Compute

1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & -1 & -3 \\ 0 & -2 & 1 & 0 & -3 \\ -1 & -2 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 0 & -2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 3 & -2 \\ -3 & 0 & -2 \end{pmatrix}.$$

Write down the matrix for AB.

Answer. $AB = \begin{pmatrix} 4 & -3 & 2 \\ 7 & 0 & 6 \\ -1 & -3 & 4 \\ -9 & 0 & -8 \end{pmatrix}.$

2. Using the basis $(x-1),(x-2)$ for $\mathscr{P}_1(\mathbb{R})$, compute $\mathscr{M}(x-5)$, the matrix for the vector $x-5$.

Answer. *The answer is* $(-3,4)$ *since* $-3(x-1)+4(x-2) = x-5$.

3. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x,y,z) = (x-2y-2z, 2x-6y-7z, -4x+10y+10z).$$

Using the basis standard basis for the domain and the standard basis for the codomain to find $\mathscr{M}(T)$

Answer. *To find the matrix for T, apply T to each of the standard basis of $\mathbb{R}^3$ and express the results as a linear combination of the basis vectors:*

$$\begin{aligned} T(1,0,0) &= (1,2,-4) = 1(1,0,0)+2(0,1,0)-4(0,0,1) \\ T(0,1,0) &= (-2,-6,10) = -2(1,0,0)+-6(0,1,0)+10(0,0,1) \\ T(0,0,1) &= (-2,-7,10) = -2(1,0,0)+-7(0,1,0)+10(0,0,1) \end{aligned}$$

and use the coeffcients of these expressions for the columns of the matrix:

$$\mathscr{M}(T) = \begin{pmatrix} 1 & -2 & -2 \\ 2 & -6 & -7 \\ -4 & 10 & 10 \end{pmatrix}.$$

4. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x,y,z) = (x-2y-2z, 2x-6y-7z, -4x+10y+10z).$$

Using the basis $(5,4,-2),(0,1,-1),\left(1,\frac{3}{2},-1\right)$ for the domain and the standard basis for the codomain to find $\mathscr{M}(T)$.

Answer. *To finad the matrix for T, apply T to each of the vectors in the given basis and express the results as a linear combination of the basis vectors:*

$$\begin{aligned} T(5,4,-2) &= (1,0,0) = 1(1,0,0)+0(0,1,0)+0(0,0,1) \\ T(0,1,-1) &= (0,1,0) = -0(1,0,0)+1(0,1,0)+0(0,0,1) \\ T(1,3/2,-1) &= (0,0,1) = 0(1,0,0)+0(0,1,0)+1(0,0,1) \end{aligned}$$

and use the coeffcients of these expressions for the columns of the matrix:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

True/False

5. Suppose $S,T \in \mathcal{L}(V)$. If $ST = I$ then $TS = I$.

Answer. *False. For example, consider $V = \mathbb{R}^\infty$, $T : V \to V$ defined by $T(x_1,x_2,x_3,\ldots) = (0,x_1,x_2,x_3,\ldots)$ and $S : V \to V$ defined by $S(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots)$. Here $ST = I$ but TS is the map $(x_1,x_2,x_3,\ldots) \mapsto (0,x_2,x_3,\ldots)$ which is not the identity.*

6. Suppose V is finite dimensional and $S,T \in \mathcal{L}(V)$. If $ST = I$ then $TS = I$.

Answer. *True! If $ST = I$, this means that T is injective and S is surjective. Since a linear operator on finite dimensional vector space is injective iff it is surjective iff it is invertible, we have both S and T are invertible. By the uniqueness of inverses, they are their own inverses: so we have both $ST = I$ and $TS = I$.*

7. Suppose $S,T \in \mathcal{L}(V)$. ST is invertible if and only if both S and T are invertible.

Answer. *False. Let's reuse the example we used earlier: $V = \mathbb{R}^\infty$, $T : V \to V$ defined by $T(x_1,x_2,x_3,\ldots) = (0,x_1,x_2,x_3,\ldots)$ and $S : V \to V$ defined by $S(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots)$. Note that neither S nor T is invertible: S isn't injective and T isn't surjective. However, ST is invertible (it's the identity map).*

8. Suppose V is finite dimensional and $S,T \in \mathcal{L}(V)$. ST is invertible if and only if both S and T are invertible.

