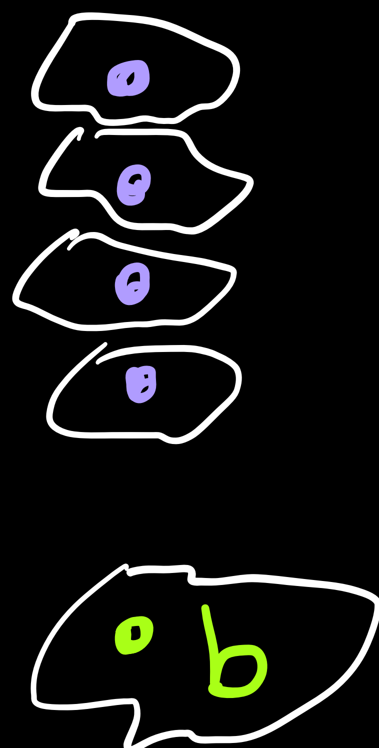
Covering Spaces

- what they are and basic properties
- Category of covers is equivalent to a nice algebraic category

Definition: A map $E \xrightarrow{p} B$ is a covering map or a cover iff

$$\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$$



$$p^{-1}(U) \cong \underbrace{F_b}_{\text{discrete}} \times U$$

Covers are quotient maps

Definition: A morphism from a cover $E \to B$ to a cover $E' \to B$ is a map

$$\begin{tikzcd} E' \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

For a base space B we have a category $\mathrm{Cov}(B)$.

Path lifting Property

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, hook] & E \arrow[d] \\
I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists ! \tilde{\gamma}"] & B
\end{tikzcd}
$$

I

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, hook] & E \arrow[d] \\
I \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\simeq"] \arrow[ur, dashed, bend left, "\tilde{\gamma}"] \arrow[ur, dashed, bend right, "\tilde{\gamma}'"'] \arrow[ur, phantom, "\simeq"] & B
\end{tikzcd}
$$

