ON THE NORMS OF p-STABILIZED ELLIPTIC NEWFORMS

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Abstract. Let $f \in S_\kappa(\Gamma_0(N))$ be a Hecke eigenform at p with eigenvalue $\lambda_f(p)$ for a prime $p \nmid N$. Let α_p and β_p be complex numbers satisfying $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = p^{\kappa-1}$. We calculate the norm of $f_p^{\alpha_p}(z) = f(z) - \beta_p f(pz)$ as well as the norm of $U_p f$, both classically and adelically. We use these results along with some convergence properties of the Euler product defining the symmetric square L-function of f to give a 'local' factorization of the Petersson norm of f.

1. Introduction

Let $\kappa \geq 2$ and $N \geq 1$ be integers and p an odd prime with $p \nmid N$. Let $f \in S_\kappa(\Gamma_0(N))$ be a newform. It is well-known that the Petersson norm $\langle f, f\rangle$ serves as a natural period for many L-functions of f [7, 15].

In this paper we focus on related periods $\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle$ (defined below) for α_p a Satake parameter of f. When f is ordinary at p, the forms $f_p^{\alpha_p}$ arise naturally in the context of Iwasawa theory as the objects which can be interpolated into a Hida family. It is in fact in the context of 'p-adic interpolation' of some automorphic lifting procedures (between two algebraic groups, one of them being GL_2) that these calculations arise (see [1] for example); however, our results apply in a more general setup as specified below.

Let $f \in S_\kappa(\Gamma_0(N))$ be an eigenform for the T_p-operator with eigenvalue $\lambda_f(p)$. Let α_p and β_p be the pair of complex numbers satisfying $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = p^{\kappa-1}$. We set $f_p^{\alpha_p}(z) = f(z) - \beta_p f(pz)$. In the case that f is ordinary at p, we can choose α_p and β_p so that α_p is a p-unit and β_p is divisible by p. In this special case $f_p^{\alpha_p}$ is the p-stabilized ordinary newform of tame level N attached to f.

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Since $f_p^{\alpha_p} = p^{1-\kappa}\beta_p(U_p - \beta_p)f$, calculating $\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle$ is in fact equivalent to calculating $\langle U_p f, U_p f\rangle$. While computation of any of these inner products does not present any difficulties (see Section 2), it is an accident resulting from the relative simplicity of the Hecke algebra on GL_2, where the T_p and the U_p operators differ by a single term. It turns out that in the higher-rank case it is the calculation of the latter inner product that provides the fastest route to computing the Petersson norm of various p-stabilizations. With these future applications in mind we present an alternative approach to calculating $\langle U_p f, U_p f\rangle$, this time working adelically (see Sections 3 and 4), as this is the method that generalizes to higher genus most readily (see [1], where this is done for the group GSp_4).

It is well-known that the Petersson norm $\langle f, f\rangle$ is closely related to the value $L(\kappa, \mathrm{Sym}^2 f)$ at κ of the symmetric square L-function of f. The absolutely convergent Euler product defining this L-function for $\mathrm{Re}(s) > \kappa$ converges (conditionally) to the value $L(s, \mathrm{Sym}^2 f)$ when $\mathrm{Re}(s) = \kappa$ (this and in fact a more general result is proved in the appendix by Keith Conrad). On the other hand our computation of $\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle$ shows that this inner product differs from $\langle f, f\rangle$ by essentially the p-Euler factor of $L(\kappa, \mathrm{Sym}^2 f)$. Combining these facts we exhibit a (conditionally convergent) factorization of $\langle f, f\rangle$ into local components defined via the inner products $\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle$ (for details, see Section 5).

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2. Classical calculation of $\langle f_p, f_p\rangle$ and $\langle U_p f, U_p f\rangle$

Let N be a positive integer. Let $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ denote the subgroup consisting of matrices whose lower-left entry is divisible by N. For a holomorphic function f on the complex upper half-plane $\mathfrak{h}$ and for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$, where $+$ denotes positive determinant, and $\kappa \in \mathbf{Z}_+$ we define the slash operator as

$$(f|_\kappa \gamma)(z) = \frac{\det(\gamma)^{\kappa/2}}{(cz+d)^\kappa} f\left(\frac{az+b}{cz+d}\right).$$

If κ is clear from the context we will simply write $f|\gamma$ instead of $f|_\kappa\gamma$. We will write $S_\kappa(\Gamma_0(N))$ for the $\mathbf{C}$-space of cusp forms of weight κ and level $\Gamma_0(N)$ (i.e., functions f as above which satisfy $f|_\kappa\gamma = f$ for all $\gamma \in \Gamma_0(N)$ and vanish at the cusps - for details see [11]).

The space $S_\kappa(\Gamma_0(N))$ is endowed with a natural inner product (the *Petersson inner product*) defined by

$$\langle f, g\rangle_N = \int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\overline{g(z)} y^{\kappa-2} dx dy$$

for $z = x + iy$ with $x, y \in \mathbf{R}$ and $y > 0$. If $\Gamma \subset \Gamma_0(N)$ is a finite index subgroup we also set

$$\langle f, g\rangle_\Gamma = \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)}y^{\kappa-2}dxdy.$$

From now on let p be a prime which does not divide N. Set $\eta = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$. We have the decomposition

$$\Gamma_0(N)\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\Gamma_0(N) = \bigsqcup_{j=0}^{p-1}\Gamma_0(N)\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} \sqcup \Gamma_0(N)\eta. \tag{2.1}$$

Recall the pth Hecke operator acting on $S_\kappa(\Gamma_0(N))$ is given by

$$T_p f = p^{\kappa/2-1}\left(\sum_{j=0}^{p-1} f|_\kappa \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} + f|_\kappa \eta\right)$$

and the pth Hecke operator acting on $S_\kappa(\Gamma_0(Np))$ is given by

$$U_p f = p^{\kappa/2-1}\left(\sum_{j=0}^{p-1} f|_\kappa \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}\right).$$

As we will be viewing $f \in S_\kappa(\Gamma_0(N))$ as an element of $S_\kappa(\Gamma_0(Np))$, we use T_p and U_p to distinguish the two Hecke operators at p defined above.