Answer. *True. In general, if V is any vector space and $S,T \in \mathcal{L}(V)$, then if S and T are both invertible, the composition ST is invertible. To see this, note that $T^{-1}S^{-1}$ is the inverse of ST, which we check by verifying that $(T^{-1}S^{-1})(ST) = I$ and $(ST)(T^{-1}S^{-1}) = I$.*

Now, in general the invertibility of the composition ST does not imply the invertibility of the factors S and T. However if V is finite dimensional, ST being invertible implies both S and T are invertible. Here's a proof. Let A be the inverse of ST. Then $A(ST) = I = (ST)A$. Note that $I = (ST)A = S(TA)$. This implies that S is surjective (since there's something you can compose S with on the right to get the identity) hence S is invertible. Similarly, $A(ST) = I \Rightarrow (AS)T = I$ which implies T is injective hence invertible.

9. For $S : \mathbb{R}^3 \to \mathbb{R}$ given by $S(x,y,z) = x+y+z$, $\text{null}(S) = \text{span}((5,-3,-2),(0,3,-3))$.

Answer. *True. Since S is a nonzero map to* $\mathbb{R}$*, the range is* 1 *dimensional. Therefore, the nullspace is two dimensional. The two vectors* $(5,-3,-2),(0,3,-3)$ *are two linearly independent vectors in the two dimensional nullspace, hence must be a basis, hence must span the nullspace.*

10. The linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, z, x+y-z)$ is surjective.

Answer. *False. Note that* $T(1,-1,0) = (0,0,0)$ *so* $\dim(\text{null}(T)) \geq 1$*. This implies the dimension of* $\text{range}(T) \leq 2$ *so T cannot be surjective.*

11. The linear map $T : \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p(x)) = (p(1), p(2), p(3))$ is surjective.

Answer. *False. Since the domain is two dimensional, the range can have dimension at most two.*

12. Let $T : \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}^3$ be the linear map defined by

$$T(p(x)) = (p(1), p(2), p(3)).$$

Using the basis $(x-1),(x-2)$ for $\mathscr{P}_1(\mathbb{R})$ and the standard basis is used for $\mathbb{R}^3$, the matrix equation $\mathscr{M}(T)\mathscr{M}(x-5) = \mathscr{M}(T(x-5))$ is the equation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}$$

Answer. *True. Since* $(x-5) = -3(x-1) + 4(x-2)$*, we know* $\mathscr{M}(x-5) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ *To see that the matrix* $\mathscr{M}(T)$ *is correct, apply T to each of the given basis vectors of* $\mathscr{P}_1(\mathbb{R})$

$$T(x-1) = (0,1,2) = 0(1,0,0) + 1(0,1,0) + 2(0,0,1)$$
$$T(x-2) = (-1,0,1) = -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Finally, note that the matrix multiplcation is correct and that the righthand side is the vector for $T(x-5) = (-4,-3,-2) = -4(1,0,0) + -3(0,1,0) - 2(0,0,1)$ *in the standard basis of* $\mathbb{R}^3$.

13. The follow homogeneous system of equations has infinitely many solutions:

$$\begin{aligned} x - 2y - 2z &= 0 \\ 2x - 6y - 7z &= 0 \\ -4x + 10y + 10z &= 0 \end{aligned}$$

Answer. *False. Note that the given statement is equivalent to the statement that there are infinitely many vectors in the nullspace of the linear map* $T:\mathbb{R}^3\to\mathbb{R}^3$ *defined by*

$$T(x,y,z)=(x-2y-2z,2x-6y-7z,-4x+10y+10z).$$

As computed earlier,

$$T(5,4,-2)=(1,0,0),\qquad T(0,1,-1)=(0,1,0),\qquad T(1,3/2,-1)=(0,0,1),$$

so $(1,0,0),(0,1,0),(0,0,1)$ *are all in the range of* T*. Thus the range of* T *must be at least three, hence three, dimensional. Therefore* $\text{null}(T)$ *is zero dimensional. Therefore, there is only one solution to the given system—the trivial solution* $(x,y,z)=(0,0,0)$.

