$\boxed{\begin{array}{c}G\text{-Sets}\\ G=\pi_1(B,b)\end{array}}$

Objects are sets S together with an action of G

$$S \times G \longrightarrow S$$

Morphisms are G-equivariant functions

$$S \xrightarrow{f} T \qquad f(sg) = f(s)g$$

$\boxed{\begin{array}{c}\text{G-Sets} \\ G = \pi_1(B, b)\end{array}}$

S is a G set

$a \in S \rightsquigarrow \boxed{\mathrm{Stab}(a)} \subseteq G$

$\rightsquigarrow \boxed{\mathrm{Orbit}(a)} = \{ag : g \in G\}$

G acts transitively $\Leftrightarrow$ S = one orbit

G-Sets

$G = \pi_1(B, b)$

Theorem: Every G-set decomposes into the disjoint union of sets on which G acts transitively

Thm: Every G-set S on which G acts transitively is isomorphic (as a G-set) to $H \backslash G$ = the set of right cosets of a subgrp H = Stabilizer(a).

Proof: G acts transitively on S. Choose $a \in S$ and let $H = \mathrm{Stab}(s)$.

$$H \backslash G \longleftrightarrow S$$

$$Hg \longleftrightarrow ag$$

$\blacksquare$

Detail: $\mathrm{Stab}(a)$ and $\mathrm{Stab}(a')$ are conjugate subgroups of G.

$a' = ag \implies \mathrm{Stab}(a') = g^{-1}\,\mathrm{Stab}(a)\,g$

$\implies H \backslash G \cong g^{-1}Hg \backslash G$

as G-sets.

Also: G acts transitively on S $\Rightarrow$

$S \cong {}_{\mathrm{Stab}(a)}\backslash G$ $\Rightarrow$

$$|\mathrm{Orbit}(a)|\,|\mathrm{Stab}(a)| \cong |G|$$

"Orbit-Stabilizer Theorem"

The category G-set

- All objects are coproducts of homogeneous G-sets (those on which G acts transitively)
- Every homogeneous G-set $S \cong H\backslash G$ for some H $(= \mathrm{stab}(a))$
- <u>Morphisms</u> ?

Every morphism $H\backslash G \xrightarrow{f} K\backslash G$ is determined

$H \longmapsto Kg$ for some $g \in G$

- well defined iff $gHg^{-1} \in K$.
- two morphisms are equal iff $Kg = Kg'$

$f \quad H \longmapsto Kg$

$f' \quad H \longmapsto Kg'$

Conclusion: $\exists$ morphism of G-sets

$$H\backslash G \longrightarrow K\backslash G \quad \text{iff}$$

$\exists g \in G$ with $g^{-1}Hg \subseteq K$

Theorem: The group of automorphisms of $H\backslash G$ as a G-set is isomorphic to $N(H)/H$.

Proof: Morphisms $H\backslash G \to H\backslash G$ correspond to $g \in G$ with $g^{-1}Hg \subseteq H$ and two are equal iff $Hg = Hg' \Rightarrow g^{-1}g' \in H$. ∎

The category G-set

$G = e\backslash G$

$\mathrm{Aut}(H\backslash G) \cong N(H)/H$

$H\backslash G$

$K\backslash G$

$\cdots$

$\{*\} = G\backslash G$