Let $f \in S_\kappa(\Gamma_0(N))$ be an eigenfunction for T_p with eigenvalue $\lambda_f(p)$. There exist (up to permutation) unique complex numbers α_p and β_p satisfying $\lambda_f(p) = \alpha_p + \beta_p$ and $\alpha_p\beta_p = p^{\kappa-1}$. We consider the following two forms:

$$f_p^{\alpha_p}(z) = f(z) - \beta_p p^{-\kappa/2}(f|_\kappa\eta)(z),$$
$$f_p^{\beta_p}(z) = f(z) - \alpha_p p^{-\kappa/2}(f|_\kappa\eta)(z).$$

One immediately obtains that $f_p^{\alpha_p} \in S_\kappa(\Gamma_0(Np))$ and that $f_p^{\alpha_p}$ is an eigenfunction for the operator U_p with eigenvalue α_p. Furthermore, if f is also an eigenform for T_ℓ for a prime $\ell \neq p$, then so is $f_p^{\alpha_p}$ and it has the same T_ℓ-eigenvalue as f. The analogous statements for $f_p^{\beta_p}$ hold as well. Note that if f is ordinary at p, then one can choose α_p and β_p so that $\mathrm{ord}_p(\alpha_p) = 0$ and then $f_p^{\alpha_p}$ is the p-stabilized newform associated to f, see [16] for example.

Theorem 2.1. *Let $f \in S_\kappa(\Gamma_0(N))$ be defined as above, where $p \nmid N$. We have*

$$\frac{\langle U_p f, U_p f\rangle_{Np}}{\langle f, f\rangle_{Np}} = p^{\kappa-2} + \frac{(p-1)\lambda_f(p)^2}{p+1}$$

and

$$\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f, f\rangle_{Np}} = \frac{p}{p+1}\left(1 - \frac{\alpha_p^2}{p^\kappa}\right)\left(1 - \frac{\beta_p^2}{p^\kappa}\right).$$

Proof. The definition of $f_p^{\alpha_p}$ and the fact that $U_p f = T_p f - p^{\kappa/2-1} f|\eta$ immediately give

$$\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np} = (1+|\beta_p|^2 p^{-\kappa})\langle f, f\rangle_{Np} - p^{-\kappa/2}(\beta_p\langle f|\eta, f\rangle_{Np} + \overline{\beta_p\langle f|\eta, f\rangle}_{Np})$$

and

$$\langle U_p f, U_p f\rangle_{Np} = (p^{\kappa-2} + \lambda_f(p)^2)\langle f, f\rangle_{Np} - p^{\kappa/2-1}\lambda_f(p)(\langle f|\eta, f\rangle_{Np} + \overline{\langle f|\eta, f\rangle}_{Np}).$$

Let us now compute $\langle f|\eta, f\rangle_{Np}$. Observe that by the definition of T_p we have

$$\langle T_p f, g\rangle_{Np} = p^{\kappa/2-1}\left(\sum_{j=0}^{p-1}\left\langle f \mid \begin{bmatrix}1 & j\\ 0 & p\end{bmatrix}, g\right\rangle_{Np} + \langle f|\eta, g\rangle_{Np}\right).$$

Using the decomposition (2.1) we can find $a_j, b_j \in \Gamma_0(N)$ so that $a_j\begin{bmatrix}1 & j\\ 0 & p\end{bmatrix} b_j = \begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix}$, and $a, b \in \Gamma_0(N)$ so that $a\begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix} b = \begin{bmatrix}p & 0\\ 0 & 1\end{bmatrix}$. Using this and the fact that $f, g \in S_\kappa(\Gamma_0(N))$, we have

$$\begin{aligned}
p^{1-\kappa/2}\langle T_p f, g\rangle_{Np} &= \sum_{j=0}^{p-1}\left\langle f \mid a_j\begin{bmatrix}1 & j\\ 0 & p\end{bmatrix}, g|b_j^{-1}\right\rangle_{Np} + \langle f|\eta, g\rangle_{Np}\\
&= \sum_{j=0}^{p-1}\left\langle f \mid a_j\begin{bmatrix}1 & j\\ 0 & p\end{bmatrix} b_j, g\right\rangle_{Np} + \langle f|\eta, g\rangle_{Np}\\
&= p\left\langle f \mid \begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix}, g\right\rangle_{Np} + \langle f|\eta, g\rangle_{Np}\\
&= p\left\langle f \mid a\begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix}, g|b^{-1}\right\rangle_{Np} + \langle f|\eta, g\rangle_{Np}\\
&= (p+1)\langle f|\eta, g\rangle_{Np}.
\end{aligned}$$

Thus, setting $g = f$ we obtain

$$\langle f|\eta, f\rangle_{Np} = p^{1-\kappa/2}\frac{\lambda_f(p)}{p+1}\langle f, f\rangle_{Np}.$$

We can now easily conclude that

$$\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f, f\rangle_{Np}} = 1 + |\beta_p|^2 p^{-\kappa} - p^{1-\kappa}(\beta_p + \overline{\beta_p})\frac{\lambda_f(p)}{p+1}$$

and

$$\frac{\langle U_p f, U_p f\rangle_{Np}}{\langle f, f\rangle_{Np}} = p^{\kappa-2} + \frac{(p-1)\lambda_f(p)^2}{p+1}.$$

Using the fact that T_p is self-adjoint with respect to the Petersson inner product we have $\alpha_p + \beta_p = \lambda_f(p) \in \mathbf{R}$. We note by Lemma 4.2 below that