14. The follow inhomogeneous system of equations has exactly one solution:

$$\begin{aligned} x-2y-2z&=\frac{1}{5}\\ 2x-6y-7z&=-17\\ -4x+10y+10z&=9 \end{aligned}$$

Answer. *True. Since the linear map* $T:\mathbb{R}^3\to\mathbb{R}^3$ *defined by*

$$T(x,y,z)=(x-2y-2z,2x-6y-7z,-4x+10y+10z)$$

is surjective, there is a solution to the given system. Since the linear map T *is injective, there is only one solution.*

15. There is a linear map $T:\mathbb{R}^5\to\mathbb{R}^2$ with

$$\text{null}(T)=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 : x_1=3x_2 \text{ and } x_3=x_4=x_5\}.$$

Answer. *False. Note that the given space is two dimensional (it is spanned by* $(3,1,0,0,0),(0,0,1,1,1)$*). If there existed a linear map with the given subspace as a nullspace, then it would have a three dimensional range, impossible for a map whose codomain is two dimensional.*

16. Suppose $T:V\to W$. If $v_1,\dots,v_n$ is independent in V, then $Tv_1,\dots,Tv_n$ is independent in W.

Answer. *False. For example, if* $T:\mathbb{R}^3\to\mathbb{R}^2$ *is zero map and* v_1,v_2 *is the independent list* $(1,0,0),(0,1,0)$, Tv_1,Tv_2 *is the dependent list* $(0,0,0),(0,0,0)$.

17. Suppose $T:V\to W$. If $v_1,\dots,v_n$ spans V, then $Tv_1,\dots,Tv_n$ spans the range(T).

Answer. *True. Every vector in* $\text{range}(T)$ *has the form* Tv *for some* $v\in V$*. If* $v_1,\dots,v_n$ *spans* V*, then there exist scalars* $a_1,\dots,a_n$ *so that* $v=a_1v_1+\cdots+a_nv_n$*. Then* $Tv=a_1Tv_1+\cdots a_nTv_n$ *and we see* Tv *is in the span of* $Tv_1,\dots,Tv_n$.

1 Extras

Here are some extra problems and theorems that we discussed in class, which I'm repeating here for your reference.

1.1 An interesting problem

18. Does there exist a quadratic polynomial q with $q(0) = 0$, $q(1) = 0$, and $\int_0^2 q = 0$?

Answer. *No. Consider the map* $T \in \mathscr{L}\left((P)_2(\mathbb{R}), \mathbb{R}^3\right)$ *defined by* $T(p) = (p(0), p(1), \int_0^2 p)$. *The question, then, is whether there is a degree two polynomial in* null(T). *I claim that* $\dim(\text{range}(T)) = 3 \Rightarrow \dim(\text{null}(T)) = 0$. *Therefore, the only polynomial* $q \in \text{null}(T)$ *is the zero polynomial. To see that* $\dim(\text{range}(T)) = 3$*, look at*

$$T(x^2) = \left(0, 1, \frac{8}{3}\right)$$
$$T(x) = (0, 1, 2)$$
$$T(1) = (1, 1, 1)$$

and note that the three vectors $\left(0, 1, \frac{8}{3}\right), (0, 1, 2), (1, 1, 1)$ *are independent vectors in the range of* T. *Therefore, the dimension of the range of* T *is at least (hence equal) to* 3.

1.2 Answering a question that was asked several times

There were some questions yesterday after class which are answered by the following theorem:

Theorem. *If* $T \in \mathscr{L}(V, W)$ *is a linear map between vector spaces with the same finite dimension, then* T *is injective iff* T *is surjective.*

Proof. The fundamental theorem of linear maps says that $\dim(V) = \dim(\text{range}(T)) + \dim(\text{null}(T))$. So, $\dim(\text{null}(T)) = 0$ if and only if $\dim(\text{range}(T)) = \dim(V) = \dim(W)$. Since $\dim(\text{range}(T)) = \dim(W)$ iff T is surjective and $\dim(\text{null}(T)) = 0$ iff T is injective, we see T is injective if and only if T is surjective. □

1.3 Injectivity, surjectivity, and left and right invertibility

You may know that for arbitrary functions between sets, *injective* and *left-invertible* are equivalent and *surjective* and *right-invertible* are equivalent. *The same is true for linear maps between vector spaces.* Let's break it down:

Theorem. *Suppose* $T \in \mathscr{L}(V, W)$. *If there exists a map* $S \in \mathscr{L}(W, V)$ *so that* $ST = I_V$ *then* T *is injective.*

Proof. Suppose $T \in \mathscr{L}(V, W)$, $S \in \mathscr{L}(W, V)$ and $ST = I_V$. To show T is injective, suppose $T(v_1) = T(v_2)$. Apply S to get $ST(v_1) = ST(v_2)$. Since $ST = I$, this says $v_1 = v_2$. □

Theorem. *Suppose* $S \in \mathscr{L}(W, V)$. *If there exists a map* $T \in \mathscr{L}(V, W)$ *so that* $ST = I_V$ *then* S *is surjective*

