Topology

A categorical approach

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Construct new spaces

Products $X \times Y$, $\prod X_i$

Subspaces $A \hookrightarrow X$

Quotients, unions, ...

Let X be a topological space and let $A \hookrightarrow X$ be a subset of X. We'd like to make A into a space.

Definition: Open sets in A

$$\Upsilon_{\text{sub}} := \{ U \cap A : \text{where } U \text{ is open in } X \}$$

Why is this a good definition?

the natural inclusion

$$A \hookrightarrow X$$

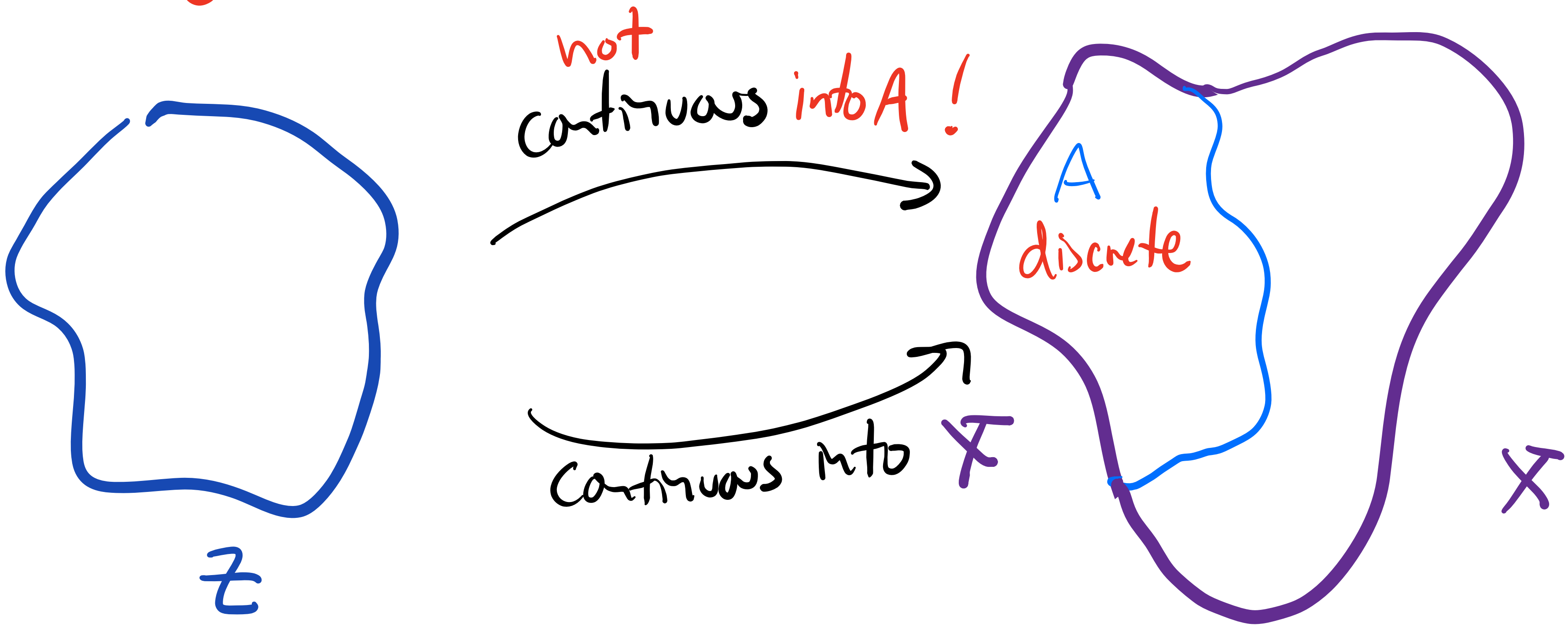
want topology here

so that this inclusion is continuous

$A \hookrightarrow X$ being continuous restricts the choices for topologies on A.

What about the discrete topology on A?

What's wrong with using the discrete topology on $A \hookrightarrow X$?



Better Definition: The subspace topology on a subset A of a topological space X is the smallest topology for which the inclusion

$$A \hookrightarrow X$$

is continuous.

We'd like to avoid

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right, "if"'] & A \arrow[r, hook, "i"] & X \end{tikzcd}$$

if is continuous

f is not continuous

Property

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right, "if"'] & A \arrow[r, hook, "i"] & X \end{tikzcd}$$

if is continuous

$\iff$

f is continuous

The subspace topology on A is uniquely characterized by

Property

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[rr, bend right, "if"'] & A \arrow[r, hook, "i"] & X
\end{tikzcd}
$$

if is continuous $\Longleftrightarrow$ f is continuous

The subspace topology on A is the only topology that has this property!

Property

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right, "if"'] & A \arrow[r, hook, "i"] & X \end{tikzcd}$$

if is continuous

$\Longleftrightarrow$

f is continuous

Suppose τ is a topology on A for which

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right, "if"'] & (A,\tau) \arrow[r, hook, "i"] & X \end{tikzcd}$$

if is continuous

$\Longleftrightarrow$

f is continuous

Suppose τ is a topology on A for which

step 1

$$\begin{tikzcd} (A,\tau) \arrow[r, "id"] \arrow[rr, bend right, "i"'] & (A,\tau) \arrow[r, hook, "i"] & X \end{tikzcd}$$

i is continuous

$\Uparrow$

id is continuous

Conclusion 1

τ is larger than τ_{subspace}

Suppose τ is a topology on A for which

step 2

$$\begin{tikzcd} (A, \tau_{\text{subspace}}) \arrow[r, "id"] \arrow[rr, bend right, "i"'] & (A, \tau) \arrow[r, hook, "i"] & X \end{tikzcd}$$

i is continuous $\Rightarrow$ id is continuous

$$\boxed{\begin{array}{l}\text{Conclusion 2} \\ \quad \tau \text{ is smaller} \\ \quad \text{than } \tau_{\text{subspace}}\end{array}}$$

Conclusion 1

Υ is larger than $\Upsilon_{subspace}$

Conclusion 2

Υ is smaller than $\Upsilon_{subspace}$

$$\Rightarrow \Upsilon = \Upsilon_{subspace}$$

$\blacksquare$

Topologies on

Products, quotients, unions, ...

have descriptions

by specifying which functions into / out of them are continuous

The end $\blacksquare$