$\overline{\alpha_p} = \beta_p$. Thus $|\alpha_p|^2 = |\beta_p|^2 = |\alpha_p||\beta_p| = p^{\kappa-1}$ and $\beta_p + \overline{\beta_p} = \lambda_f(p)$. This allows us to simplify the formula for $\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f,f\rangle_{Np}}$ to

$$\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f,f\rangle_{Np}} = 1 + \frac{1}{p} - p^{1-\kappa}\frac{\lambda_f(p)^2}{p+1}.$$

Again using that $\lambda_f(p) = \alpha_p + \beta_p$ we obtain

$$\begin{aligned}\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f,f\rangle_{Np}} &= \frac{1}{p+1}\left(p + 1 + \frac{p+1}{p} - p^{1-\kappa}(\alpha_p^2 + \beta_p^2 + 2p^{\kappa-1})\right) \\ &= \frac{p}{p+1}\left(1 - \frac{\alpha_p^2}{p^\kappa}\right)\left(1 - \frac{\beta_p^2}{p^\kappa}\right).\end{aligned}$$

□

Corollary 2.2. *We have*

$$\lim_{\substack{p\to\infty \\ p \text{ is prime}}} \frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{p+1} = \langle f,f\rangle_N.$$

Proof. Using Theorem 2.1 and the fact that $\langle f,f\rangle_{Np} = (p+1)\langle f,f\rangle_N$, we have for every prime $p \nmid N$ that

$$\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{p+1} = \frac{p}{p+1}\left(1 - \frac{\alpha_p^2}{p^\kappa}\right)\left(1 - \frac{\beta_p^2}{p^\kappa}\right)\langle f,f\rangle_N.$$

Since $|\alpha_p|^2 = |\beta_p|^2 = p^{\kappa-1}$, we see that the first three factors on the right tend to 1 as p tends to infinity. □

3. Relation between the classical and adelic inner products

While the classical calculations for $\langle U_p f, U_p f\rangle$ are rather elementary, it is also useful to note that one can perform these calculations adelically. The problem of calculating $\langle U_p f, U_p f\rangle$ is one that is local in nature, so it lends itself nicely to such an approach. Moreover, in a higher genus setting such as when working with Siegel modular forms, it is the adelic approach that generalizes most readily [1]. In this section we provide the necessary background relating the adelic and classical inner products that is needed to relate the adelic inner product calculated in Section 4 to the calculation given in the previous section.

In this and the following sections p will denote a prime number and v will denote an arbitrary place of $\mathbf{Q}$ including the Archimedean one, which we will denote by ∞. Let $G = \mathrm{GL}_2$ and fix $N \geq 1$. By strong approximation (see for example [2, Theorem 3.3.1, p. 293]) we have

$$G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})\prod_p K_p, \tag{3.1}$$

where K_p is a compact subgroup of $G(\mathbf{Q}_p)$ such that $\det K_p = \mathbf{Z}_p^\times$. One example would be to take $K_p = K_0(N)_p$, where

$$K_0(N)_p = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G(\mathbf{Z}_p) : c \equiv 0 \bmod N \right\}.$$

Note that $K_0(N)_p = G(\mathbf{Z}_p)$ if $p \nmid N$. We will also set

$$K_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G(\hat{\mathbf{Z}}) : c \equiv 0 \bmod N \right\} = \prod_p K_0(N)_p.$$

The decomposition (3.1) implies that

$$G(\mathbf{Q}) \setminus G(\mathbf{A}) = G^+(\mathbf{R}) \prod_p K_p, \tag{3.2}$$

where $+$ indicates positive determinant.

Let $Z \subset G$ denote the center. For every p there is a unique Haar measure dg_p on $G(\mathbf{Q}_p)$ normalized so that the volume of any maximal compact subgroup of $G(\mathbf{Q}_p)$ is one. We use the standard Haar measure on $G(\mathbf{R})$ as defined in [2, § 2.1]. Define the adelic analogue of the Petersson inner product:

$$\langle \phi_1, \phi_2 \rangle = \int_{Z(\mathbf{A})G(\mathbf{Q})\setminus G(\mathbf{A})} \phi_1(g)\overline{\phi_2(g)}dg,$$

where ϕ_1 and ϕ_2 lie in $L^2(Z(\mathbf{A})G(\mathbf{Q}) \setminus G(\mathbf{A}))$ and have the same central character and dg is the Haar measure on $Z(\mathbf{A})G(\mathbf{Q}) \setminus G(\mathbf{A})$ corresponding to our choice of local Haar measures.

Let $f \in S_\kappa(\Gamma_0(N))$ be an eigenform. For $g = \gamma g_\infty k \in G(\mathbf{A})$ with $\gamma \in G(\mathbf{Q})$, $g_\infty = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G^+(\mathbf{R})$ and $k \in K_0(N)$, set

$$\phi_f(g) = \frac{(\det g_\infty)^{\kappa/2}}{(ci+d)^\kappa} f(g_\infty i). \tag{3.3}$$

Then ϕ_f is an automorphic form on $G(\mathbf{A})$ and it is easy to see (using the bijection in [5, Equation 5.13]) that one has

$$\langle \phi_f, \phi_f \rangle = \frac{1}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} \langle f, f \rangle_N. \tag{3.4}$$

Let $\pi_f \cong \otimes \pi_{f,v}$ be the automorphic representation generated by ϕ_f. If f is a newform, then we can write $\phi_f = \otimes_v \phi_{f,v}$ for $\phi_{f,v} \in \pi_{f,v}$ and $\phi_{f,v}$ are spherical vectors for all $v \nmid N$, $v \neq \infty$. For every v we can choose a $G(\mathbf{Q}_v)$-invariant inner product $\langle \cdot, \cdot \rangle_v$ (and any two such are scalar multiples of each other) so that $\langle \phi_{f,v}, \phi_{f,v} \rangle_v = 1$ for all $v \nmid N$, $v \neq \infty$. It follows that there is constant c so that

$$\langle \phi_f, \phi_f \rangle = c \prod_v \langle \phi_{f,v}, \phi_{f,v} \rangle_v. \tag{3.5}$$

We are now in a position to relate the ratio $\frac{\langle U_p f, U_p f\rangle_{Np}}{\langle f,f\rangle_{Np}}$ to something that can be calculated locally. In fact since we are only interested in this ratio, the precise value of the constant c in (3.5) will be irrelevant.

