Topology

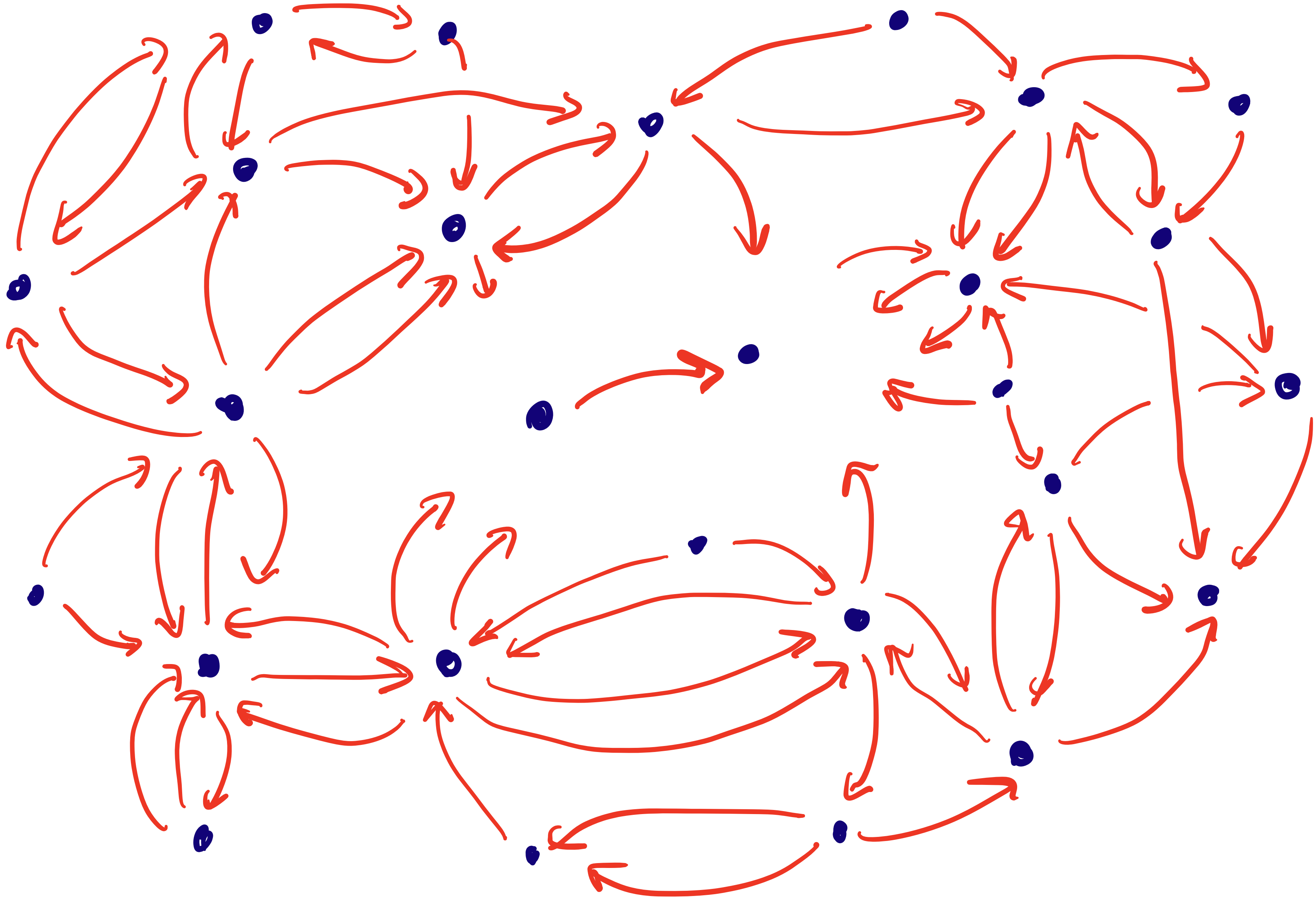
A categorical approach

Bradley, Bryson, Terilla

What is Topology?

Objects $X, Y, Z, \ldots$

Morphisms $X \longrightarrow Y$



Morphisms

Composable:

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & Z \end{tikzcd}$$

identities:

$$X \xrightarrow[\mathrm{id}]{} X$$

Composition, identities

$\rightsquigarrow$ invertible morphism

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

Definition

"isomorphism"

in Topology

objects: topological spaces

morphisms: continuous functions

isomorphisms: "homeomorphisms"

When are two objects considered the same?

