1. Prove: If some vector in a list of vectors in a vector space V is a linear combination of the other vectors, then the list is linearly dependent.

Answer. Suppose that $v_k = \lambda_1 v_1 + \lambda_2 v_2 + \cdots \lambda_n v_n$. Subtracting v_k from both sides gives $0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + (-1)v_k + \cdots \lambda_n v_n$ proving that $v_1, \ldots, v_n$ is dependent.

2. Does $(1, 2, 3, -5), (4, 5, 8, 3), (9, 6, 7, -1)$ span $\mathbb{R}^4$? Explain.

Answer. No. Since $\mathbb{R}^4$ is four dimensional, no set with fewer than four vectors can span $\mathbb{R}^4$.

3. Is the list $(1, 2, 3), (4, 5, 8), (9, 6, 7), (-3, 2, 8)$ linearly independent in $\mathbb{R}^3$? Explain.

Answer. No. Since $\mathbb{R}^3$ is three dimensional, no set with greater than three vectors can be independent.

4. Prove that F^∞ is infinite-dimensional.

Answer. Let $s_1, s_2, \ldots, s_n$ be any finite list of vectors in F^∞. I'll prove that this list cannot span F^∞. I can make an independent list in F^∞ that's longer. Specifically, $(1, 0, 0, 0 \ldots)$, $(0, 1, 0, 0, \ldots)$, $\ldots$, $(0, 0, \ldots, 0, 1, 0, \ldots)$. Since any spanning list must be at least as long as an independent list, we know $s_1, \ldots, s_n$ cannot span F^∞.

5. Suppose that v_1, v_2, v_3 is a basis for a vector space V. Prove or disprove $v_1 + v_2, v_1 - v_2, v_3$ is also a basis for V.

Answer. If v_1, v_2, v_3 is a basis for V, then V is three-dimensional, so it suffices to check whether the list $v_1 + v_2, v_1 - v_2, v_3$ spans V. Note that v_1, v_2, and v_3 are in the span of $v_1 + v_2, v_1 - v_2, v_3$ since:

$$\begin{aligned}
v_1 &= \frac{1}{2}(v_1 + v_2) - \frac{1}{2}(v_1 - v_2) + 0v_3 \\
v_2 &= \frac{1}{2}(v_1 + v_2) + \frac{1}{2}(v_1 - v_2) + 0v_3 \\
v_3 &= 0(v_1 + v_2) + 0(v_1 - v_2) + 1v_3
\end{aligned}$$

Therefore, $V = \operatorname{span}(v_1, v_2, v_3) \subseteq \operatorname{span}(v_1 + v_2, v_1 - v_2, v_3)$. We conclude the list $v_1 + v_2, v_1 - v_2, v_3$ spans V.

6. Prove or disprove: Let $p_0, p_1, \ldots, p_n$ be polynomials in $\mathcal{P}(F)$ and suppose $\deg(p_i) = i$ for $i = 0, 1, \ldots, n$. Then $p_0 p_1, \ldots, p_n$ is a basis for $\mathcal{P}_n(F)$.

Answer. If $p_0, \ldots, p_n$ were dependent, one of the p_i would be a linear combination of $p_0, \ldots, p_{i-1}$. But every linear combination of $p_0, \ldots, p_{i-1}$ will have degree at most $i - 1$ and p_i has degree i. So, $p_0, \ldots, p_n$ must be independent.

7. Suppose that p_1, p_2, p_3, p_4, p_5 is a list polynomials in $\mathcal{P}_4(\mathbb{R})$ that all vanish at $x = 3$. Prove that p_1, p_2, p_3, p_4, p_5 is linearly dependent.

Answer. The space $\mathcal{P}_4(\mathbb{R})$ of all polynomials of degree less than or equal to 4 is 5 dimensional since $\{1, x, x^2, x^3, x^4\}$ is a basis. The space U of polynomials that vanish at $x = 3$ is a proper subspace of this 5 dimensional space. Therefore, U has dimension at most 4. So, any list of 5 or more polynomials in this space must be dependent.

8. Let $U = \{(a, b, c) \in \mathbb{R}^3 : a + b + c = 0\}$.

(a) Find a basis for U.

