Covering Spaces II

"The Big Picture"

Reminder about based covers:

If there exists a base-point preserving map of covers then it is unique.

$$\begin{tikzcd} & E, e \arrow[d, "p"] \\ E', e' \arrow[ur, dashed] \arrow[r, "p'"'] & B, b \end{tikzcd}$$

when E' is path connected.

So the automorphism group

$$\mathrm{Aut}((E,e),(B,b)) = \{\mathrm{id}\}$$

of a based cover isn't very interesting.

And the entire category of based, connected covers is a poset equivalent to the poset of (conjugacy classes) of subgroups of $\pi_1(B,b)$:

$$
\begin{tikzcd}
(\tilde{E},\tilde{e}) \arrow[d] \arrow[dr] \arrow[rr] & & (E,e) \arrow[dr] & & \\
{} & {} & & \cdots \arrow[dr] & \\
& & & & (B,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(\tilde{E},\tilde{e}) = \{*\} \arrow[d] \arrow[dr] \arrow[rr] & & \pi_1(B,b) \arrow[dr] & & \\
{} & {} & & \cdots \arrow[dr] & \\
& & & & \pi_1(B,b)
\end{tikzcd}
$$

However, restricting to based covers of B conceals the very interesting algebraic action that $\pi_1(B)$ has on the fibers of a covering map that is at work behind the scenes!

Fix a space B. We have three interesting categories associated to B:

$\mathrm{Cov}(B)$

The category of unbased covers of B

$\mathrm{Set}^{\pi_1(B)}$

Set-valued functors on $\pi_1(B)$, the fundamental groupoid of B.

G-Sets

$G = \pi_1(B,b)$

The category of $\pi_1(B,b)$-Sets = Sets with an action of the fundamental group.

Theorem: Under mild topological hypotheses on B these three categories are equivalent.

$\mathrm{Cov}(B)$

$\mathrm{Set}^{\pi_1(B)}$

G-Sets, $G = \pi_1(B,b)$

everything about covering spaces

is encoded in these two (equivalent) algebraic categories.

Excellent Theorem

inclusion is an equivalence of categories

$$
\begin{tikzcd}
\mathbf{Set}_{\pi_1(B)} \arrow[r, bend right, "\text{pick point } b \in B"'] & G\text{-}\mathbf{Sets},\ G = \pi_1(B,b) \arrow[l, bend right, "\text{if } B \text{ is path connected}"']
\end{tikzcd}
$$

$$\boxed{\mathrm{Cov}(B)} \xrightarrow{\text{Key Passage}} \boxed{\mathrm{Set}^{\pi_1(B)}}$$

$$\boxed{\mathrm{Cov}(B)}$$

objects are covers $\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}$

Morphisms are

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

$$\boxed{\mathrm{Set}^{\pi_1(B)}}$$

objects are functors

$$F : \pi_1(B) \longrightarrow \mathrm{Sets}$$

point $b \in B \longrightarrow$ set S_b

(homotopy classes of) path $b \to b'$ $\longrightarrow$ function $S_b \to S_{b'}$

