Name:

1	d
2	b
3	d
4	b
5	e
6	T
7	T
8	T
9	T
10	F

11

Part 1: Multiple choice. One point each.

1. Which one of the following equals $\dfrac{10+5i}{1+2i}$?

(a) 2 (b) $3i$ (c) $3+4i$ (d) $4-3i$ (e) $\frac{1}{3}+\frac{4}{3}i$

Answer. (d) is correct. Look, $\frac{10+5i}{1+2i} = 4-3i$ if and only if $10+5i = (1+2i)(4-3i)$, which is true: $(1+2i)(4-3i) = 4-(-6)+(-3+8)i = 10+5i$.

2. Which one of the following is *not* a field?

(a) The numbers $\{0,1\}$ with $+$ and $\times$ defined "mod 2"

(b) The integers $\mathbb{Z}$

(c) The rational numbers $\mathbb{Q}$

(d) The real numbers $\mathbb{R}$

(e) The complex numbers $\mathbb{C}$

Answer. (b). The integers $\mathbb{Z}$ are not a field since there are nonzero integers, such as 3, that have no multiplicative inverse.

3. Which one of the following statements is false?

(a) The set $A = \{(a,b,c) \in \mathbb{R}^3 : a+b=c\}$ is a subspace of $\mathbb{R}^3$.

(b) The set $B = \{p \in \mathcal{P}(\mathbb{R}) : p''(1) = 0\}$ is a subspace of $\mathcal{P}(\mathbb{R})$.

(c) The set $C = \{f \in \mathbb{R}^{\mathbb{R}} : f(x) = f(x+1) \text{ for all } x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(d) The set $D = \{(a,b) \in \mathbb{R}^2 : ab = 0\}$ is a subspace of $\mathbb{R}^2$.

(e) The set $E = \{f \in \mathbb{R}^{[0,1]} : \int_0^1 f(x)\,dx = 0\}$ is a subspace of $\mathbb{R}^{[0,1]}$.

Answer. (d). The set D is not closed under addition. $(2,0)$ and $(0,3)$ are in D but the sum $(2,3)$ is not.

4. Which one of the following lists of vectors is a basis for $\mathbb{C}^2$?

(a) $(1,i),(i,-1)$

(b) $(1,i),(i,0)$

(c) $(1,i),(i,0),(0,1)$

(d) $(1,1)$

(e) $(1,0)$

Answer. (b) is correct. $\mathbb{C}^2$ is two dimensional so any linearly independent list of two vectors is a basis. $(1,i),(i,0)$ is independent since the second isn't a multiple of the first. Notice that (c) is wrong since it has too many vectors to be independent and (d) and (e) are wrong because they have too few vectors to span. For (a), these two vectors are not independent, the second is i times the first.

5. Let U, V, and W be the following subspace of $\mathbb{R}^3$:

$$\begin{aligned} U &= \{(a,b,c) \in \mathbb{R}^3 : a = b\} \\ V &= \{(a,b,c) \in \mathbb{R}^3 : a = 0 \text{ and } b = 0\} \\ W &= \{(a,b,c) \in \mathbb{R}^3 : a + b + c = 0\} \end{aligned}$$

Which one of the following statements is true?

(a) $U + V = W$

(b) $U + V = \mathbb{R}^3$

(c) $U \oplus V = \mathbb{R}^3$

(d) $U \oplus W = \mathbb{R}^3$

(e) $V \oplus W = \mathbb{R}^3$

Answer. (e) is correct. First, let's see why the others are wrong. To see that (a) is false, notice that the sum of any vector (a, a, b) in U and any vector $(0, 0, c)$ in V will again be in U. So $(1, -1, 0) \in W$ is not in the sum $U + V$ and we see $U + V \neq W$. This also shows that $U + V \neq \mathbb{R}^3$ since $(1, -1, 0) \notin U + V$ so (b) is false. Therefore, it cannot be that $U \oplus V = \mathbb{R}^3$ so (c) is false. To see (d) is false, observe that $(1, 1, -2) \in U \cap W$ so $U \cap W \neq \{(0, 0, 0)\}$.

