Answers.

1	D
2	C
3	C
4	D
5	C
6	F
7	T
8	T
9	T
10	T
11	F

Part 1: Multiple choice. One point each.

1. Suppose $T : \mathscr{P}(\mathbb{R}) \to \mathbb{R}^3$ is a linear map and that $T(x^2) = (10,5,2)$, $T(x^4) = (0,0,2)$, and $T(1+x) = (-5,1,1)$. Then $T(3+3x+2x^2-x^4) =$

(a) $(-15,3,3)$ (b) $(20,10,4)$ (c) $(0,0,-2)$ (d) $(5,13,5)$ (e) $(2,5,0,3)$

Answer. (d). $T(3+3x+2x^2-x^4) = 3T(1+x)+2T(x^2)-T(x^4) = 3(-5,1,1)+2(10,5,2)-(0,0,2) = (5,13,5)$.

2. Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $S(x,y) = (x+y, 5y, x-y, 2x+y)$. Which is a basis for the range of S?

(a) $(2,5,0,3)$

(b) $(1,0,0,0),(0,1,0,0)$

(c) $(1,0,1,2),(1,5,-1,1)$

(d) $(2,5,0,3),(1,5,-1,0),(0,-5,2,1)$

(e) $(1,0,0,0),(0,1,0,0),(0,0,1,0)$

Answer. (c) Since $(1,0)$ and $(0,1)$ span $\mathbb{R}^2$, their images span the range of S. We have $S(1,0) = (1,0,1,2)$ and $S(0,1) = (1,5,-1,1)$, which are independent, hence a basis for range(S).

3. Let $T : \mathscr{P}_4(\mathbb{R}) \to \mathbb{R}$ be defined by $T(p) = p''(0)$. Which of the following is a basis for null(T)?

(a) $1,x,x^2$

(b) $1+x, 1+x^2, 1+x+x^2$

(c) $1,x,x^3,x^4$

(d) x^4,x^3,x^2

(e) x,x^2,x^3

Answer. (c) Notice that T is not the zero map, so range$(T) = \mathbb{R}$, which is one-dimensional. Therefore null(T) has dimension 4. By inspection, the list in (c) is a list of four independent vectors in null(T), hence is a basis. *Note: no other list has four elements so no other answer could be correct.*

4. Suppose that $S \in \mathscr{L}(\mathbb{R}^2,\mathbb{R}^4)$ satisfies range$(S) = \{(a,b,c,d) : a = c \text{ and } b = d\}$. Then, which of one of the following statements is false.

(a) S is injective.

(b) There exists a map $T \in \mathscr{L}(\mathbb{R}^4,\mathbb{R}^2)$ with $TS = I_{\mathbb{R}^2}$.

(c) $\dim(\text{range}(S)) = 2$.

(d) $\dim(\text{null}(S)) = 2$.

(e) $\dim(\text{range}(S)) + \dim(\text{null}(S)) = 2$.

Answer. (d) Notice that $\text{range}(S) = \text{span}((1,0,1,0),(0,1,0,1))$ is two dimensional so (c) is correct. Since $\dim(\text{range}(S)) + \dim(\text{null}(S)) = \dim(\mathbb{R}^2) = 2$ (and we see (e) is correct) we have $\dim(\text{null}(S)) = 0$. So (d) is wrong. $\text{null}(T) = \{0\}$ implies S is injective and (a) is correct which is equivalent to S being left invertible so (b) is correct.

Note: if only one answer is correct, you can narrow it down to (a) or (d) since these two are contradictory for any map S.

5. Consider the map $T : \mathscr{P}_3(\mathbb{R}) \to \mathbb{R}^4$ defined by $T(p) = (p(0), p(1), p'(0), p'(1))$ Which is the matrix for T using the basis $1, x, x^2, x^3$ for the domain and the standard basis for the codomain?