Morphisms in $\mathrm{Set}^{\pi_1(B)}$ $\qquad F \Rightarrow G$

$$\{ S_b \longrightarrow T_b \}_{b \in B}$$

satisfying

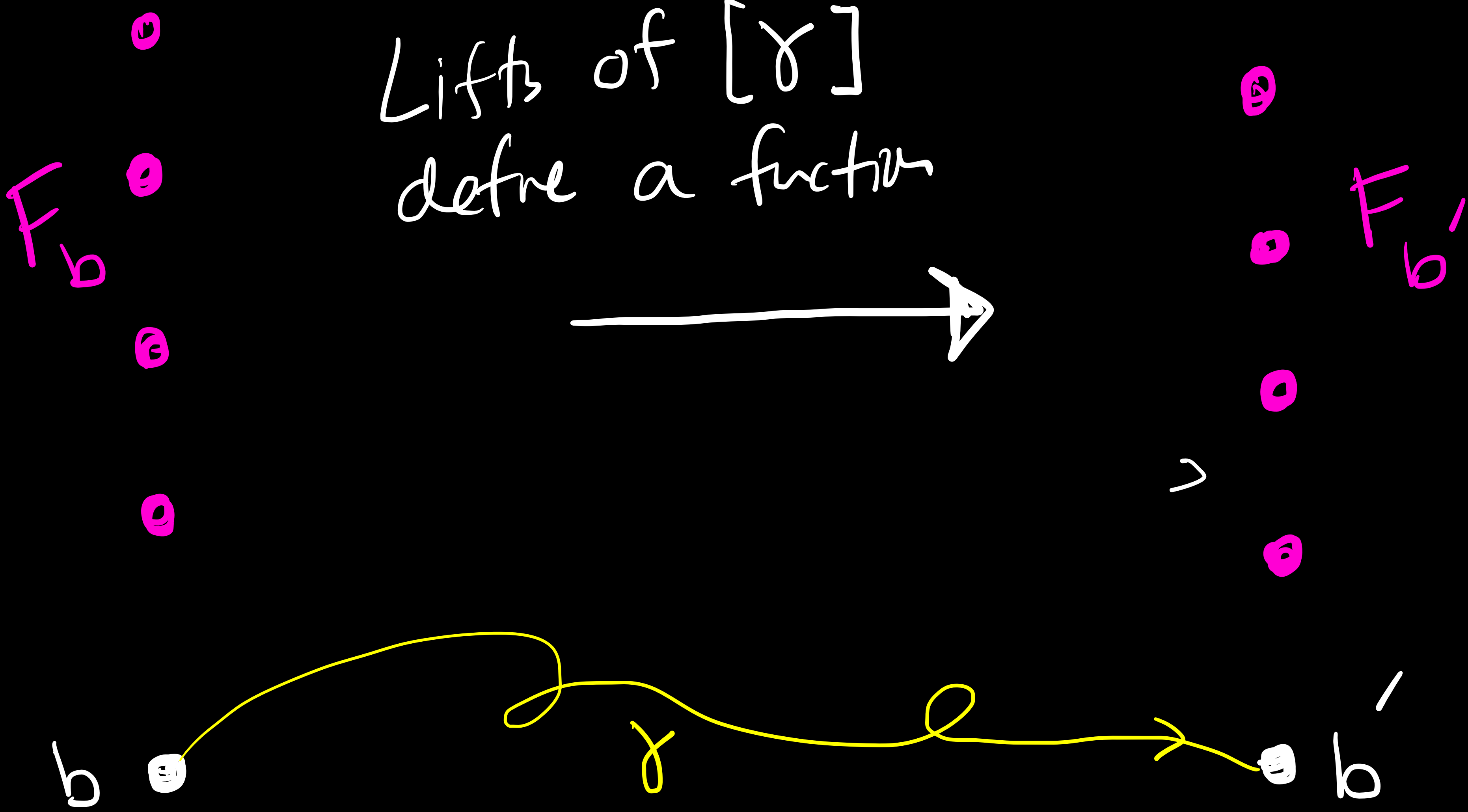
$$\begin{tikzcd} S_b \arrow[r] \arrow[d] & T_b \arrow[d] \\ S_a \arrow[r] & T_a \end{tikzcd}$$

$$\boxed{\mathrm{Cov}(B)} \xrightarrow[\text{objects}]{} \boxed{\mathrm{Set}^{\pi_1(B)}}$$

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array} \longmapsto \quad b \longmapsto F_b := p^{-1}(b)$$

$$b \quad b' \longmapsto F_b \to F_{b'}$$

Lifts of $[\gamma]$ define a function



$$\boxed{\mathrm{Cov}(B)} \xrightarrow[\text{morphisms}]{} \boxed{\mathrm{Set}^{\pi_1(B)}}$$

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr] & & E' \arrow[dl] \\
& B &
\end{tikzcd}
\qquad
F_b \xrightarrow{\varphi} F'_b$$

$$\begin{tikzcd}
F_b \arrow[r] & F_a \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi"'] \arrow[d, "\rho"'] & F'_b \arrow[r] \arrow[dl, "\rho'"] & F'_a \\
& b \overset{\gamma}{\frown} a & &
\end{tikzcd}$$

B needs a universal cover,

locally path connected, connected.

$$\begin{tikzcd} \boxed{\mathrm{Cov}(B)} \arrow[r, bend right, "\text{just explained this}"'] & \boxed{\mathrm{Set}^{\pi_1(B)}} \arrow[r, bend right, "\text{choose } b\in B"'] & \boxed{\begin{array}{c} G\text{-Sets} \\ G=\pi_1(B,b) \end{array}} \arrow[ll, bend right] \end{tikzcd}$$

Everything about covering spaces

$\mathrm{Cov}(B)$

G-Sets

$G = \pi_1(B, b)$

is encoded in this category

The category G-set

See the video on G-set!