Now, to see that (e) is correct, notice that $V \cap W = \{(0, 0, 0)\}$. So, the statement is true if $V + W = \mathbb{R}^3$. To see this, notice that $\dim(V) = 1$ since $(0, 0, 1)$ is a basis for V and $\dim(W) = 2$ since $(1, -1, 0)$ and $(1, 0, -1)$ is a basis for W. So $V + W$ is a $\dim(V) + \dim(W) - \dim(V \cap W) = 1 + 2 - 0 = 3$ dimensional subspace of $\mathbb{R}^3$, hence must be all of $\mathbb{R}^3$.

Part II: True or False. One point each.

6. Every polynomial of degree three can be expressed as a linear combination of the polynomials $1, 2 + 4x, 11x^2 + 2x + 3, 7x^3 + 5x^2 - 1$.

Answer. True. Notice that this is an independent list of 4 vectors in the four dimensional vector space $\mathcal{P}_3(\mathbb{R})$. So, this list is a basis and hence spans. So every polynomial in $\mathcal{P}_3(\mathbb{R})$, in particular those of degree three, can be written as a linear combination of the given polynomials.

7. If p_1, p_2, p_3, p_4 is a list polynomials in $\mathcal{P}_3(\mathbb{R})$ that satisfy $\displaystyle\int_0^1 p_i(x)\,dx = 0$ then p_1, p_2, p_3, p_4 is dependent.

Answer. True. Note that $U = \{p \in \mathcal{P}_3(\mathbb{R}) : \int_0^1 p(x)\,dx = 0\}$ is a three dimensional space since U is a proper subspace of $\mathcal{P}_3(\mathbb{R})$ so the dimension is 3 or less, and $2x - 1$, $3x^2 - 1$, and $4x^3 - 1$ is an independent list in U. Therefore, any list of length four in U must be dependent.

8. The space $\{p \in \mathcal{P}_3(\mathbb{R}) : p(0) = p(1)\}$ is three dimensional.

Answer. True. Call this space U. Note that U is a proper subspace of $\mathcal{P}_3(\mathbb{R})$ and so it has dimension at most three. The list $1, (x)(x-1), (x^2)(x-1)$ is an independent list of length 3 in U, so the dimension of U is at least three.

9. If U and W are subspaces of a vector space V, then $U \cap W$ is a subspace of V.

Answer. True. To prove it, suppose U and W are subspaces and look at $U \cap W$. Notice $0 \in U \cap V$ since $0 \in U$ and $0 \in V$. Now, to see that $U \cap V$ is closed under vector addition, suppose $x \in U \cap V$ and $y \in U \cap V$. Since U is a subspace and $x \in U$ and $y \in U$, we know $x + y \in U$. Similarly, $x + y \in V$. So, $x + y \in U \cap V$. Finally, to see that $U \cap V$ is closed under scalar multiplication, suppose $x \in U \cap V$ and α is in the ground field F. Since U is a subspace, $x \in U$ and $\alpha \in F$, we know $\alpha x \in U$. Also, since V is a subspace, $x \in V$ and $\alpha \in F$, we know $\alpha x \in V$. So, $\alpha x \in U \cap V$.

10. For vector spaces U, V, and W; if $U \oplus V = U \oplus W$ then $V = W$.

Answer. False. For example, let $U = \{(a, 0) \in \mathbb{R}^2\}$, $V = \{(0, b) \in R^2\}$, and $W = \{(a, a) \in \mathbb{R}^2\}$. Then $U \oplus V = \mathbb{R}^2$ and $U \oplus W = \mathbb{R}^2$ but $V \neq W$.

Part III: Short answer. Two points.

11. Choose one of the true/false problems and explain your answer. Neatness counts.