Fix $p \nmid N$. As noted above, we normalize our Haar measure so that $\mathrm{vol}(K_0(1)_p) = 1$. For a vector v_p inside the space of $\pi_{f,p}$, we set

$$T_p v_p = \int_{K_0(1)_p \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] K_0(1)_p} \pi_{f,p}(g) v_p dg$$

and

$$V_p v_p = \int_{\left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right] K_0(1)_p} \pi_{f,p}(g) v_p dg.$$

Note that we have the decompositions

$$K_0(1)_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_0(1)_p = \bigsqcup_{b=0}^{p-1} \begin{bmatrix} p & b \\ 0 & 1 \end{bmatrix} K_0(1)_p \sqcup \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} K_0(1)_p \tag{3.6}$$

and

$$K_0(p)_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_0(p)_p = \bigsqcup_{b=0}^{p-1} \begin{bmatrix} p & b \\ 0 & 1 \end{bmatrix} K_0(p)_p. \tag{3.7}$$

The adelic operator corresponding to the U_p-operator acting on classical modular forms as defined in Section 2 is given by

$$U_p v_p := T_p v_p - V_p v_p.$$

Lemma 3.1. *We have*

$$\frac{\langle U_p^{\mathrm{cl}} f, U_p^{\mathrm{cl}} f\rangle_{Np}}{\langle f, f\rangle_{Np}} = p^{\kappa-2} \langle U_p \phi_{f,p}, U_p \phi_{f,p}\rangle_p$$

for any local inner product pairing $\langle , \rangle_p$ *so that* $\langle \phi_{f,p}, \phi_{f,p}\rangle_p = 1$ *and* U_p^{cl} *is the classical* U_p*-operator as defined in Section 2.*

Proof. If we set $U_p\phi_f := (U_p \phi_{f,p}) \otimes \otimes_{v \neq p} \phi_{f,v}$ then it follows by the same argument as the one in the proof of [5, Lemma 3.7] that

$$\phi_{U_p^{\mathrm{cl}} f} = p^{\kappa/2-1} U_p \phi_f. \tag{3.8}$$

The lemma is now immediate from (3.4) and (3.5). □

It only remains to calculate $\langle U_p \phi_{f,p}, U_p \phi_{f,p}\rangle$, which is done in the next section.

4. Local calculation of $\langle U_p\phi_{f,p}, U_p\phi_{f,p}\rangle$

We will now give a calculation that, when combined with the results of the previous section, provides a local way to calculate $\langle U_p f, U_p f\rangle$ in terms of $\langle f, f\rangle$. As in the previous section we fix $f \in S_k(\Gamma_0(N))$ a newform and a prime p not dividing N. We again let $\pi_f = \otimes_v \pi_{f,v}$ be the automorphic representation associated to f. Note that since $p \nmid N$, we can take the principal series representation $\pi_p(\chi_1, \chi_2)$ to be the model for $\pi_{f,p}$ and for functions $\psi, \psi' \in \pi_p(\chi_1, \chi_2)$ define the local inner product by

$$\langle \psi, \psi' \rangle_p := \int_{K_0(N)_p} \psi(g)\overline{\psi'(g)} dg,$$

where the Haar measure is normalized so that $\mathrm{vol}(K_0(N)_p) = 1$. Then the vector $\phi_{f,p} \in \pi_p$ corresponds to the function (which we will also denote by $\phi_{f,p} \in \pi_p(\chi_1, \chi_2)$) which can be described explicitly as

$$\phi_{f,p}\left(\begin{bmatrix} a & * \\ 0 & b \end{bmatrix} k\right) = \chi_1(a)\chi_2(b)|ab^{-1}|_p^{1/2},$$

where $|\cdot|_p$ denotes the standard p-adic norm ($|p|_p = p^{-1}$) and $k \in K_0(N)_p$.

As this section is focused on the calculation of $\langle U_p\phi_{f,p}, U_p\phi_{f,p}\rangle$, we will from now on write ϕ for $\phi_{f,p}$ and $K_0(1)$ (resp., $K_0(p)$) for $K_0(1)_p$ (resp., $K_0(p)_p$).

Remark 4.1. We note here that the calculation which follows can also be performed using the MacDonald formula for matrix coefficients (see [3, § 4]). However, in the relatively simple case of GL_2 the elementary approach which we present below does not add any computational difficulty and is perhaps more transparent.

Set

$$\mathcal{B} := \left\{ \begin{bmatrix} p & b \\ 0 & 1 \end{bmatrix} : b \in \{0, 1, \dots, p-1\} \right\}, \quad \mathcal{B}' := \mathcal{B} \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \right\}.$$

If $g \in K_0(1)$ and $\beta \in \mathcal{B}'$, there is a permutation σ_g of $\mathcal{B}'$ and elements $k(g, \beta) \in K_0(1)$ such that $g\beta = \sigma_g(\beta)k(g, \beta)$. Furthermore, note that if $g \in K_0(1) - K_0(p)$, then the corresponding permutation cannot fix $\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$. This implies that for such a g, there exists $\beta \in \mathcal{B}$ such that $\sigma_g(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$. Since in the computation of $U_p\phi$ only matrices in $\mathcal{B}$ are used, we are interested in the restriction of σ to $\mathcal{B}$. For such a g there are $p-1$ matrices in the image of σ_g which have $(p, 1)$ on the diagonal and one that has $(1, p)$ on the diagonal. Set $\mathcal{B}_1(g) = \{\beta \in \mathcal{B} : \sigma_g(\beta) \in \mathcal{B}\}$ and $\mathcal{B}_2(g) = \{\beta \in \mathcal{B} : \sigma_g(\beta) \in \mathcal{B}' - \mathcal{B}\}$. So for $g \in K_0(1) - K_0(p)$ we have (note that our ϕ is right-$K_0(1)$-invariant and

