1. Which of the following is *not* a field? Explain.

(a) The integers $\mathbb{Z}$

(b) The rational numbers $\mathbb{Q}$

(c) The real numbers $\mathbb{R}$

(d) The complex numbers $\mathbb{C}$

Answer. The integers $\mathbb{Z}$ are not a field since not all integers have multiplicative inverses. For example, $2 \in \mathbb{Z}$ and there is no integer $k \in \mathbb{Z}$ so that $2 \times k = 1$.

2. Which of the following is *not* a field? Explain.

(a) The numbers $\{0, 1\}$ with $+$ and $\times$ defined "mod 2".

(b) The numbers $\{0, 1, 2\}$ with $+$ and $\times$ defined "mod 3".

(c) The numbers $\{0, 1, 2, 3\}$ with $+$ and $\times$ defined "mod 4".

(d) The numbers $\{0, 1, 2, 3, 4\}$ with $+$ and $\times$ defined "mod 5".

Answer. The numbers $\{0, 1, 2, 3\}$ with $+$ and $\times$ defined "mod 4" is not a field because 2 does not have a multiplicative inverse. To see this, look at the products of 2 with every other number in the set:

$$2 \times 0 = 0 \qquad 2 \times 1 = 2 \qquad 2 \times 2 = 0 \qquad 2 \times 3 = 2$$

and observe that there is no number so that when multiplied by 2 results in 1.

3. Let $\alpha \in \mathbb{C}$ be nonzero. Define the number $\frac{1}{\alpha}$ and prove that $\frac{1}{\frac{1}{\alpha}} = \alpha$.

Answer. For any nonzero complex number α, the number $\frac{1}{\alpha}$ is *by definition* the unique complex number so that

$$(\alpha)\left(\frac{1}{\alpha}\right) = 1.$$

The number $\dfrac{1}{\frac{1}{\alpha}}$ is, by definition, the unique complex number so that when multiplied by $\frac{1}{\alpha}$ the result is 1, and that number is α.

4. Express $\dfrac{1}{4 + 5i}$ in the form $a + bi$ for real numbers a, b.

Answer. Here's one way to do it: write $\frac{1}{4+5i} = a + bi$

$$\begin{aligned}\left(\frac{1}{4+5i}\right)(4+5i) = 1 &\Rightarrow (a+bi)(4+5i) = 1\\ &\Rightarrow (4a-5b) + (5a+4b)i = 1 + 0i\\ &\Rightarrow (4a-5b) = 1 \text{ and } 5a+4b = 0\\ &\Rightarrow (4a-5b) = 1 \text{ and } b = -\frac{5}{4}a\\ &\Rightarrow 4a - 5\left(-\frac{5}{4}a\right) = 1 \text{ and } b = -\frac{5}{4}a\\ &\Rightarrow \frac{41}{4}a = 1 \text{ and } b = -\frac{5}{41}a\\ &\Rightarrow a = \frac{4}{41} \text{ and } b = -\frac{5}{41}\end{aligned}$$

So, the answer (which a quick computation will verify) is $\dfrac{1}{4+5i} = \dfrac{4}{41} - \dfrac{5}{41}i$

Answer. Here's another way:

$$\left(\frac{1}{4+5i}\right) = \left(\frac{1}{4+5i}\right)\left(\frac{4-5i}{4-5i}\right) = \frac{4-5i}{16-(-25)} = \frac{4}{41} - \frac{5}{41}i.$$

5. True or False:

(a) There exists a number $\alpha \in \mathbb{R}$ so that $\alpha^2 = -2$.

Answer. False

(b) There exits a number $\alpha \in \mathbb{C}$ so that $\alpha^2 = -2$.

Answer. True. There are two distinct numbers in $\mathbb{C}$ whose square is -2. Namely, $\alpha = \sqrt{2}i$ and $\alpha = -\sqrt{2}i$.

6. True or False:

(a) There is only one number $\alpha \in \mathbb{R}$ so that $\alpha^3 = 2$.

Answer. True

(b) There is only one number $\alpha \in \mathbb{C}$ so that $\alpha^3 = 2$.