(a) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix}$

Answer. (c) We compute

$$\begin{aligned} T(1) &= (1,1,0,0) = 1(1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1) \\ T(x) &= (0,1,1,1) = 0(1,0,0,0) + 1(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1) \\ T(x^2) &= (0,1,0,2) = 0(1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 2(0,0,0,1) \\ T(x^3) &= (0,1,0,3) = 0(1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 3(0,0,0,1) \end{aligned}$$

to obtain the columns of $\mathscr{M}(T)$.

Part II: True or False. One point each.

6. Let $T \in \mathscr{L}(V, W)$ be a linear map between finite dimensional vector space. For every subspace Y of W, we have $\dim(T^{-1}(Y)) \geq \dim Y$.

Answer. False. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x,y) = (x,y,0)$. Let $Y = \{(0,0,z) : z \in \mathbb{R}\}$. Then $Y = \text{span}((0,0,1))$ is one dimensional and $T^{-1}(Y) = \{(0,0)\}$ is zero dimensional.

7. The map $T : \mathscr{P}_3(\mathbb{R}) \to \mathbb{R}^4$ defined by $T(p) = (p(0), p(1), p'(0), p'(1))$ is an isomorphism.

Answer. True. First note that the dimension of the domain and the range are both four. So, it suffices to prove that T is injective. Applying T to the basis $1, x, x^2, x^3$ gives a list $(1,1,0,0), (0,1,1,1), (0,1,0,2), (0,1,0,3)$ that is independent, hence T is injective.

Note: to see that the list is injective, observe that $(0,1,0,2), (0,1,0,3)$ *aren't multiples of each other. All vectors in the span of* $(0,1,0,2), (0,1,0,3)$ *have a zero in the third entry, so the* $(0,1,1,1)$ *isn't in the span of* $(0,1,0,2), (0,1,0,3)$. *The span of* $(0,1,1,1), (0,1,0,2), (0,1,0,3)$ *doesn't contain* $(1,1,0,0)$ *since it has a nonzero first entry.*

8. There exists a nonzero polynomial $p \in \mathscr{P}_3(\mathbb{R})$ so that $p(0) = 0$, $\int_0^2 e^x p(x) = 0$, and $p'(1) = 0$.

Answer. True. To prove it, consider the function $S : \mathscr{P}_3(\mathbb{R}) \to \mathbb{R}^3$ defined by $S(p) = \left(p(0), \int_0^2 e^x p(x), p'(1) \right)$. Note that S is a linear map from a four dimensional space to a three dimensional space so it must have a nonzero polynomial in its nullspace. That is, there must be a nonzero p with $p(0) = 0$, $\int_0^2 e^x p(x) = 0$, and $p'(1) = 0$.

9. For any map $T \in \mathscr{L}(V)$, if $T^2 = 0$ then $\operatorname{range}(T) \subseteq \operatorname{null}(T)$.

Answer. True. To prove it, let $v \in \operatorname{range}(T)$. Then $v = Tv'$ for some $v' \in V$. Apply T again to get $Tv = T^2 v' = 0$ since $T^2 = 0$, so $v \in \operatorname{null}(T)$.

10. There exists a map $T \in \mathscr{L}(\mathbb{R}^2)$ with $\operatorname{null}(T) = \operatorname{range}(T)$.

Answer. True. For example, there is a linear map that sends the y-axis to the x-axis and the x-axis to the origin. Explicitly, let $T(x,y) = (y,0)$. Then the $\operatorname{null}(T) = \{(x,0) : x \in \mathbb{R}\}$. This is the same set as $\operatorname{range}(T)$.

11. There exists a map $T \in \mathscr{L}(\mathbb{R}^3)$ with $\operatorname{null}(T) = \operatorname{range}(T)$.

Answer. False. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map. We have $3 = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T))$. So, it's impossible for $\dim(\operatorname{null}(T)) = \dim(\operatorname{range}(T))$ since 3 is odd.