$\mathrm{vol}(K_0(1)) = 1$)

$$\begin{aligned}(U_p\phi)(g) &= \mathrm{vol}(K_0(1)) \sum_{\beta\in\mathcal{B}} \phi(g\beta) \\ &= \sum_{\beta\in\mathcal{B}} \phi(\sigma_g(\beta)k(g,\beta)) \\ &= \sum_{\beta\in\mathcal{B}_1(g)} \phi(\sigma_g(\beta)) + \sum_{\beta\in\mathcal{B}_2(g)} \phi(\sigma_g(\beta)) \\ &= (p-1)\chi_1(p)p^{-1/2} + \chi_2(p)p^{1/2}.\end{aligned}$$

If $g \in K_0(p)$, then the permutation σ fixes $\begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix}$, hence we obtain

$$(U_p\phi)(g) = \mathrm{vol}(K_0(1)) \sum_{\beta\in\mathcal{B}} \phi(g\beta) = \sum_{\beta\in\mathcal{B}} \phi(\sigma(\beta)) = p\chi_1(p)p^{-1/2} = \chi_1(p)p^{1/2}.$$

Now let us compute the integral:

$$\langle U_p\phi, U_p\phi\rangle_{K_0(1)} = \int_{K_0(p)} U_p\phi(g)\overline{U_p\phi(g)}dg + \int_{K_0(1)-K_0(p)} U_p\phi(g)\overline{U_p\phi(g)}dg.$$

We have

$$\begin{aligned}\int_{K_0(p)} U_p\phi(g)\overline{U_p\phi(g)}dg &= \int_{K_0(p)} p|\chi_1(p)|^2 dh \\ &= \mathrm{vol}(K_0(p))p|\chi_1(p)|^2 \\ &= \frac{p|\chi_1(p)|^2}{p+1}\end{aligned} \tag{4.1}$$

and, since $\mathrm{vol}(K_0(1) - K_0(p)) = p/(p+1)$,

$$\begin{aligned}&\int_{K_0(1)-K_0(p)} U_p\phi(g)\overline{U_p\phi(g)}dg \\ &= \int_{K_0(1)-K_0(p)} \left[\frac{(p-1)^2}{p}|\chi_1(p)|^2 + (p-1)\mathrm{tr}\,(\chi_1(p)\overline{\chi_2(p)}) + p|\chi_2(p)|^2\right] dg \\ &= \frac{p}{p+1}\left[\frac{(p-1)^2}{p}|\chi_1(p)|^2 + (p-1)\mathrm{tr}\,(\chi_1(p)\overline{\chi_2(p)}) + p|\chi_2(p)|^2\right].\end{aligned} \tag{4.2}$$

Putting (4.1) and (4.2) together we get

$$\langle U_p\phi, U_p\phi\rangle = \frac{p^2-p+1}{p+1}|\chi_1(p)|^2 + \frac{p^2}{p+1}|\chi_2(p)|^2 + \frac{p^2-p}{p+1}\mathrm{tr}\,(\chi_1(p)\overline{\chi_2(p)}). \tag{4.3}$$

Lemma 4.2. *We have $\chi_j(p) = p^{s_j}$ for $j = 1, 2$, where s_j is a purely imaginary number. In particular, $|\chi_j(p)| = 1$ for $j = 1, 2$. Moreover, we have $\overline{\alpha_p} = \beta_p$.*

Proof. The first part follows from [5, p. 92] and is a direct consequence of the fact that cusp forms on GL_2 satisfy the Ramanujan conjecture. Observe that $\alpha_p = p^{(\kappa-1)/2}\chi_1(p)$ and $\beta_p = p^{(\kappa-1)/2}\chi_2(p)$. Using that $\alpha_p\beta_p = p^{\kappa-1}$, we obtain $\chi_1(p)\chi_2(p) = 1$. This, combined with the fact that $\chi_j(p) = p^{s_j}$ with s_j purely imaginary implies $\chi_1(p) = \overline{\chi_2(p)}$. Thus $\overline{\alpha_p} = \beta_p$. □

Using Lemma 4.2 we can simplify (4.3) to

$$\langle U_p\phi, U_p\phi\rangle = \frac{2p^2-p+1}{p+1} + \frac{p^2-p}{p+1}\mathrm{tr}\,(\chi_1(p)\overline{\chi_2(p)}).$$

Moreover using that $\alpha_p = p^{(\kappa-1)/2}\chi_1(p)$, $\beta_p = p^{(\kappa-1)/2}\chi_2(p)$ and $\chi_1(p) = \overline{\chi_2(p)}$ we have

$$\mathrm{tr}\,(\chi_1(p)\overline{\chi_2(p)}) = (\chi_1(p)+\chi_2(p))^2 - 2\chi_1(p)\chi_2(p) = p^{1-\kappa}\lambda_f(p)^2 - 2.$$

Thus we obtain

$$\begin{aligned}\langle U_p\phi, U_p\phi\rangle &= \frac{p+1}{p+1} + \frac{p(p-1)p^{1-k}\lambda_f(p)^2}{p+1}\\ &= 1 + \frac{(p-1)\lambda_f(p)^2}{p+1}p^{2-k},\end{aligned}$$

hence we see that by Lemma 3.1 this recovers the classical formula from Theorem 2.1.