Answer. False. There are in fact three complex numbers whose cubes are 2. In addition to $\sqrt[3]{2}$, there are:

$$\sqrt[3]{2}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \text{ and } \sqrt[3]{2}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

You can get the idea about this from exercise 2 in section 1A of the book revealed cube roots of 1, which when multiplied by $\sqrt[3]{2}$ yields cube roots of 2. Another way is to think geometrically about multiplication in $\mathbb{C}$.

7. Does there exist a number $\alpha \in \mathbb{C}$ so that $\alpha(1+i, 2, 2+2i, 3-2i) = (2, 2-2i, 4, 1-5i)$?

Answer. Yes, a quick computation shows that $\alpha = (1-i)$ works:

$$(1-i)\,(1+i, 2, 2+2i, 3-2i) = (2, 2-2i, 4, 1-5i).$$

8. Let V be a vector space over a field F. Prove that

(a) For all $v \in V$, $0v = 0$.

Note: the zero on the left is the zero scalar in F and the zero on the right is the zero vector in V.

Answer. Since $0 = 0+0$, we have $0v = (0+0)v$. Using the distributive property yields $0v = 0v + 0v$. Adding $-0v$ to both sides gives $0v - 0v = (0v + 0v) - 0v$. On the left, we have the zero vector and on the right, we use associativity to get $0 = 0v + (0v - 0v)$. Using the fact that $0v - 0v = 0$ again, gives $0 = 0v + 0 \Rightarrow 0 = 0v$.

(b) For all $v \in V$, $(-1)v = -v$.

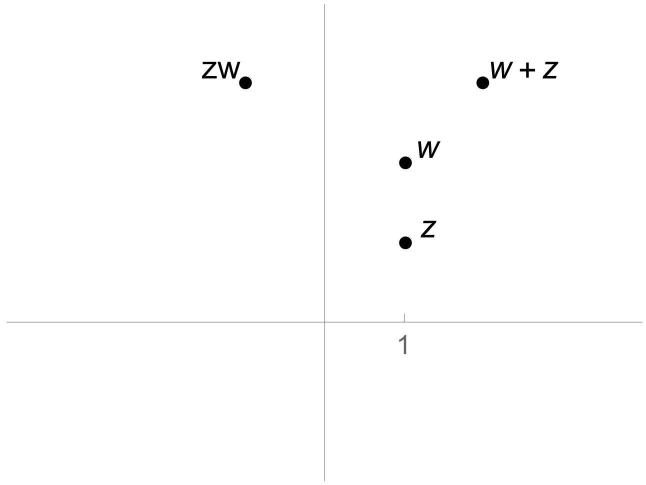
Note: the -1 on the left is a scalar in the field F, the $-v$ on the right is the additive inverse of the vector $v \in V$.

Answer. We need to show that $(-1)v + v = 0$. So,

$$(-1) + v = (-1)v + 1v = (-1+1)v = 0v = 0.$$

9. Using the correspondence $a+bi \longleftrightarrow (a,b)$ complex numbers can be identified with points in the Cartesian plane. The four points pictured below correspond to $z, w, z+w$, and zw for two complex numbers $z, w \in \mathbb{C}$. Which are which?

Answer. Here's the answer. It is also correct to swap z and w.



10. Consider the vector space $\mathbb{R}^3$. Which of the following subsets are subspaces?

(a) $\{(a,b,c,d) \in \mathbb{R}^4 : a+b+c=0\}$

Answer. Subspace.

(b) $\{(a,b,c,d) \in \mathbb{R}^4 : abc=0\}$

Answer. Not a subspace. It's not closed under addition: the point $(1,1,0,0)$ and $(0,1,1,0)$ are in the set, but the sum $(1,2,1,0)$ is not.

(c) $\{(a,b,c,d) \in \mathbb{R}^4 : a \geq 0\}$

Answer. Not a subspace. It's not closed under scalar multiplication: the point $(1,0,0,0)$ is in the set but $-3(1,0,0,0) = (-3,0,0,0)$ is not in the set.

(d) $\{(a,b,c,d) \in \mathbb{R}^4 : a=2\}$

Answer. Not a subspace. The zero vector isn't in the set.

