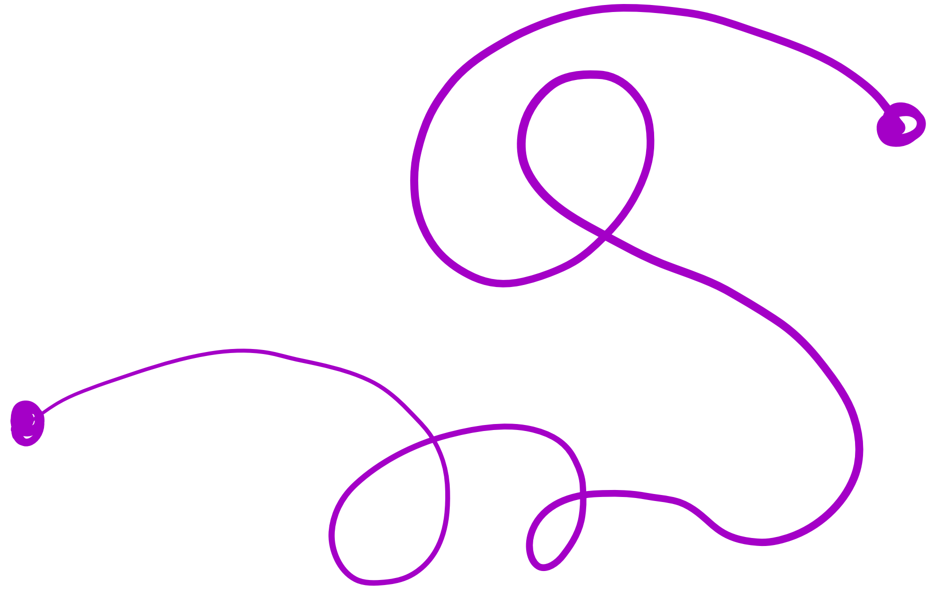
The interval

$$I = [0,1]$$

paths $[0,1] \longrightarrow X$



$$\pi_0 : \mathrm{Top} \longrightarrow \mathrm{Sets}$$
$$X \longmapsto \text{path components}$$

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Groups}$$
$$X, x \longmapsto \pi_1(X, x)$$

Homotopy

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\simeq"] & Y \end{tikzcd}$$

$$X \times I \xrightarrow{H} Y$$

$$(x, 0) \longmapsto fx$$

$$(x, 1) \longmapsto gx$$

Why $[0,1]$?

Why is the interval $[0,1]$ so important?

$0 \qquad\qquad 1$

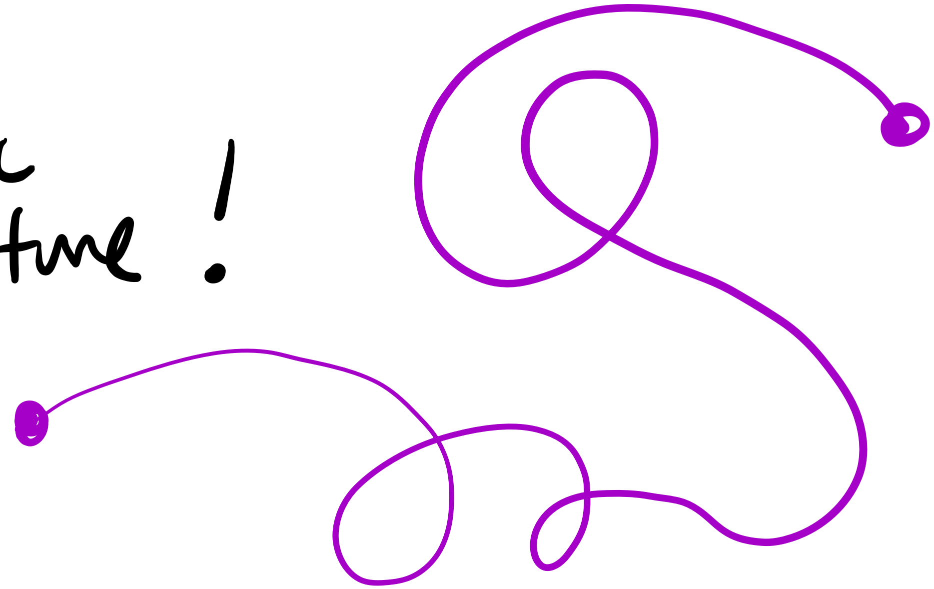
$$[0,1] \xrightarrow{\cong} [0,1] \vee [0,1]$$