5. Applications to L-values

Let $f \in S_\kappa(\Gamma_0(N))$ be a newform. In this section we apply the results of the previous sections to give a 'local' decomposition of the Petersson norm of f. This depends on showing that value $L^N(k, \mathrm{Sym}^2 f)$ obtained by meromorphic continuation of $L(s, \mathrm{Sym}^2 f)$ can be expressed as a conditionally convergent Euler product.

Recall that the (partial) *symmetric square L-function* of f is defined by the Euler product

$$L^N(s, \mathrm{Sym}^2 f) = \prod_{p\nmid N} \frac{1}{L_p(s, \mathrm{Sym}^2 f)}, \tag{5.1}$$

where

$$L_p(s, \mathrm{Sym}^2 f) := \left(1 - \frac{\alpha_p^2}{p^s}\right)\left(1 - \frac{\alpha_p\beta_p}{p^s}\right)\left(1 - \frac{\beta_p^2}{p^s}\right).$$

The product (5.1) converges absolutely for $\mathrm{Re}\, s > \kappa$. It is well-known that $L^N(s, \mathrm{Sym}^2 f)$ admits meromorphic continuation to the entire complex plane with possible poles only at $s = \kappa$ and $\kappa - 1$, of order at most one [14, Theorem 1]. In our case (since f is assumed to have trivial character), the L-function does not have a pole at $s = \kappa$ [14, Theorem 2]. We will continue to denote this extended function by $L^N(s, \mathrm{Sym}^2 f)$. Using that $\alpha_p\beta_p = p^{\kappa-1}$ we conclude that

$$\frac{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}}{\langle f, f\rangle_{Np}} = \frac{p^2}{p^2-1}\frac{1}{L_p(\kappa, \mathrm{Sym}^2 f)} = \frac{\zeta_p(2)}{L_p(\kappa, \mathrm{Sym}^2 f)}, \tag{5.2}$$

where $\zeta_p(s) = 1/(1-1/p^s)$.

Corollary 5.1. *We have*

$$\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np} = \langle f_p^{\beta_p}, f_p^{\beta_p}\rangle_{Np}.$$

Set

$$\langle f, f\rangle_N^{(p)} := \frac{\langle f, f\rangle_{Np}}{\langle f_p^{\alpha_p}, f_p^{\alpha_p}\rangle_{Np}} = \frac{\langle f, f\rangle_{Np}}{\langle f_p^{\beta_p}, f_p^{\beta_p}\rangle_{Np}}.$$

We will now show that $\langle f, f\rangle_N^{(p)}$ can in some sense be regarded as a 'local' (at p) period for the symmetric square L-function.

Theorem 5.2. *The value $L^N(\kappa, \mathrm{Sym}^2 f)$ given by the meromorphic continuation is equal to the conditionally convergent Euler product*

$$\prod_{p\nmid N}\frac{1}{L_p(\kappa, \mathrm{Sym}^2 f)}$$

when we order the factors according to increasing p.

Proof. Let ϕ_f be defined as in (3.3) and let $\chi_1(p) = \alpha_p/p^{(\kappa-1)/2}$ and $\chi_2(p) = \beta_p/p^{(\kappa-1)/2}$ be its Satake parameters for $p \nmid N$ as in Section 4. For $p \nmid N$ define

$$L_p(s, \mathrm{Sym}^2 \phi_f) := \left(1 - \frac{\chi_1(p)^2}{p^s}\right)\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{\chi_2(p)^2}{p^s}\right)$$

and note that $L_p(s, \mathrm{Sym}^2 \phi_f) = L_p(s+\kappa-1, \mathrm{Sym}^2 f)$. Thus the Euler product

$$L^N(s, \mathrm{Sym}^2 \phi_f) := \prod_{p\nmid N}\frac{1}{L_p(s, \mathrm{Sym}^2 \phi_f)}$$

converges absolutely for $\mathrm{Re}\, s > 1$ and inherits all the corresponding properties (in particular the meromorphic continuation and the lack of a pole at $s = 1$) from $L^N(s, \mathrm{Sym}^2 f)$. As before we will continue to denote this extended function by $L^N(s, \mathrm{Sym}^2 \phi_f)$.

Let π be the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with ϕ_f. It is known [6, Theorem 9.3] that there exists an automorphic representation σ of $\mathrm{GL}_3(\mathbf{A})$ such that the (partial) standard L-function $L^N(s,\sigma)$ coincides with $L^N(s, \mathrm{Sym}^2 \pi) := L^N(s, \mathrm{Sym}^2 \phi_f)$. Also $L^N(s,\sigma)$ does not vanish on the line $\mathrm{Re}\, s = 1$ by a result of Jacquet and Shalika (see [8, Theorem 1]; see also [9]). Finally note that by Lemma 4.2, we have $|\chi_1(p)| = |\chi_2(p)| = 1$ if $p \nmid N$. Thus we are in a position to apply Theorem A.1 in the appendix with $K = \mathbf{Q}$ and $d = 3$ to $L^N(s, \mathrm{Sym}^2 \phi_f)$ and the theorem follows. □

By Theorem 5.2 and (5.2) we have

$$\frac{L^N(\kappa, \mathrm{Sym}^2 f)}{\prod_{p\nmid N}\langle f, f\rangle_N^{(p)}} = \zeta^N(2),$$

where the superscript means that we omit the Euler factors at primes dividing N, and the product $\prod_{p\nmid N}$ (here and below) is ordered according to increasing p. Using [7, Theorem 5.1] we have

$$L^N(\kappa, \mathrm{Sym}^2 f) = \prod_{p|N}\left(1 - \frac{\lambda_f(p)^2}{p^\kappa}\right) \times \frac{2^{2\kappa}\pi^{\kappa+1}}{(\kappa-1)!\delta(N)N\phi(N)}\langle f, f\rangle_N,$$

where $\delta(N) = 2$ or 1 according as $N \leq 2$ or not. Using this we obtain the following corollary that can be viewed as a factorization of the 'global' period $\langle f, f\rangle_N$ in terms of the 'local' periods $\langle f, f\rangle_N^{(p)}$.

