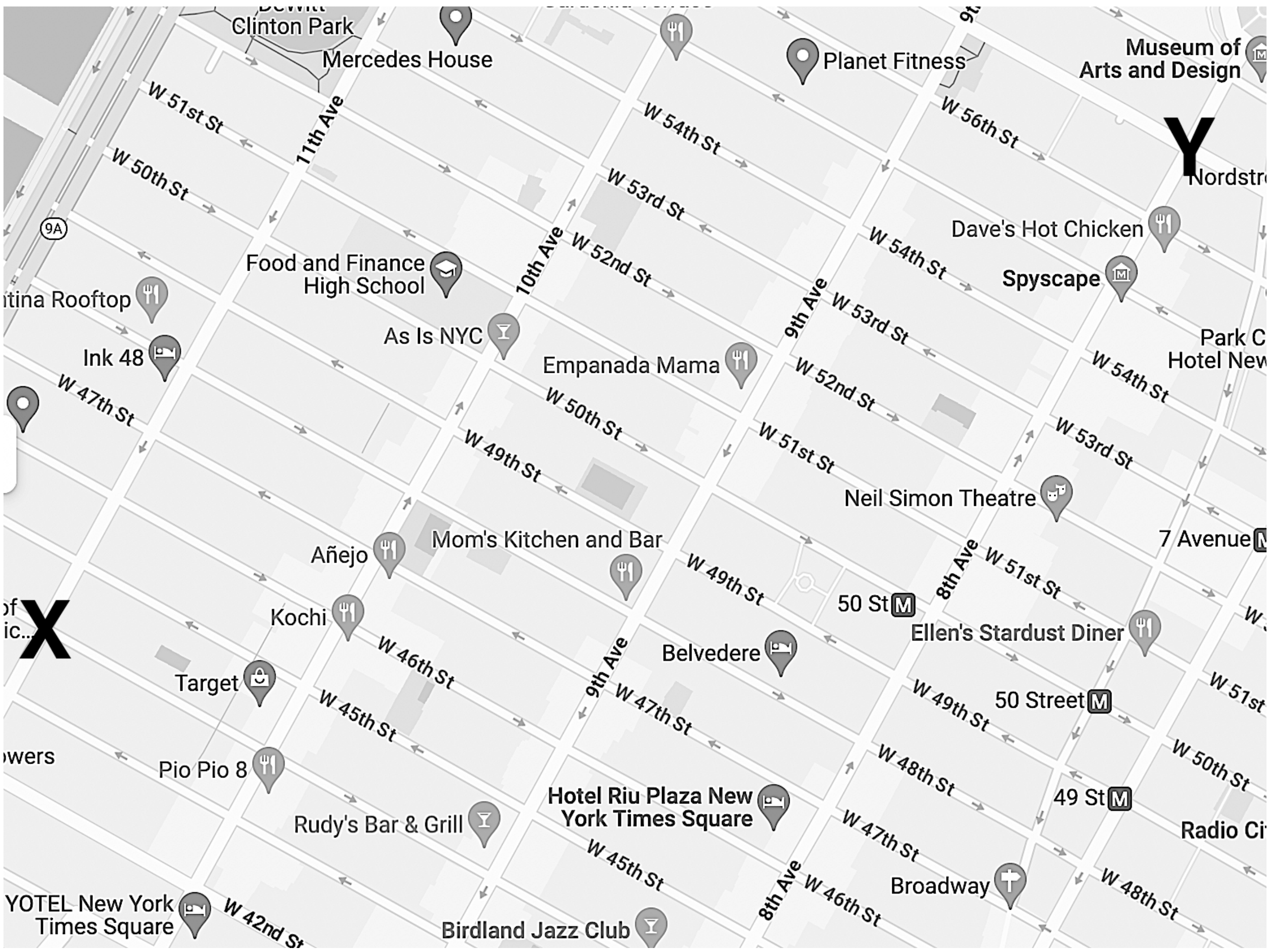
1 Functions and Counting

1. [2 points] Suppose you are trying to get from the corner of eleventh avenue and 44th street to the corner of eight avenue and 57th street (from the X to the Y on the map). How many different ways are there to walk there along the streets and avenues, assuming you don't go out of your way?



Answer. The answer is $\binom{16}{3} = 560$ different paths and here's how I got that. You need to walk up 13 blocks and over three avenues. That's 16 total blocks and you need to choose three occasions to go over on an avenue. For example, you can denote the path where you walk over three avenues over then up 13 blocks by $AAABBBBBBBBBBBBB$ or use $BBBAABBBBBBBBBBA$ to denote the path that goes three blocks up, over two avenues, up ten blocks, then over one avenue.

2. [3 points] Let $X = \{a, e, i, o, u\}$ and $Y = \{\text{red}, \text{green}, \text{blue}, \text{purple}, \text{yellow}, \text{orange}\}$.

a) How many different functions are there $X \longrightarrow Y$?

Answer. 6^5

b) How many functions $X \longrightarrow Y$ are injective ?

Answer. 6!

c) How many functions $X \longrightarrow Y$ are surjective ?

Answer. 0

3. [5 points] Let $X = \{a, e, i, o, u\}$ and $Y = \{\text{red}, \text{green}, \text{blue}, \text{purple}, \text{yellow}, \text{orange}\}$ and consider $f : X \longrightarrow Y$ defined by

$$a \mapsto \text{green} \qquad e \mapsto \text{green} \qquad i \mapsto \text{blue} \qquad o \mapsto \text{green} \qquad u \mapsto \text{red}$$

a) $f(e) = \text{green}$.

b) $f(\{e, i\}) = \{\text{green, blue}\}$

c) $f^{-1}\left(\{\text{red, purple, blue}\}\right) = \{i, u\}$

d) $f^{-1}(f(\{e\})) = f^{-1}(\{green\}) = \{a, e, o\}$

e) Find two sets $A, B \subseteq X$ for which $f(A \cap B) \neq f(A) \cap f(B)$.