$$t \xrightarrow{\cong} 2t$$

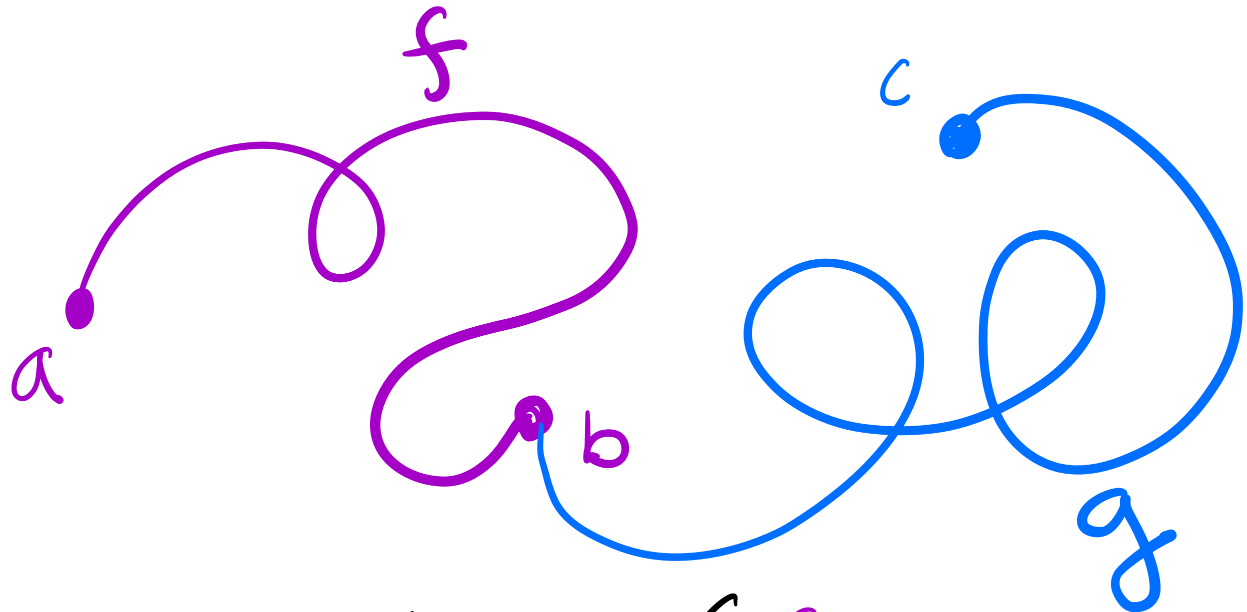
(Peter Freyd 2008)

$$\left\{ \text{paths } [0,1] \longrightarrow X \right\}$$

has algebraic structure !



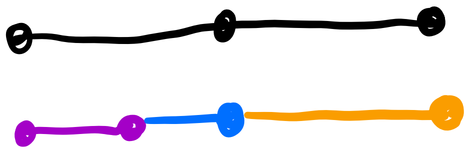
Paths can be concatenated

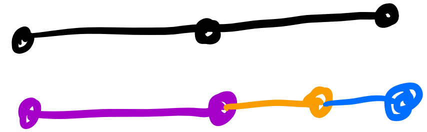


$$(gf)(t) := \begin{cases} f(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