$\tilde{\gamma}(1) = \tilde{\gamma}'(1)$

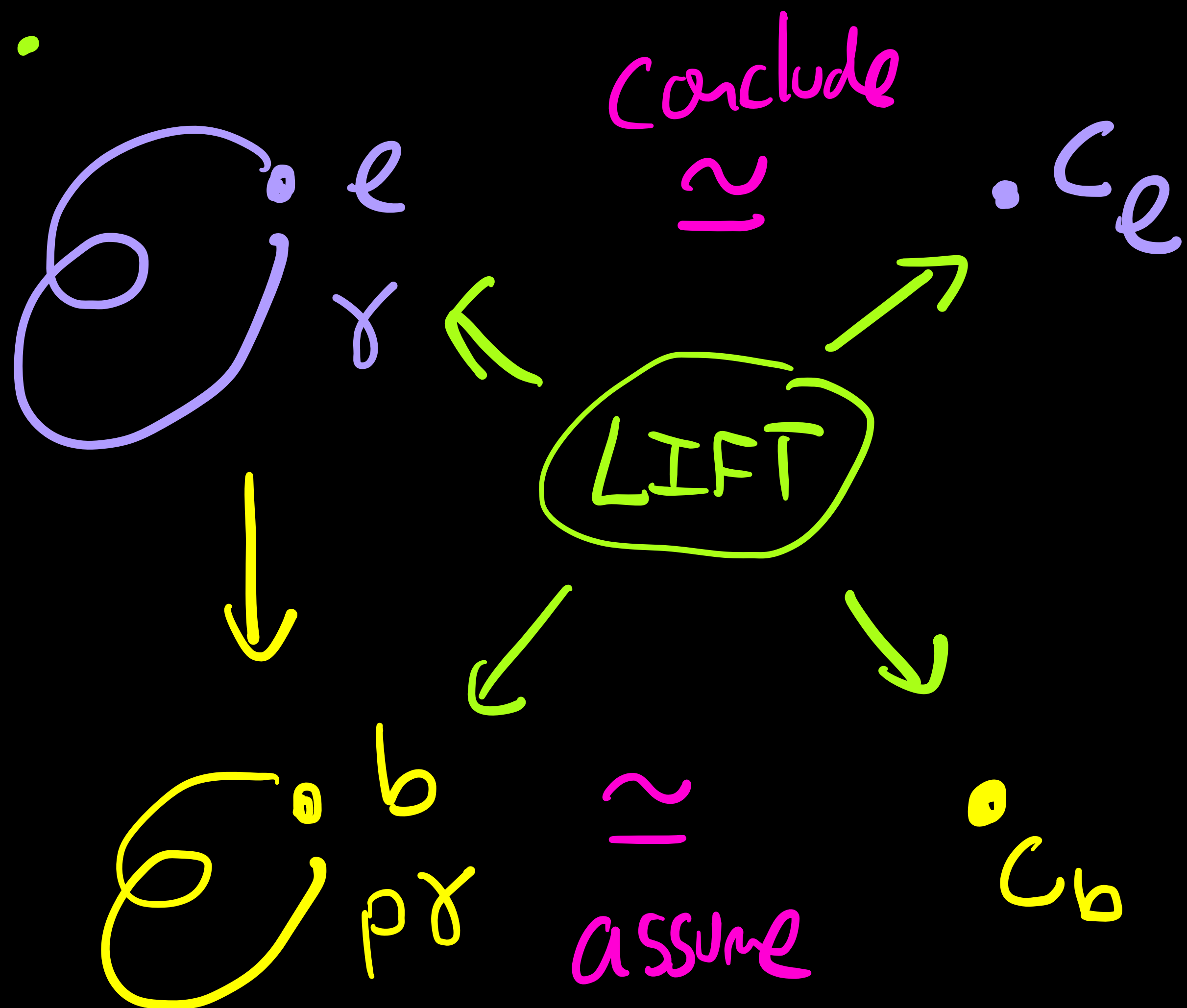
II

e_0
$\tilde{\gamma}'$
$\tilde{\gamma}$
same endpoints!
γ
γ'
b_0
B

Path Lifting Property $\Longrightarrow$

$$\begin{array}{c} \pi_1(E,e) \\ \Big\downarrow \ \text{injective} \\ \pi_1(B,b) \end{array}$$

Proof:



Conclusion:

view

$$\pi_1(E,e) \leq \pi_1(B,b)$$

subgroup

General Map Lifting of Covers:

check:

$\tilde{X}$ connected $\implies$ lift is unique if it exists.

Does $\exists$ lift?

$$
\begin{tikzcd}
 & E \arrow[d] \\
\tilde{X} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
$$

Does there exist a lift?

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Easy necessary condition: Suppose lift exists and then apply π_1

$$\begin{tikzcd} & \pi_1(E,e) \arrow[d] \\ \pi_1(X,x) \arrow[r] \arrow[ur] & \pi_1(B,b) \end{tikzcd}$$

Necessary condition for a lift to exist:

$$\pi_1(X,x) \subseteq \pi_1(E,e)$$

(viewed inside $\pi_1(B,b)$)

__Theorem:__ X is locally path connected and connected.

$$\begin{tikzcd} & E,e \arrow[d] \\ X,x \arrow[r] \arrow[ur, "\exists!"] & B,b \end{tikzcd} \iff \pi_1(X,x) \to \pi_1(E,e) \subseteq \pi_1(B,b)$$

Morphisms of covers are solutions to the lifting problem !!!

$\exists$ morphism of covers $\iff$ $\pi_1(E',e') \subseteq \pi_1(E,e)$

$$\begin{tikzcd} & E,e \arrow[d] \\ E',e' \arrow[ur, dashed] \arrow[r] & B,b \end{tikzcd}$$

Picture of the category of covers

(based, connected, locally path connected)

$$\begin{tikzcd}[column sep=small]
 & & (\tilde{E},\tilde{e}) \arrow[dl, dotted] \arrow[d] \arrow[ll] & & & & & \{e\} \arrow[dl, dotted] \arrow[d] \arrow[ll] \\
(E'',e'') \arrow[d] & & (E,e) \arrow[dl] & & H'' \arrow[d] & & & H \arrow[dl] \\
(E',e') \arrow[r] & (B,b) & & & H' \arrow[r] & G=\pi_1(B,b) & &
\end{tikzcd}$$

Category, Cov(B) is quite thin.

