Instructions: Again, I pulled problems from old qualifying exams that you should be able to do. It would be good to solve them all, but just write up and turn in solutions to 1, 2c, 4, 7, 8, 10, 11, 14 (most of these have short solutions!) Due on Monday, November 14.

1 The compact open topology

1. [Problem 1 from Spring 2015]

Consider $\mathsf{Top}([0,1],\mathbb{R})$ with the compact open topology. Define a family of functions $\mathcal{F} \subset \mathsf{Top}([0,1],\mathbb{R})$ by $\mathcal{F} = \{f_a : 0 < a < 1\}$ where $f_a(x) = 1 - \frac{x}{a}$. Is $\mathcal{F}$ compact?

2. [Problem 4 from Fall 2017 and 3 from Spring 2021]

Let $X = \mathsf{Top}([0,1],[0,1])$ with the compact open topology.

(a) Is the function $ev : X \times [0,1] \to [0,1]$ defined by $(f,x) \mapsto f(x)$ continuous?

(b) Is the map $I : X \to [0,1]$ defined by $f \mapsto \displaystyle\int_0^1 f$ continuous?

(c) Is the evaluation map $ev : X \times [0,1] \to [0,1]$ defined by $(f,x) \mapsto f(x)$ continuous if X is given the product topology.

2 Compactifications

3. [Problem 7 from Fall 2021]

Suppose X and Y are noncompact, locally compact, and Hausdorff. Prove that $(X \times Y)^* \cong X^* \wedge Y^*$. Here $(-)^*$ means the one-point compactification and $\wedge$ means the smash product of pointed spaces, with the base point being the point added in the compactification.

4. [Problem 4 from Fall 2015]

Prove that any compact Hausdorff space is a retract of its Stone-Čech compactification.

5. [Problem 7 from Fall 2015]

Prove that the Stone-Čech compactification of the rationals is uncountable.

6. [Problem 4c and 4d from Spring 2016]

Look up what the projective space $\mathbb{RP}^n$ are.

(a) Prove or disprove: the one-point compactification of $\mathbb{R}$ is $\mathbb{RP}^1$.

(b) Prove or disprove: the one-point compactification of $\mathbb{R}^2$ is $\mathbb{RP}^2$.

3 CW complexes and Euler characteristic

7. [Problem 12a from Spring 2015]

Show that $S^1 \vee S^1 \vee S^2$ and the torus T both have CW-structures with four cells: one 0-cell, two 1-cells, and one 2-cell.

8. [Problem 12 from Fall 2016 15c from Spring 2021]

The number of cells is a CW structure on a space X is not an invariant of the space. It turns out, however, that the alternating sum is and invariant called the Euler characteristic of X.

$$\chi(X) := \sum (-1)^i d_i$$

where d_i is the number of i cells of a CW structure on X.

(a) Compute $\chi(T^n)$ where $T^n = (S^1)^{\times n}$.

(b) Compute $\chi(S^2 \times T^2)$.

4 Suspensions

9. [Problems 4 and 7 from Spring 2000]

Let ΣX be the suspension of X. Prove that

(a) If X is connected, then ΣX is simply connected.

(b) If X is contractible, then ΣX is contractible.

(c) X is compact if and only if ΣX is compact.

(d) X is Hausdorff if and only if ΣX is Hausdorff.

10. [Problem 4 from Spring 2017] Let $\mathbf{Top}_*$ be the category of pointed topological spaces and let $\mathbf{\Sigma} : \mathbf{Top}_* \to \mathbf{Top}_*$ be the functor that sends a pointed space X to its reduced suspension $\Sigma X := S^1 \wedge X$. Let $\mathbf{\Omega} : \mathbf{Top}_* \to \mathbf{Top}_*$ be the functor that sends a pointed space X to its based loop space $\Omega X := \mathbf{Top}_*(S^1, X)$. The setup

$$\mathbf{\Sigma} : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : \mathbf{\Omega}$$

defines an adjunction. Your problem: describe the unit and counit of this adjunction.

5 And a few miscellaneous problems...

11. [Problem 4 from Fall 2000]

Let $1 \leq n \leq \infty$ and let $X_n = S^1 \vee S^1 \vee \cdots \vee S^1$ be the wedge sum of n copies of S^1. If $n = \infty$, then X_∞ is the colimit of $S^1 \subset S^1 \vee S^1 \subset S^1 \vee S^1 \vee S^1 \subset \cdots$

(a) Prove that X_n is compact if and only if $n < \infty$.

(b) Prove that X_n is connected for all n.

12. [Problem 5a from Spring 2017]

Let X be the torus and Y be the two-sphere with three points removed.

(a) Are X and Y homotopic?

(b) Are X and Y homeomorphic?

13. [Problem 1 from Part II of Fall 2005]

Let (X, x) and (Y, y) be two based spaces. Prove that $\pi_1((X, x) \times (Y, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$.

14. [Problem 5a from Spring 2002]

Let A be a retract of X. This means that A is a subset of X and the inclusion $i : A \to X$ is left-invertible. Prove that the inclusion $i : A \to X$ induces a one-to-one map in homology.

Note: For this problem, you only need to know that homology is a functor.

15. [Problem 7 from Spring 2021 and 8 from Fall 2021]

Does the functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ have a left or right adjoint $F : \mathsf{Grp} \to \mathsf{Top}_*$? Is π_1 representable?