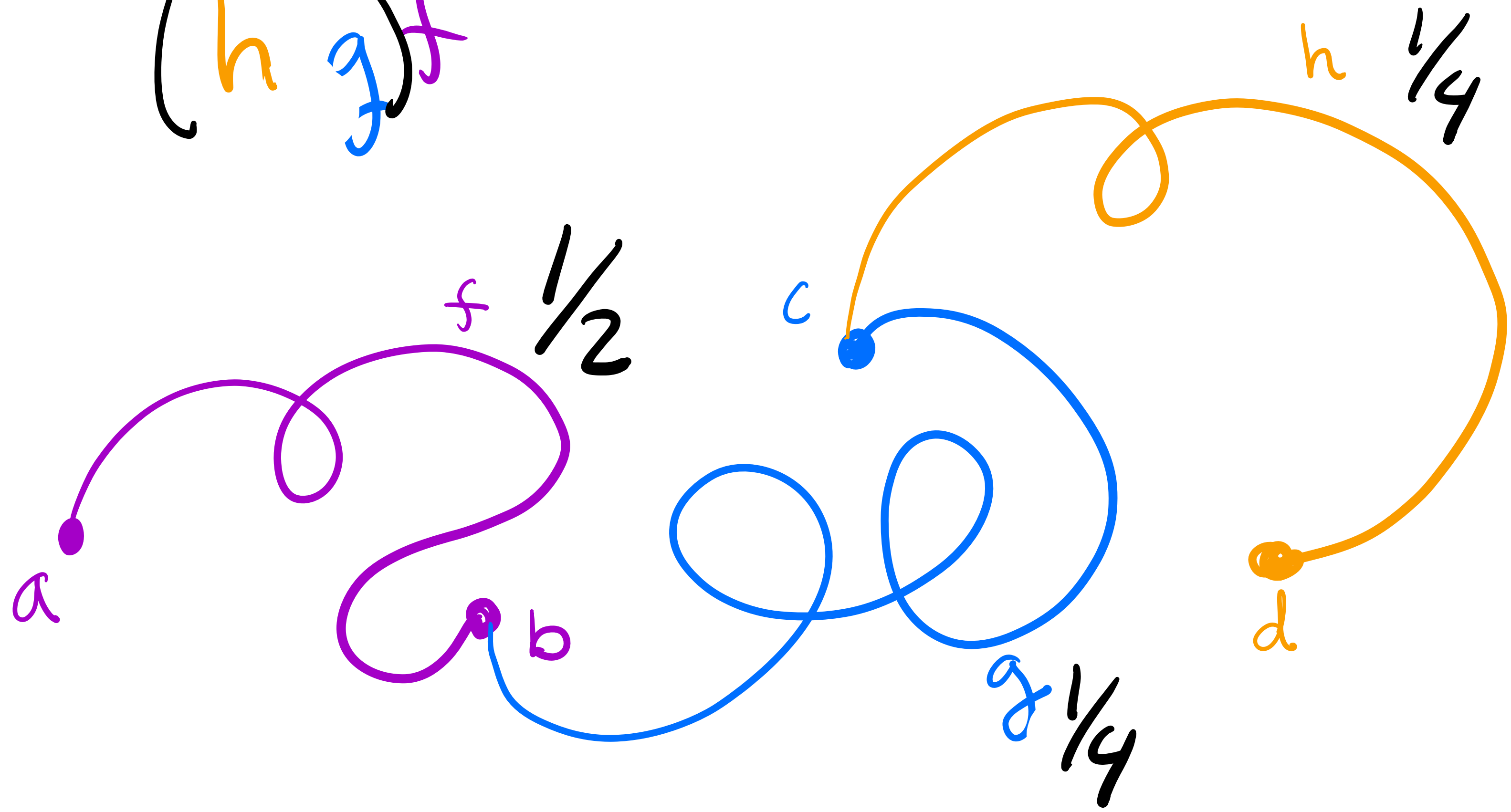
path concatenation isn't associative

$h(gf)$ vs $(hg)f$





$(h\,g)f$



Define $\sim$ on a space X by

$x \sim x' \iff \exists$ path from x to x'.

reflexive, symmetric, transitive

Equivalence classes of $\sim$ are called <u>path components</u>.

Definition: If every two points in X can be connected by a path, then we call X path connected.