Corollary 5.3. *We have*

$$\langle f, f\rangle_N = \frac{(\kappa-1)!\delta(N)N\phi(N)\zeta^N(2)}{2^{2\kappa}\pi^{\kappa+1}}\prod_{p|N}\frac{1}{1-\lambda_f(p)^2/p^\kappa}\prod_{p\nmid N}\langle f, f\rangle_N^{(p)}.$$

Appendix A. Convergence of Euler products on $\mathrm{Re}(s) = 1$ by Keith Conrad[3]

Let K be a number field. A degree d Euler product over K is a product

$$L(s) = \prod_{\mathfrak{p}} \frac{1}{(1-\alpha_{\mathfrak{p},1}\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\mathrm{N}\mathfrak{p}^{-s})},$$

where $|\alpha_{\mathfrak{p},j}| \leq 1$ for all nonzero prime ideals $\mathfrak{p}$ in the integers of K and $1 \leq j \leq d$. On the half-plane $\mathrm{Re}(s) > 1$ this converges absolutely and is nonvanishing. Combining factors at prime ideals lying over a common prime number, $L(s)$ is also an Euler product over $\mathbf{Q}$ of degree $d[K : \mathbf{Q}]$.

We want to prove a general theorem about the representability of $L(s)$ by its Euler product on the line $\mathrm{Re}(s) = 1$. If $L(s)$ is the L-function of a nontrivial Dirichlet character, this is in [4, pp. 57–58], [10, § 109], and [12, p. 124] if $s = 1$ and [10, § 121] if $\mathrm{Re}(s) = 1$.

Theorem A.1. *If $L(s)$ is a degree d Euler product over K and it admits an analytic continuation to $\mathrm{Re}(s) = 1$ where it is nonvanishing, then $L(s)$ is equal to its Euler product on $\mathrm{Re}(s) = 1$ when factors are ordered according to prime ideals of increasing norm: if $\mathrm{Re}(s) = 1$ then*

$$L(s) = \lim_{x\to\infty}\prod_{\mathrm{N}\mathfrak{p}\leq x}\frac{1}{(1-\alpha_{\mathfrak{p},1}\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\mathrm{N}\mathfrak{p}^{-s})}.$$

The proof is based on the following lemma about representability of a Dirichlet series on the line $\mathrm{Re}(s) = 1$.

Lemma A.2. *Suppose* $g(s) = \sum_{n\geq 1} b_n n^{-s}$ *has bounded Dirichlet coefficients. If* $g(s)$ *admits an analytic continuation from* $\mathrm{Re}(s) > 1$ *to* $\mathrm{Re}(s) \geq 1$, *then* $g(s)$ *is still represented by its Dirichlet series on the line* $\mathrm{Re}(s) = 1$.

Proof. See [13]. □

Here is the proof of Theorem A.1.

Proof. We will apply Lemma A.2 to a logarithm of $L(s)$, namely the absolutely convergent Dirichlet series

$$(\log L)(s) := \sum_{\mathfrak{p}} \sum_{k\geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}},$$

where $\mathrm{Re}(s) > 1$. The coefficient of $1/\mathrm{N}\mathfrak{p}^{ks}$ has absolute value at most $d/k \leq d$, so if we collect terms and write $(\log L)(s)$ as a Dirichlet series indexed by the positive integers, say $\sum_{n\geq 1} c_n/n^s$, then $c_n = 0$ if n is not a prime power and $|c_n| \leq d[K : \mathbf{Q}]$ if n is a prime power. Therefore the coefficients of $(\log L)(s)$ as a Dirichlet series over $\mathbf{Z}^+$ are bounded.

Since $L(s)$ is assumed to have an analytic continuation to a nonvanishing function on $\mathrm{Re}(s) \geq 1$, $(\log L)(s)$ has an analytic continuation to $\mathrm{Re}(s) \geq 1$, so Lemma A.2 implies that

$$(\log L)(s) = \sum_{\mathfrak{p}^k} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} \tag{A.1}$$

for $\mathrm{Re}(s) = 1$, where the terms in the series are collected in order of increasing values of $\mathrm{N}(\mathfrak{p}^k)$.

Although a rearrangement of terms in a conditionally convergent series can change its value, one particular rearrangement of the series in (A.1) doesn't change the sum:

$$\sum_{\mathfrak{p}^k} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\mathfrak{p}} \sum_{k\geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} \tag{A.2}$$

when $\mathrm{Re}(s) = 1$, where the sum on the left is in order of increasing values of $\mathrm{N}(\mathfrak{p}^k)$ and the outer sum on the right is in order of increasing values of $\mathrm{N}(\mathfrak{p})$. To prove (A.2), we rewrite it as

$$\sum_{\mathrm{N}(\mathfrak{p}^k)\leq x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\mathrm{N}(\mathfrak{p})\leq x} \sum_{k\geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} + o(1) \tag{A.3}$$

as $x \to \infty$, and we will prove (A.3) when $\mathrm{Re}(s) > 1/2$, not just $\mathrm{Re}(s) = 1$. For $\mathrm{Re}(s) = 1$ we can pass to the limit in (A.3) as $x \to \infty$ and conclude (A.2).