Answer. If $A = \{a, e, i\}$ and $B = \{i, o, u\}$ then $f(A \cap B) = f(\{i\}) = \{\text{blue}\}$. But $f(A) \cap f(B) = \{\text{green, blue}\} \cap \{\text{blue,green,red}\} = \{\text{green, blue}\}$.

2 Short Answer: 1 point each

4. What are the values of $\lfloor 41.23 \rfloor$ and $\lfloor -2.3 \rfloor$?

Answer. $\lfloor 41.23 \rfloor = 41$ and $\lfloor -2.3 \rfloor = -3$.

5. Define a function $F : \mathbb{N} \to \mathbb{N}$ recursively by setting $F(1) = 1$, $F(2) = 1$, and for $n \geq 2$, setting $F(n) = F(n-1) + F(n-2)$. What is $F(6)$?

Answer. $F(3) = F(2) + F(1) = 1 + 1 = 2$, $F(4) = F(3) + F(2) = 2 + 1 = 3$, $F(5) = F(4) + F(3) = 3 + 2 = 5$, and $F(6) = F(5) + F(4) = 5 + 3 = 8$.

6. What is the quotient and remainder when 57 is divided by 4?

Answer. $57 = 4 * 14 + 1$ so the quotient is 14 and the remainder is 1.

7. Simplify $\dfrac{6^5}{3^6}$.

Answer. $\dfrac{6^5}{3^6} = \dfrac{2^5 3^5}{3^6} = \dfrac{2^5}{3} = \dfrac{32}{3}$.

8. Write $\log_2(a^4) + \log_2(b^2) - \log_2(ab)$ as a single, simple expression.

Answer. $\log_2(a^4) + \log_2(b^2) - \log_2(ab) = \log_2\left(\dfrac{a^4 b^2}{ab}\right) = \log_2\left(a^3 b\right)$

9. Write $\log_2(703)$ using only $\log_{10}$.

Answer. $\log_2(703) = \dfrac{\log_{10}(703)}{\log_{10}(2)}$

10. Compute $\log_5\left(\dfrac{1}{5}\right) \times \log_{\frac{1}{5}}(5)$.

Answer. $\log_5\left(\dfrac{1}{5}\right) \times \log_{\frac{1}{5}}(5) = (-1) \times (-1) = 1$.

11. Simplify $\dfrac{402!}{401!}$.

Answer. $\dfrac{402!}{401!} = \dfrac{402 \cdot 401 \cdot 400 \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{401 \cdot 400 \cdot 399 \cdot \cdots \cdot 3 \cdot 2 \cdot 1} = 402$.

12. True or false: A function $\{a, b, c, d\} \to \{a, b, c, d\}$ is injective if and only if it is surjective.

Answer. This is true! If a function $\{a, b, c, d\} \to \{a, b, c, d\}$ is injective, then a, b, c, d are all mapped to distinct elements. Therefore, the range must have four elements in it. The only subset of $\{a, b, c, d\}$ with four elements is the set $\{a, b, c, d\}$ itself, which means the function is surjective.

13. True or false: A function $\mathbb{N} \to \mathbb{N}$ is injective if and only if it is surjective.

Answer. This is false! For example, the shift map $n \mapsto n + 1$ defines an injective function $\mathbb{N} \to \mathbb{N}$ that is not surjective.

3 Bonus

14. [2 points] $\lfloor \log_{10}(12345678901234567890123456789012345678901234567890) \rfloor =$

Answer. The answer is 49. To see this, note that the argument of the log here has fifty digits. So, it lies between the number 1 followed by forty nine zeroes and the number 1 followed by fifty zeros:

$$\begin{aligned} 10^{49} &= 10000000000000000000000000000000000000000000000000 \\ &< 12345678901234567890123456789012345678901234567890 \\ &< 100000000000000000000000000000000000000000000000000 = 10^{50} \end{aligned}$$

Applying $\log_{10}$ yields $49 < \log_{10}(12345678901234567890123456789012345678901234567890) < 50$. So, the floor function applied to this number is 49.

15. [2 points] Let $X = \{a, e, i, o, u\}$ and $Y = \{\text{red}, \text{green}, \text{blue}\}$. How many functions $X \to Y$ are surjective?

Answer. The answer is 150. To see this, first carefully count the functions $X \to Y$ that are not surjective. They come in three types: let A be the set of functions $X \to Y$ for which *red* is not in the range, let B be the set of functions for which *green* is not in the range, and let C be the set of functions without *blue* in the range. So, the number of non-surjective functions $X \to Y$ will be $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Note, the functions $X \to Y$ for which *red* is not in the range are the same as functions $\{a, e, i, o, u\} \to \{green, blue\}$. So, $|A| = 2^5$. Similarly $|B| = 2^5$ and $|C| = 2^5$. A function in $A \cap B$ has neither *red* nor *green* in the range, and so must be the constant function that sends every vowel to *blue*. So $|A \cap B| = 1$. Similarly, $|A \cap C| = 1$ and $|B \cap C| = 1$. There are no functions in $A \cap B \cap C$ since every function must have at least one element in its range. Therefore, there are

$$2^5 + 2^5 + 2^5 - 1 - 1 - 1 = 3 * 32 - 3 = 93$$

non surjective functions. Since there are $3^5 = 243$ functions from $X \to Y$ altogether, there are $243 - 93 = 150$ surjective functions $X \to Y$.