Path connected is preserved by continuous surjections:

$$I \longrightarrow X \longrightarrow\!\!\!\!\!\rightarrow Y$$

$$X \longrightarrow Y \quad \rightsquigarrow$$

$$\left\{ \begin{array}{c} \text{path components} \\ \text{of } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{path components} \\ \text{of } Y \end{array} \right\}$$

$$A \longmapsto \begin{array}{l} \text{the path component} \\ \text{containing } fA \end{array}$$

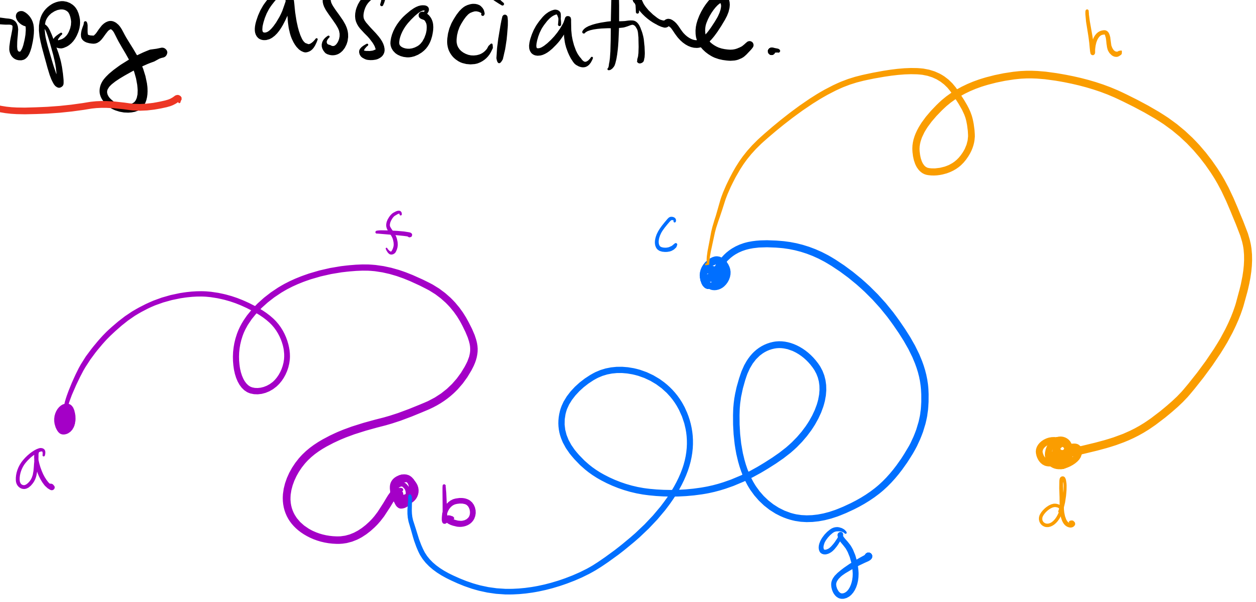
$$\pi_0 : \mathbf{Top} \longrightarrow \mathbf{Sets}$$

$$\begin{tikzcd}
X \arrow[d] & & \{\text{path components of } X\} \arrow[d] & A \arrow[d, mapsto] \\
Y \arrow[r, phantom, "\longrightarrow"] & & \{\text{path components of } Y\} & \text{the path component containing } fA
\end{tikzcd}$$

$\tilde{X}$ path connected

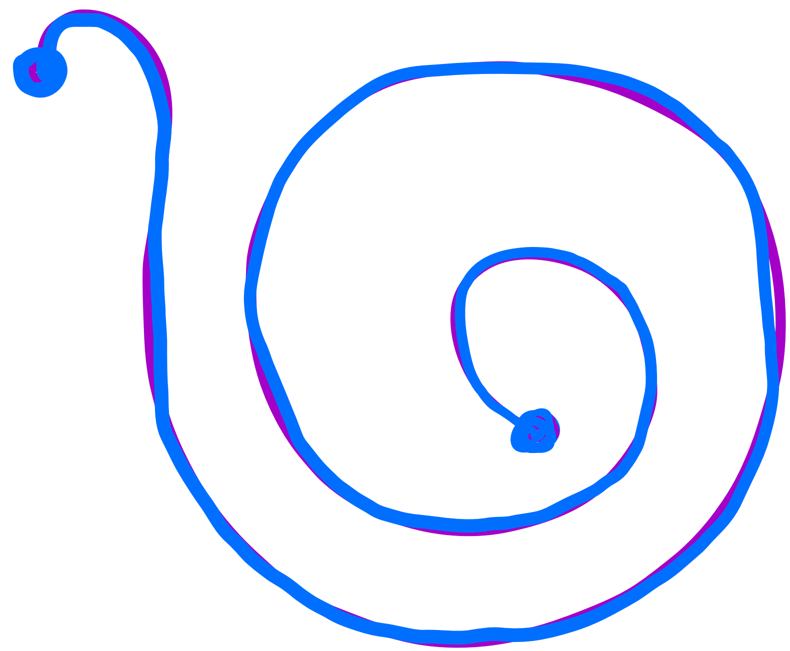
$\Leftarrow$ $\tilde{X}$ connected, locally path connected

