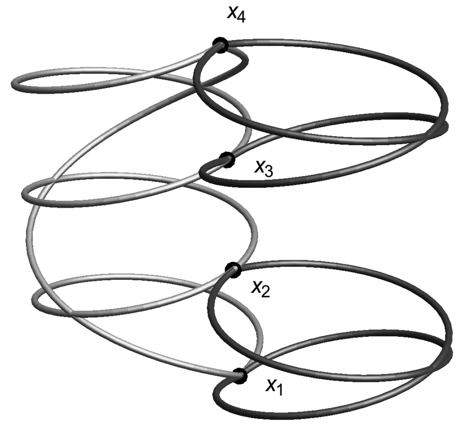
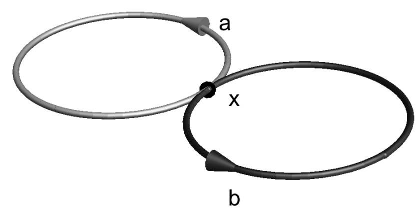
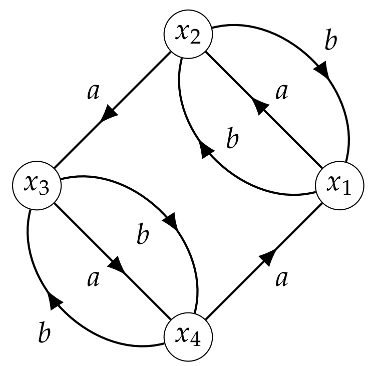
1. Let $B = S^1 \vee S^1$ and consider the cover $p : E \to B$ pictured below





I've tried to sketch the covering space in $\mathbb{R}^3$ and the base space in the xy-plane with the covering map being project $(x, y, z) \mapsto (x, y)$. The seeming self intersections in E are an illusion of the way I'm picturing this, like the way a drawing of a Klein bottle appears to intersect itself. You might find it more helpful to have this other picture of E below. Here, the space E is pictured as a graph and the map $E \to B$ is described by the labelling and arrows.



(a) Compute $\pi_1(E, x_1)$ and $\pi_1(E, x_2)$ as subgroups of $\pi_1(B, x)$.

Hint: Use the picture. Let a be the class of the loop in B starting at x and proceeding counterclockwise once around the circle on the left, and let b be the class of the loop in B starting at x and proceeding counterclockwise once around the circle on the right. So, $\pi_1(B, x) = F(a, b)$. Now, both $\pi_1(E, x_1)$ and $\pi_1(E, x_2)$ are isomorphic to a free group on 5 generators, since E is homotopic to $S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$, which you can see by contracting the "a" edges.

(b) Describe the action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$.

(c) Compute Aut(E), the group of automorphisms of the cover $E \xrightarrow{p} B$.

2. The free group on two generators $F(a, b)$ is a subgroup of the free group on three generators $F(c, d, e)$ and vice versa. Find covering maps $p : E \longrightarrow X$ with

(a) $F(c, d, e) = \pi_1(B, b)$ and $F(a, b) = \pi_1(E, e)$

(b) $F(a, b) = \pi_1(B, b)$ and $F(c, d, e) = \pi_1(B, e)$

3. Consider the topological space $B = \{a, b, c, d\}$ with topology

$$\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{c, d, a\}, \{a, b, c, d\}\}.$$

The universal cover of B is the space $\tilde{E} = \{a_i, b_i, c_i, d_i : i \in \mathbb{Z}\}$ with topology generated by the basic open sets $\{a_i\}$, $\{c_i\}$, $\{a_i, b_i, c_i\}$ and $\{c_i, d_{i+1}, a_{i+1}\}$. The covering map is given by

$$\begin{aligned} p :& \tilde{E} \longrightarrow B \\ & x_i \longmapsto x \end{aligned}$$

Use a covering space argument to compute the fundamental group of B.

4. Suppose $(E, e) \xrightarrow{p} (B, b)$ is any based cover. Prove that the induced map in homotopy groups $p_* : \pi_n(E, e) \longrightarrow \pi_n(B, b)$ is an isomorphism for $n \geq 2$.

5. [Problem 4 Spring 2014]

Find all connected three-fold covers of the wedge sum of two projective planes, carefully justifying your answer.

6. Problem 3 Spring 2011]

Use a covering space argument to prove that any map $\mathbb{R}P^2 \times \mathbb{R}P^2 \longrightarrow S^1$ is null-homotopic.