1

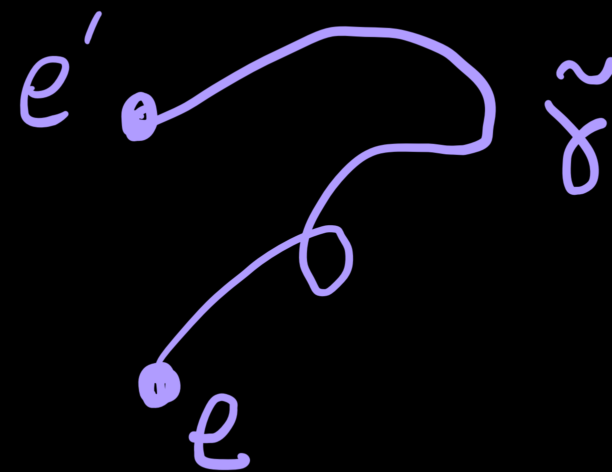
based
connected
locally path connected

$$\begin{tikzcd}
E', e' \arrow[rr] \arrow[dr] & & E, e \arrow[dl] \\
& B, b &
\end{tikzcd}$$

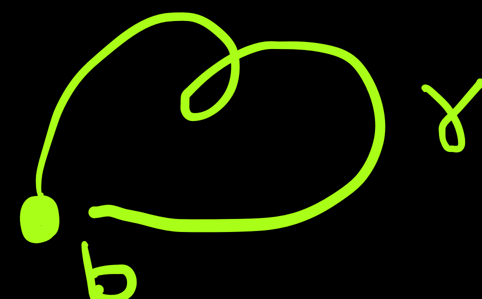
If $\exists$ cover $(\tilde{E}, \hat{e}) \longrightarrow (B, b)$ with $\pi_1(\tilde{E}, \hat{e}) = \{*\}$ then it is universal meaning it maps to every other cover uniquely.

Moreover, for every subgroup $H \subseteq \pi_1(B, b)$ and every pt $e \in F_b$, there exists a cover $(E, e) \longrightarrow (B, b)$ obtained as a quotient of $(\tilde{E}, \hat{e})$:

$$\begin{array}{c} \tilde{E}, \tilde{e} \\ \big\downarrow \\ B, b \end{array}$$



$E := \tilde{E}/_{e \sim e'}$



$[\gamma] \in H$

Picture of $Cov(B)$ when B has a universal cover:

simplified picture since based connected

$$\begin{tikzcd}
& & {(\tilde{E},\tilde{e})} \arrow[dl] \arrow[d] & \\
& {(E'',e'')} \arrow[dl] & \cdots & \\
{(E,e)} \arrow[dr] & & {(E',e')} \arrow[dl] & \\
& {(B,b)} & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& & \{e\} \arrow[dl] \arrow[d] & \\
& H'' \arrow[dl] & \cdots & \\
H \arrow[dr] & & H' \arrow[dl] & \\
& \pi_1(B,b) & &
\end{tikzcd}$$

When Does a space B have a universal cover?

~~Necessary~~ Condition:

Necessary + Sufficient!

$$\pi_1(\tilde{E}, \tilde{e}) = \{*\}$$

$$\begin{tikzcd}
V, \tilde{e} \arrow[r] & \tilde{E}, \tilde{e} \arrow[d] \\
U, b \arrow[u, "\text{locally iso}"] \arrow[r] \arrow[r, bend left=60] & B, b
\end{tikzcd}$$

$\Rightarrow$ $\pi_1(U, b) \to$ trivial subgrp of $\pi_1 B, b$

$$\tilde{E} = \rho_b B / \nu$$