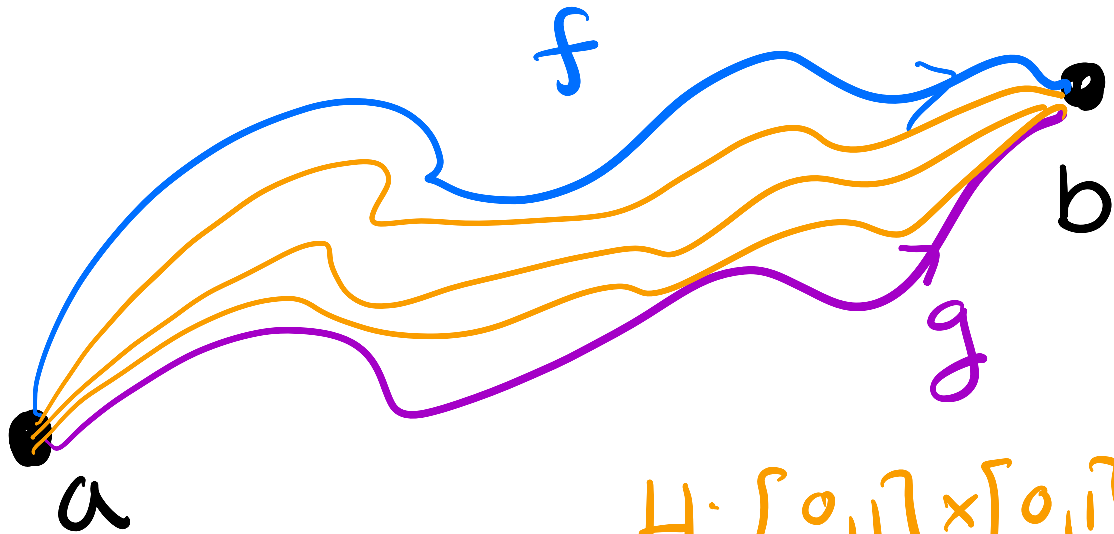
Concatenation of paths is homotopy associative.



$$h(gf) \simeq (hg)f$$

Definition: Two paths are homotopic iff there exists a homotopy between them that fixes the endpoints.





$$
\begin{aligned}
H : [0,1] \times [0,1] &\longrightarrow X \\
(t,0) &\longmapsto ft \\
(t,1) &\longmapsto gt \\
(0,s) &\longmapsto a \\
(1,s) &\longmapsto b
\end{aligned}
$$

$$h(g f) \simeq (h g) f$$

$$c_b f \simeq f \simeq f c_a$$

$$f^{-1} f \simeq c_a \qquad f f^{-1} \simeq c_b$$

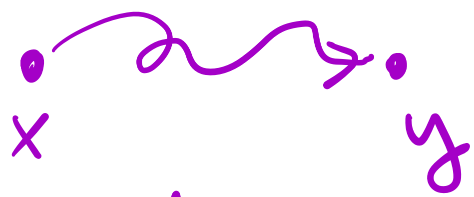
The algebraic structure on homotopy classes of paths is a <u>Groupoid</u>.

Category $\pi_1(X)$

objects $x, y, z, \ldots$

points in X

morphisms $\pi_1(X)(x,y)$



homotopy classes of paths from x to y.

composition path concatenation