(e) $\{(a,b,c,d) \in \mathbb{R}^4 : a=d\}$

Answer. Subspace.

(f) $\{(a,b,c,d) \in \mathbb{R}^4 : a+b+1=c\}$

Answer. Not a subspace. The zero vector isn't in the set.

(g) $\{(a,b,c,d) \in \mathbb{R}^4 : a+b=2c\}$

Answer. Subspace.

11. Consider the vector space $\mathbb{R}^{\mathbb{R}}$. Which of the following subsets are subspaces?

(a) $\{f : \mathbb{R} \to \mathbb{R} : f(1) = 1\}$

Answer. Not a subspace. The zero vector isn't in the set.

(b) $\{f : \mathbb{R} \to \mathbb{R} : f(1) = 0\}$

Answer. Subspace.

(c) $\{f : \mathbb{R} \to \mathbb{R} : f \text{ is onto}\}$

Answer. Not a subspace. The zero vector isn't in the set.

(d) $\{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$

Answer. Subspace.

(e) $\{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$

Answer. Subspace.

(f) $\{f : \mathbb{R} \to \mathbb{R} : f''(x) = f(x)\}$

Answer. Subspace. Here, we'll go through and verify. First, note that the zero function is in the set. Second, note that if $f''(x) = f(x)$ and $g''(x) = g(x)$ then $(f+g)''(x) = f''(x) + g''(x) = f(x) + g(x) = (f+g)(x)$ so the set is closed under addition. Third, note that if $f''(x) = f(x)$ and $\alpha \in \mathbb{R}$ then $(\alpha f)''(x) = \alpha(f''(x)) = \alpha f(x)$ so the set is closed under scalar multiplication.

12. Let $V = \mathbb{R}^3$. Consider the following three subspaces of V

$$\begin{aligned} W &= \{(0,0,a) \in V : a \in \mathbb{R}\} \\ X &= \{(a,a,a) \in V : a \in \mathbb{R}\} \\ Y &= \{(a,b,c) \in V : a+b+c = 0\} \\ Z &= \{(a,a,b) \in V : a,b \in \mathbb{R}\} \end{aligned}$$

True or False:

(a) $(1,1,-2) \in W$ False

(b) $(1,1,-2) \in X$ False

(c) $(1,1,-2) \in Y$ True

(d) $(1,1,-2) \in Z$ True

(e) W is a subspace of X False

(f) W is a subspace of Y False

(g) W is a subspace of Z True

(h) X is a subspace of Z True

(i) W is a subspace of Z True

(j) $W \cap X = \{(0,0,0)\}$ True

(k) $X \cap Z = X$ True.

(l) $Z = W + X$ True

(m) $Z = W \oplus X$ True

(n) $V = Y + Z$ True

(o) $V = Y \oplus Z$ False

Footnotes

(j) To see that $W \cap X = \{(0,0,0)\}$ note that if $(x,y,z) \in W$, then $x = y = 0$. If $(x,y,z) \in X$ then $z = x = y$. So, if (x,y,z) is a vector in both X and W then $x = y = 0$ and $x = y = z$, which together mean that $(x,y,z) = (0,0,0)$.

(l) The statement $W + X = Z$ means that every vector in Z can be expressed as a sum of a vector in W and a vector in X. For example, $(4,4,2) \in Z$ can be written $(4,4,2) = (0,0,-2) + (4,4,4)$. To see that this is always possible, suppose $(a,a,b) \in Z$. Consider $(a,a,a) \in X$ and $(0,0,b-a) \in W$ and observe that $(a,a,b) = (a,a,a) + (0,0,b-a)$.

(m) To see that $Z = W \oplus X$, it suffices to know that $Z = W + X$ and that $W \cap X = \{0\}$, which are explained above. This is "Blue Box 1.45: Direct Sum of Two Subspaces".

(o) It is true that $V = Y + Z$, every vector $(a,b,c) \in V$ can be written as the sum of a vector in Y and a vector in Z, but not uniquely, so the sum isn't a direct sum. For example $(1,2,3) = (-2,-1,3) + (3,3,0)$ and $(1,2,3) = (0,1,-1) + (1,1,4)$