In plane geometry, these two triangles are the same:

Def: X and Y are isomorphic if and only if there exists an isomorphism.

$$X \xrightarrow{f} Y$$

Notation:

$$X \cong Y$$

$$X \xrightarrow{\cong} Y$$

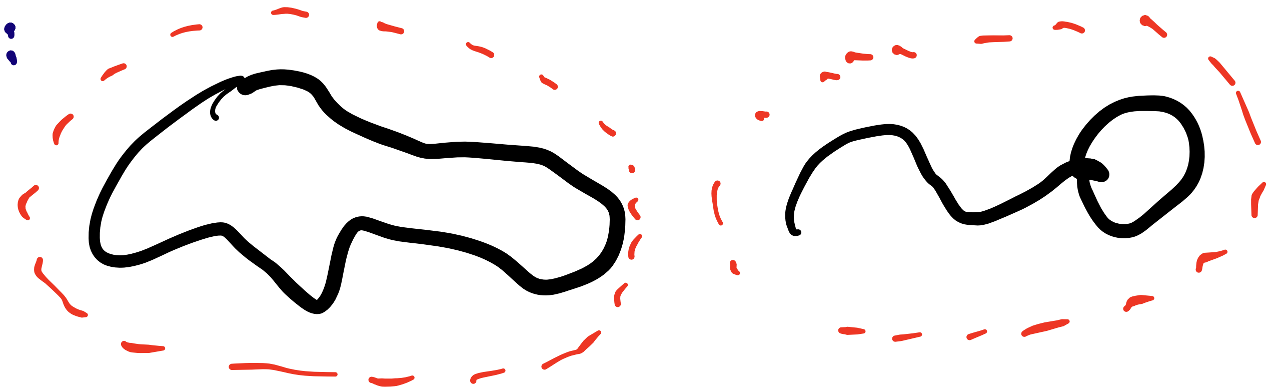
In Topology we are concerned with properties that are preserved by homeomorphisms.

Definition: A topological property is a property that is preserved by homeomorphisms.

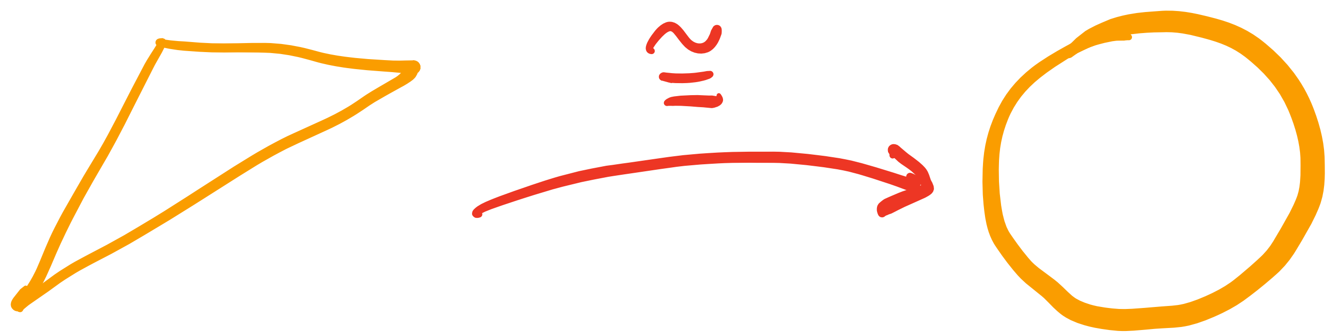
$$X \xrightarrow{\cong} Y$$

has property P   has property P

Being connected is a topological property:

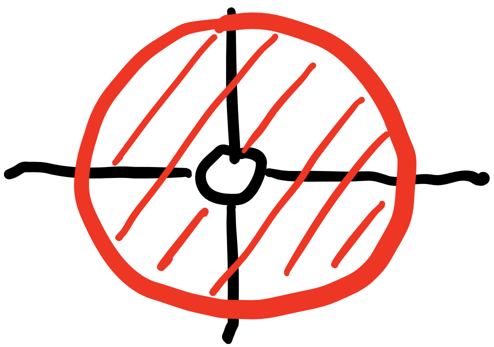


Being a triangle is not a topological property



Being orange is not a topological property.

Being bounded or unbounded is **not** a topological property.



$$\{z\in\mathbb{R}^2 : 0<|z|\le 1\} \cong \{z\in\mathbb{R}^2 : |z|\ge 1\}$$

$$z \longleftrightarrow 1/z$$

How can we study topological properties?

Functors!

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$

$$
\begin{tikzcd}
X \arrow[d, "f"'] & & FX \arrow[d, "Ff"] \\
Y \arrow[r, phantom, "\longmapsto"] & & FY
\end{tikzcd}
$$

Functors respect composition and identities and therefore map isomorphisms to isomorphisms

Therefore: Every functor $F : \mathrm{Top} \to \mathcal{C}$ defines a topological property!

Algebraic Topology is concerned with functors

the category Top $\longrightarrow$ algebraic category

homotopy groups, homology & cohomology, K-theory, ...