The sum on the right in (A.3) the sum on the left in (A.3) is equal to

$$\sum_{\mathrm{N}(\mathfrak{p})\leq x}\ \sum_{\substack{k\geq 2\\ \mathrm{N}(\mathfrak{p})^k>x}} \frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}},$$

which is equal to

$$\sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\ \sum_{k\geq 2} \frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}} + \sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\ \sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}} \frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}}. \tag{A.4}$$

The absolute value of the first sum in (A.4) is bounded above by

$$\begin{aligned}
\sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\ \sum_{k\geq 2}\left|\frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}}\right| &\leq \sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\ \sum_{k\geq 2}\frac{d}{k\mathrm{N}\mathfrak{p}^{k\sigma}} \quad \text{where } \sigma=\mathrm{Re}(s)\\
&< \frac{d}{2}\sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\ \sum_{k\geq 2}\frac{1}{\mathrm{N}\mathfrak{p}^{k\sigma}}\\
&= \frac{d}{2}\sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\frac{1}{\mathrm{N}\mathfrak{p}^{\sigma}(\mathrm{N}\mathfrak{p}^{\sigma}-1)}\\
&< \frac{d}{2}\sum_{\sqrt{x}<\mathrm{N}(\mathfrak{p})\leq x}\frac{4}{\mathrm{N}\mathfrak{p}^{2\sigma}} \quad \text{since } \mathrm{N}(\mathfrak{p})^{\sigma}>\sqrt{2}>\frac{4}{3},
\end{aligned}$$

which tends to 0 as $x\to\infty$ since $\sum_{\mathfrak{p}} 1/\mathrm{N}\mathfrak{p}^{2\sigma}$ converges. The absolute value of the second sum in (A.4) is bounded above by

$$\begin{aligned}
\sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\ \sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}}\frac{d}{k\mathrm{N}\mathfrak{p}^{k\sigma}} &< \sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\ \sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}}\frac{d}{\log_{\mathrm{N}\mathfrak{p}}(x)\mathrm{N}\mathfrak{p}^{k\sigma}}\\
&= \sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\frac{d\log \mathrm{N}(\mathfrak{p})}{\log x}\ \sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}}\frac{1}{\mathrm{N}\mathfrak{p}^{k\sigma}}.
\end{aligned}$$

Letting n be the least integer above $\log_{\mathrm{N}\mathfrak{p}}(x)$,

$$\sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}}\frac{1}{\mathrm{N}\mathfrak{p}^{k\sigma}} = \frac{1/\mathrm{N}(\mathfrak{p})^{n\sigma}}{1-1/\mathrm{N}(\mathfrak{p})^{\sigma}} < \frac{1/x^{\sigma}}{1/4} = \frac{4}{x^{\sigma}},$$

so

$$\sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\ \sum_{\substack{k\geq 3\\ \mathrm{N}(\mathfrak{p})^k>x}}\frac{d}{k\mathrm{N}\mathfrak{p}^{k\sigma}} < \sum_{\mathrm{N}(\mathfrak{p})\leq \sqrt{x}}\frac{4d\log \mathrm{N}(\mathfrak{p})}{x^{\sigma}\log x} = O\left(\frac{\sqrt{x}}{x^{\sigma}\log x}\right),$$

which tends to 0 as $x\to\infty$ since $\sigma>1/2$.

Now that we established (A.2), take the exponential of the right side: if $\mathrm{Re}(s) = 1$, then

$$\prod_{\mathfrak{p}} \exp\left(\sum_{k\geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\mathrm{N}\mathfrak{p}^{ks}}\right) = \prod_{\mathfrak{p}} \frac{1}{(1-\alpha_{\mathfrak{p},1}\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\mathrm{N}\mathfrak{p}^{-s})},$$

where the products run over $\mathfrak{p}$ in order of increasing norms and the last calculation is justified since $|\alpha_{\mathfrak{p},j}/\mathrm{N}(\mathfrak{p})^s| \leq 1/\mathrm{N}(\mathfrak{p}) < 1$. Since $L(s) = e^{(\log L)(s)}$ for $\mathrm{Re}(s) \geq 1$, by (A.1) and (A.2) we have

$$L(s) = \prod_{\mathfrak{p}} \frac{1}{(1-\alpha_{\mathfrak{p},1}\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\mathrm{N}\mathfrak{p}^{-s})}$$

for $\mathrm{Re}(s) = 1$, where the product is in order of increasing values of $\mathrm{N}\mathfrak{p}$. □

Example A.3. Let $L(s)$ be the L-function of the elliptic curve $y^2 = x^3 - x$ over $\mathbf{Q}$. For $\mathrm{Re}(s) > 3/2$ it has an Euler product over the odd primes of the form

$$L(s) = \prod_{p\neq 2} \frac{1}{1-a_p p^{-s} + p\cdot p^{-2s}} = \prod_{p\neq 2} \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}, \tag{A.5}$$

where $|\alpha_p| = \sqrt{p}$ and $|\beta_p| = \sqrt{p}$ for $p \neq 2$. Since $y^2 = x^3 - x$ has CM by $\mathbf{Z}[i]$, $L(s)$ is also the L-function of a Hecke character χ on $\mathbf{Q}(i)$ such that $|\chi((\alpha))| = |\alpha| = |\mathrm{N}(\alpha)|^{1/2}$ for all nonzero α in $\mathbf{Z}[i]$ with odd norm. Therefore $L(s)$ also has an Euler product over the nonzero prime ideals of $\mathbf{Z}[i]$ of odd norm: for $\mathrm{Re}(s) > 3/2$,

$$L(s) = \prod_{(\pi)\neq(1+i)} \frac{1}{1-\chi(\pi)/\mathrm{N}(\pi)^s}. \tag{A.6}$$

The function $L(s)$ is entire and is nonvanishing on the line $\mathrm{Re}(s) = 3/2$, so $L(s + 1/2)$ fits the conditions of Theorem A.1 using $K = \mathbf{Q}$ and $d = 2$ for (A.5), and $K = \mathbf{Q}(i)$ and $d = 1$ for (A.6). Therefore (A.5) and (A.6) are both true on the line $\mathrm{Re}(s) = 3/2$. For instance, $L(3/2) \approx .826348$, the partial Euler product for (A.5) at $s = 3/2$ over prime numbers up to 100,000 is $\approx .826290$, and the partial Euler product for (A.6) at $s = 3/2$ over nonzero prime ideals in $\mathbf{Z}[i]$ with norm up to 100,000 is $\approx .826480$.

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