Answer. As a proper subspace of $\mathbb{R}^3$, the dimension of U is at most 2. The vectors $(1, 0, -1), (0, 1, -1)$ are an independent list in U, hence are a basis.

(b) Extend your basis to a basis of $\mathbb{R}^3$.

Answer. Adding any vector not in U works. For example, $(1, 0, -1), (0, 1, -1), (1, 2, 3)$.

(c) Find a subspace W of $\mathbb{R}^3$ so that $\mathbb{R}^3 = U \oplus W$.

Answer. Let W be the span of $(1, 2, 3)$.

9. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5)\}$.

(a) Find a basis for U.

Answer. Notice that since U is a proper subspace of of the five dimensional space $\mathcal{P}_4(\mathbb{R})$, we know the dimension of U is at most 4. Here's an independent list of four polynomials in U:

$$1, (x - 2)(x - 5), (x - 2)^2(x - 5), (x - 2)^2(x - 5)^2$$

and so it is a basis for U. To see that the list is independent, note that no polynomial in this list can be a linear combination of the previous polynomials since the degree of each polynomial is strictly greater than the degrees of the polynomials that preceed it.

(b) Extend your basis to a basis of $\mathcal{P}_4(\mathbb{R})$

Answer. It suffices to add any polynomial not in U. For example x works since it has a different value at 2 and at 5. So

$$1, (x - 2)(x - 5), (x - 2)^2(x - 5), (x - 2)^2(x - 5)^2, x$$

is a basis for $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ so that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Answer. Let W be the span of x.

10. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$.

(a) Find a basis for U.

Answer. As a proper subspace of the five dimensional space $\mathcal{P}_4(\mathbb{R})$, the dimension of U is at most four. The list $x, 3x^2 - 1, x^3, 5x^4 - 1$ is a list of four independent polynomials in U, hence is a basis. To see that the list is independent, note that no polynomial in this list can be a linear combination of the previous polynomials since the degree of each polynomial is strictly greater than the degrees of the polynomials that preceed it.

(b) Extend your basis to a basis of $\mathcal{P}_4(\mathbb{R})$

Answer. It suffices to add any polynomial not in U. For example, the polynomial 1 works: $x, 3x^2 - 1, x^3, 5x^4 - 1, 1$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ so that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Answer. Let W be the span of 1 — that's the space of constant polynomials.

11. Prove that any two three dimensional subspaces of $\mathbb{R}^5$ must have a nonzero vector in their intersection.

Answer. Let U_1, U_2 be two three dimensional subspaces of $\mathbb{R}^5$. We know

$$\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) = \dim(U_1 + U_2)$$

Therefore, $3 + 3 - \dim(U_1 \cap U_2) \leq 5 \Rightarrow \dim(U_1 \cap U_2) > 0$ and therefore must contain a nonzero vector.

12. A function $f \in \mathbb{R}^{\mathbb{R}}$ is called *even* iff $f(-x) = f(x)$ for all $x \in \mathbb{R}$. A function $f \in \mathbb{R}^{\mathbb{R}}$ is called *odd* iff $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Check that

$$U = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is even } \} \text{ and } W = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is odd } \}$$

are subspaces of $\mathbb{R}^{\mathbb{R}}$. Prove or disprove $\mathbb{R}^{\mathbb{R}} = U \oplus W$.

Answer. I'll prove that $\mathbb{R}^{\mathbb{R}} = U \oplus W$. First, we'll show $\mathbb{R}^{\mathbb{R}} = U + W$. We have to prove that every function can be written as the sum of an even function and an odd function. So, let $f : \mathbb{R}^{\mathbb{R}}$ be an arbitrary function. Define $g \in \mathbb{R}^{\mathbb{R}}$ and $h \in \mathbb{R}^{\mathbb{R}}$ by

$$g(x) = \frac{1}{2}\left(f(x) + f(-x)\right) \text{ and } h(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

Note that g is even, h is odd, and $f(x) = g(x) + h(x)$. This proves that $\mathbb{R}^{\mathbb{R}} = U + W$. To see that ths um is direct, just observe that if $f \in U \cap W$, then $f(x) = f(-x) = -f(x) \Rightarrow f(x) = 0$ for all x. So, $U \cap W = \{0\}$ proving that the sum is direct: $\mathbb{R}^{\mathbb{R}} = U \oplus W$.