divide both 8 and 15, so $|H \cap K| = 1$.

8. The groups $(\mathbb{Z}/12\mathbb{Z})^*$ and $(\mathbb{Z}/5\mathbb{Z})^*$ are isomorphic.

Answer. False. $\mathbb{Z}/12\mathbb{Z} = \{1, 5, 7, 11]\}$ and $\mathbb{Z}/5\mathbb{Z} = \{1, 2, 3, 4\}$. In $\mathbb{Z}/12\mathbb{Z}$, every element squares to 1, but in $\mathbb{Z}/5\mathbb{Z}$ we have $2^2 = 4 \neq 1$.

9. If $\phi : G \to G'$ is a group homomorphism then $\phi(a) = \phi(b)$ if and only if a and b are in the same coset of $\ker(\phi)$.

Answer. True.

$$\phi(a) = \phi(b) \Leftrightarrow \phi(a)^{-1}\phi(b) = e \Leftrightarrow \phi(a^{-1}b) = e \Leftrightarrow a^{-1}b \in \ker(\phi) \Leftrightarrow a\ker(\phi) = b\ker(\phi)$$

Mathematical writing [3 pts]

Choose *one* of the following problems.

10. The subgroup $K = \{1, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. Your problem: compute the cosets of K and compute a multiplication table for S_4/K.

Answer. Here are the cosets of K:

$$\begin{aligned} K &= \{1, (12)(34), (13)(24), (14)(23)\} \\ (12)K &= \{(12), (34), (1423), (1324)\} \\ (13)K &= \{(13), (1432), (24), (1234)\} \\ (23)K &= \{(23), (1243), (1342), (14)\} \\ (123)K &= \{(123), (243), (142), (134)\} \\ (132)K &= \{(132), (143), (234), (124)\} \end{aligned}$$

Here is the multiplication table:

	K	$(12)K$	$(13)K$	$(23)K$	$(123)K$	$(132)K$
K	K	$(12)K$	$(13)K$	$(23)K$	$(123)K$	$(132)K$
$(12)K$	$(12)K$	K	$(123)K$	$(132)K$	$(13)K$	$(23)K$
$(13)K$	$(13)K$	$(132)K$	K	$(123)K$	$(23)K$	$(13)K$
$(23)K$	$(23)K$	$(123)K$	$(132)K$	K	$(12)K$	$(13)K$
$(123)K$	$(123)K$	$(23)K$	$(12)K$	$(13)K$	$(132)K$	K
$(132)K$	$(132)K$	$(13)K$	$(23)K$	$(12)K$	K	$(132)K$

The way this works out should be a comfort. In class, we described a surjective group homomorphism $S_4 \to S_3$ with kernel K. So, it must be the case that $S_4/K \cong S_3$, as this multiplication table illustrates.

11. Define what it means for a subgroup N of a group G to be *normal*. Give an example of a group G with a subgroup N that is normal and a subgroup H that is not normal. Justify your answer.

Answer. A subgroup N of a group G is normal if and only if $gN = Ng$ for all $g \in G$.

For an example, consider $N = \{e, (123), (132)\} \subset S_3$. Here, N is normal since $gN = \{(12), (13), (23)\} = Ng$ for any $g = (12), (13), (23)$ and $gN = N = Ng$ for $g = e, (123), (132)$. The subgroup $H = \{e, (12)\}$ is not normal since $(123)H = \{(123), (23)\}$ and $H(123) = \{(123), (13)\}$ are different.

12. Let $\phi : G \to G'$ be a group homomorphism. Define what it means for ϕ to be *monic* and for ϕ to be *left-invertible*. Prove that if ϕ is left invertible then ϕ is monic, but not conversely.

Answer. A group homomorphism $\phi : G \to G'$ is monic if and only if for all group homomorphisms $\alpha : G'' \to G$ and $\beta : G'' \to G$, we have $\phi\alpha = \phi\beta \Rightarrow \alpha = \beta$. A group homomorphism $\phi : G \to G'$ is left-invertible if and only if there exists a group homomorphism $\psi : G' \to G$ with $\psi\phi = \mathrm{id}_G$.

To prove that left invertible implies monic, suppose $\phi : G \to G'$ is left invertible and we have group homomorphisms $\alpha : G'' \to G$ and $\beta : G'' \to G$ with $\phi\alpha = \phi\beta$. Since ϕ is left invertible, there exists a group homomorphism $\psi : G' \to G$ with $\psi\phi = \mathrm{id}_G$. Composing $\phi\alpha = \phi\beta$ with ψ yields

$$\psi\phi\alpha = \psi\phi\beta \Rightarrow \mathrm{id}_G\,\alpha = \mathrm{id}_G\,\beta \Rightarrow \alpha = \beta.$$

To see that it is possible to be monic and not left invertible, consider the homomorphism $\phi : \mathbb{Z}/3\mathbb{Z} \to S_3$ defined by $0 \mapsto e$, $1 \mapsto (123)$, $2 \mapsto (132)$. Here, ϕ is monic. To prove that, suppose $\alpha : G'' \to \mathbb{Z}/3\mathbb{Z}$ and $\beta : G'' \to \mathbb{Z}/3\mathbb{Z}$ are homomorphisms that are not equal. Then there will be an element $g \in G''$ with $\alpha(g) \neq \beta(g)$. Then $\phi\alpha(g) \neq \phi\beta(g)$ since ϕ is injective.

To see that ϕ is not left invertible, suppose $\psi : S_3 \to \mathbb{Z}/3\mathbb{Z}$ satisfies $\psi\phi = \mathrm{id}_{\mathbb{Z}/3\mathbb{Z}}$. Then $\psi(e) = 0$, $\psi(123) = 1$ and $\psi(132) = 2$, so ψ is surjective. Hence $S_3/\ker(\psi) \cong \mathbb{Z}/3\mathbb{Z}$. This would require $|\ker\psi| = 2$, but S_3 has no normal subgroups of order 2.