Proof. Suppose $S \in \mathscr{L}(W, V)$, $T \in \mathscr{L}(V, W)$ and $ST = I_V$. To show $S : W \to V$ is surjective, let $v \in V$. Look at $w := Tv$. We have $Sw = STv = Iv = v$. □

Theorem. *Suppose $T \in \mathscr{L}(V,W)$. If T is injective then there exists a linear map $S \in \mathscr{L}(W,V)$ with $ST = I_V$.*

Proof. This is true without any hypotheses on the dimensions of V and W. However, I will prove it assuming $\dim(V) = m$ and $\dim(W) = n$. Suppose $T \in \mathscr{L}(V,W)$ is injective and let $v_1,\dots,v_n$ be a basis for V. Then $Tv_1,\dots,Tv_m$ is an independent list of vectors in W. Extend this list to a basis $Tv_1,\dots,Tv_m,w_{n+1},\dots,w_n$ of W. The map $S \in \mathscr{L}(W,V)$ defined by $Tv_1 \mapsto v_1$, $\dots$, $Tv_m \mapsto v_m$, $w_{m+1} \mapsto 0$, $\dots$, $w_n \mapsto 0$ satisfies $ST = I_V$. □

Theorem. *Suppose $S \in \mathscr{L}(W,V)$. If S is surjective then there exists a linear map $T \in \mathscr{L}(V,W)$ with $ST = I_V$.*

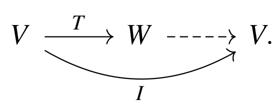
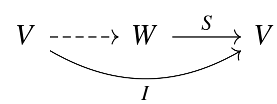
Proof. This is true without any hypotheses on the dimensions of V and W. However, I will prove it assuming $\dim(V) = m$ and $\dim(W) = n$. Suppose $S \in \mathscr{L}(W,V)$ is surjective and let $v_1,\dots,v_n$ be a basis for V. There exist $w_1,\dots,w_n$ with $Sw_1 = v_1,\dots,Sw_n = v_n$. Becasue the list $v_1,\dots,v_n$ is independent in V, the list $w_1,\dots,w_n$ is independent in W. Extend to a basis $w_1,\dots,w_n,w_{n+1},\dots,w_m$ of W. Then $T \in \mathscr{L}(V,W)$ defined by $Tw_1 = v_1,\dots,Tw_n = v_n,Tw_{n+1} = 0,\dots,Tw_m = 0$ satisfies $ST = I_V$. □

I emphasize that these theorems are true even if V and W are infinite dimensional. I proved them only for finite dimensions since having a basis makes it easy to define a linear map. In infinite dimensions, one needs to use the axiom of choice to construct the one-sided inverses. To summarize we have

Theorem. *Suppose $T \in \mathscr{L}(V,W)$. The map T is injective iff there exists $S \in \mathscr{L}(W,V)$ with $ST = I_V$.*

Theorem. *Suppose $S \in \mathscr{L}(W,V)$. The map S is surjective iff there exists $T \in \mathscr{L}(V,W)$ with $ST = I_V$.*

Using pictures, where dashed arrows means "there exists a map", we have

T is injective if and only if  and S is surjective if and only if 

Theorem. *A map $T \in \mathscr{L}(V,W)$ is invertible if and only if T is bijective.*

Proof. Suppose $T \in \mathscr{L}(V,W)$ is invertible. Then there exists a map S with $ST = I_V$ and $TS = I_W$. The first equation implies T is injective, the second implies T is surjective.

Now, suppose T is bijective. Because T is injective, there exists a map S with $ST = I_V$. Because T is surjective, there exists a map R so that $TR = I_W$. If we show $R = S$, then we are finished. Look:

$$R = I_V R = (ST)R = S(TR) = SI_W = S$$

□

Be aware that there can be many different one-sided inverses for a map. If $T : V \to W$ is injective but not surjective, there are infinitely many different maps $S : W \to V$ with $ST = I_V$. Likewise, if $S : W \to V$ is surjective but not injective, there are infinitely many different maps $T : V \to W$ with $ST = I_V$. However, the situation collapses for bijections, as the proof above shows: if a map is both left and right invertible, then any left inverse has to equal any right inverse and so there is only a single map that serves as a two sided inverse, which we call *the* inverse.

As a final remark, to pinpoint a difference between maps between finite dimensional spaces and maps between infinite dimensional spaces, we only have the equality

$$\dim(\text{Domain}\,T) = \dim(\text{range}\,T) + \dim(\text{null}\,T)$$

for maps between finite dimensional spaces.