$G = e\backslash G$

$H\backslash G$

$K\backslash G$

$\cdots$

$\{*\} = G\backslash G$

$\mathrm{Aut}(H\backslash G) \cong N(H)/H$

Let's complete the picture:

For a base space B

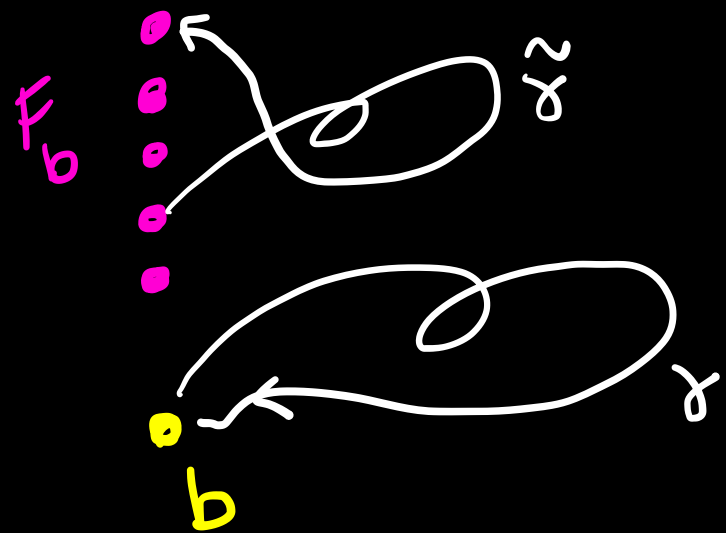
path connected
locally path connected
locally simply connected

choose a point $b \in B$

$$G := \pi_1(B, b)$$

To each cover

$$\begin{array}{c} E \\ \downarrow \\ B \end{array} \rightsquigarrow \text{Action of } G = \pi_1(B,b) \text{ on } F_b$$



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For any $e \in F_b$

$$\mathrm{Stab}(e) = \pi_1(E,e) \subseteq \pi_1(B,b).$$

If E is connected then the action of $\pi_1(B,b)$ on F_b is isomorphic to

$$\pi_1(E,e) \backslash \pi_1(B,b)$$

②

$$\begin{array}{c} E \\ \downarrow \\ B \end{array} \text{ connected} \Rightarrow |F_b| \cong |F_{b'}| \quad \forall\, b, b' \in B$$

Because both are equal to index $\pi_1(E, e)$ in $\pi_1(B, b)$

③

Every cover $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ decomposes into the coproduct of connected covers $E \cong \coprod E_\alpha$ $\begin{array}{c} E_\alpha \\ \downarrow \\ B \end{array}$.

④ And the other way around!

For every subgroup H of $G := \pi_1(B,b)$

$\exists$ cover E and a point $e \in E$ so
connected

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

that $\pi_1(E,e) = H$

Unique up to iso

(5)

Every morphism of covers

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[dl] \\ B & \end{tikzcd}$$

induces a G-equivariant map

$$F_b \xrightarrow{f} F_b'$$

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There exists a morphism of covers

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr] & E' \arrow[d] \\ & B \end{tikzcd}$$

sending $e \overset{f}{\longmapsto} e'$

$\Longleftrightarrow$

$\pi_1(E,e) \subseteq \pi_1(E,e')$

in $G = \pi_1(B,b)$

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The automorphism group of a fixed connected cover is isomorphic to

$$\mathrm{Aut}\left(\begin{array}{c} E \\ \downarrow \\ B \end{array}\right) \cong N(H)/H \qquad H := \pi_1(E, e) \text{ for } e \in F_b.$$

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Special Case: If $\tilde{E}$ is the universal cover of B, then

$$\operatorname{Aut}\left(\begin{array}{c}\tilde{E}\\ \downarrow\\ B\end{array}\right) \cong \pi_1(B,b).$$

(You can use this to compute $\pi_1(\mathbb{RP}^2)$, $\pi_1(S^1)$, $\pi_1(SO(3))$, ...)

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Moreover, if $H = \pi_1(E,e)$ is normal in $G = \pi_1(B,b)$ then

$$\mathrm{Aut}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \cong G/H$$

and for any $e_1, e_2 \in F_b$ there exists $f \in \mathrm{Aut}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix}$ with $f(e_1) = e_2$.

terminology: $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ "normal" or "regular".

The category Cov(B)

$$\begin{tikzcd} \tilde{E} \arrow[d] \arrow[rr, bend left] \arrow[r] & E \arrow[d] \arrow[r, phantom, "\cong"] \arrow[r, leftrightarrow] & E' \arrow[d] \\ B & B & B \end{tikzcd}$$

$\pi_1(\tilde{E}) = \{*\}$

$\pi_1(E,e) \overset{\text{conj}}{\longleftrightarrow} \pi_1(E',e')$

$G \circlearrowright\ \subseteq \pi_1(B,b) \qquad \subseteq \pi_1(B,b)$

$$\mathrm{Aut}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \cong N(\pi_1(E,e)) / \pi_1(E,e)$$

connected covers

all covers = coproducts of these.

....

$$\begin{array}{c} B \\ \downarrow \\ B \end{array}$$