**Problem 4 Section 3A** Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in $V$ such that $Tv_1, \ldots, Tv_m$ is a linear independent list in $W$. Then $v_1, \ldots, v_m$ is independent in $V$.

*Proof.* Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in $V$ such that $Tv_1, \ldots, Tv_m$ is a linear independent list in $W$. To show that $v_1, \ldots, v_m$ is independent, suppose that $0 = \alpha_a v_1 + \cdots + \alpha_m v_m$. Apply $T$ to get

$$\begin{aligned} 0 &= T(\alpha_a v_1 + \cdots + \alpha_m v_m) \\ &= \alpha_1 Tv_1 + \cdots \alpha_m Tv_m. \end{aligned}$$

Since $Tv_1, \ldots, Tv_m$ is independent, $\alpha_1 = \cdots = \alpha_m = 0$ as needed to prove $v_1, \ldots, v_m$ is independent. □

**Problem 3 Section 3B** Suppose $V$ is a vector space over a field $F$. Given any list of vectors $v_1, \ldots, v_m$ in $V$, one can define a linear map $T : F^m \to V$ by

$$T(\alpha_1, \ldots, \alpha_m) = \alpha_1 v_1 + \cdots + \alpha_m v_m.$$

Properties of the list $v_1, \ldots, v_m$ translate into properties of the linear map $T$. Specifically, $T$ is surjective if and only if $v_1, \ldots, v_m$ spans $V$ and $T$ is injective if and only if $v_1, \ldots, v_m$ is independent.

**Problem 9 Section 3B** Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_m$ is an independent list of vectors in $V$. Then $Tv_1, \ldots, Tv_m$ is a linear independent list in $W$.

*Proof.* Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_m$ is independent. To see that $Tv_1, \ldots, Tv_m$ is a linear independent list in $W$ suppose that $0 = \alpha_a Tv_1 + \cdots + \alpha_m Tv_m$. We have

$$\begin{aligned} 0 &= \alpha_a Tv_1 + \cdots + \alpha_m Tv_m \\ &= T(\alpha_a v_1 + \cdots + \alpha_m v_m) \end{aligned}$$

Since $T$ is injective, this implies $\alpha_1 v_1 + \cdots + Tv_m = 0$. Since $v_1, \ldots, v_m$ is independent, $\alpha_1 = \cdots = \alpha_m = 0$ as needed to prove $Tv_1, \ldots, Tv_m$ is independent. □

**Problem 10 Section 3B** Suppose $T \in \mathcal{L}(V, W)$ and that $v_1, \ldots, v_m$ spans $V$. Then $Tv_1, \ldots, Tv_m$ spans the range of $T$.

*Proof.* Suppose $T \in \mathcal{L}(V, W)$ and that $v_1, \ldots, v_m$ spans $V$. First notice that $\text{span}(Tv_1, \ldots, Tv_m) \subseteq \text{range}(T)$ since each vector $Tv_1, \ldots, Tv_m \in \text{range}(T)$ and $\text{range}(T)$ is a subspace closed under addition and scalar multiplication.

To complete the proof that $\text{span}(Tv_1, \ldots, Tv_m) = \text{range}(T)$, we will show $\text{range}(T) \subseteq \text{span}(Tv_1, \ldots, Tv_m)$. To do so, let $w \in \text{range}(T)$. This means that there exists a vector $v \in V$ with $Tv = w$.. Since $v_1, \ldots, v_m$ spans $V$, there exists $\alpha_1, \ldots, \alpha_m \in F$ with $v = \alpha_1 v_1 + \cdots \alpha_m v_m$. Apply $T$ to get

$$\begin{aligned} w &= Tv \\ &= T\alpha_1 v_1 + \cdots \alpha_m v_m \\ &= \alpha_1 Tv_1 + \cdots + \alpha_m Tv_m. \end{aligned}$$

This proves that $w$ is in the span of $Tv_1, \ldots, Tv_m$ as needed. □

**Problem 13 Section 3B** Suppose $T \in \mathcal{L}(F^4, F^2)$ such that

$$(T) = \{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 5x_2 \text{ and } x_3 = 7x^4\}.$$

Then $T$ is surjective.

*Proof.* The fact that $T$ is surjecive follows from the fact that $(T)$ has dimension 2 since $\dim(F^4) = 4 = \dim((T)) + \dim(\text{range}(T))$. To see that $(T)$ is two dimensional, notice that every vector $v \in (T)$ has the form

$$v = (5x_2, x_2, 7x_4, x_4) = x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1).$$

This says that $(T)$ is the span of $(5, 1, 0, 0), (0, 0, 7, 1)$ and since $(5, 1, 0, 0), (0, 0, 7, 1)$ is independent, it's a basis for $(T)$. $\square$