AN $R = T$ THEOREM FOR IMAGINARY QUADRATIC FIELDS

TOBIAS BERGER[1] AND KRZYSZTOF KLOSIN[2]

ABSTRACT. We prove the modularity of certain residually reducible p-adic Galois representations of an imaginary quadratic field assuming the uniqueness of the residual representation. We obtain an $R = T$ theorem using a new commutative algebra criterion that might be of independent interest. To apply the criterion one needs to show that the quotient of R by its ideal of reducibility is cyclic Artinian of order no greater than the order of the congruence module $\mathbf{T}/J$, where J is an Eisenstein ideal in the local Hecke algebra $\mathbf{T}$. The inequality is proven by applying the Main conjecture of Iwasawa Theory for Hecke characters and using a result of [Ber09]. This strengthens our previous result [BK09] to include the cases of an arbitrary p-adic valuation of the L-value, in particular, cases where R is not a discrete valuation ring. As a consequence we show that the Eisenstein ideal is principal and that $\mathbf{T}$ is a complete intersection.

1. INTRODUCTION

Let K be a number field and $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ a continuous irreducible representation. It has been a subject of much interest and effort lately to determine which such Galois representations are *modular*, i.e., which such ρ's have their L-function equal to an L-function of an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$. Since the ground-breaking work of Wiles there has been a lot of progress in answering this question [Wil95, TW95, BCDT01, SW97, Fuj99, SW99, SW01, Tay02, Kis07].

This article is a continuation of our efforts to prove the modularity of continuous representations of $\mathrm{Gal}(\overline{K}/K)$ for K an imaginary quadratic field. This case is substantially different from other situations for which modularity has been proven so far since the associated symmetric space is a hyperbolic 3-manifold, so in particular tools from algebraic geometry are not available.

In a previous paper [BK09] we proved modularity of irreducible Galois representations of an imaginary quadratic K (in the minimal case) when the residual representation ρ_0 (i.e., the composite of ρ with the map $\overline{\mathbf{Z}}_p \twoheadrightarrow \overline{\mathbf{F}}_p$ after choosing an integral lattice in the space of ρ) is reducible under an additional condition that a certain L-value associated to the determinant of ρ has small p-adic valuation. As we showed in [BK09] this condition implies that the universal deformation ring of ρ_0 is a discrete valuation ring. This article removes this condition.

We achieve this by developing a new commutative algebra criterion to prove "$R = T$" theorems, applicable also in other situations (see Remark 1.2 and the discussion

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at the end of Section 6.4). For this fix a reducible, non-semisimple residual two-dimensional Galois representation and study its minimal ordinary deformations with corresponding universal deformation ring R and universal deformation ρ_R. Following [BC06, Cal06] we define the ideal of reducibility $I_{\rm re}$ to be the smallest ideal I in R such that $\mathrm{tr}(\rho_R) \mod I$ is the sum of two characters.

Assuming that there is only one (up to isomorphism) non-split extension of the two characters (in either order) which are the constituents of the semisimplification of the residual representation [BC06, Cal06] prove that $I_{\rm re}$ is principal. In this case our new commutative algebra criterion allows us to prove that R is isomorphic to $\mathbf{T}$, a Hecke algebra, provided that $R/I_{\rm re}$ is a cyclic Artinian module and its order is no greater than the order of $\mathbf{T}/J$ for J an Eisenstein ideal of $\mathbf{T}$ (see Proposition 6.9 and Theorem 6.10). Lower bounds for the order of $\mathbf{T}/J$ have been proven in many cases starting with Mazur and Wiles [MW84]. For our case of imaginary quadratic fields we refer to [Ber09] which bounds $\mathrm{val}_p(\#\mathbf{T}/J)$ from below by the p-valuation (say n) of a Hecke L-value. Since we can rule out non-trivial reducible infinitesimal deformations by the uniqueness of our residual representation the cyclicity of $R/I_{\rm re}$ and an upper bound for its p-order (which again equals n) can be obtained using the Main Conjecture for Hecke characters of imaginary quadratic fields, proven by Rubin.

There are several consequences of the $R = T$ theorem for both the Hecke algebra and the universal deformation ring. On the one hand the principality of the reducibility ideal implies that the Eisenstein ideal is also principal. On the other hand using our knowledge of the structure of the universal deformation ring we conclude that the Hecke algebra is a complete intersection (Corollary 6.14). Finally, the lower bound on the size of the Hecke congruence module translates under the isomorphism $R \cong \mathbf{T}$ into a statement about cyclicity of a certain Galois group and the existence of certain reducible deformations (Corollary 6.16).

To give a more precise account, let c be the non-trivial automorphism of F, and let $p > 3$ be a prime split in the extension $F/\mathbf{Q}$. Fix embeddings $F \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. Let F_Σ be the maximal extension of F unramified outside a finite set of places Σ and put $G_\Sigma = \mathrm{Gal}(F_\Sigma/F)$. Suppose $\mathbf{F}$ is a finite field of characteristic p and that $\chi_0 : G_\Sigma \to \mathbf{F}^\times$ is anticyclotomic (in general a weaker assumption of Σ-admissibility is enough - see Section 3) character ramified at the places dividing p. Suppose also that $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ is a continuous representation of the form

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi_0 \end{pmatrix} \tag{1.1}$$

and having scalar centralizer. In [BK09] we proved that under certain conditions on χ_0 and Σ the residual representation ρ_0 is unique up to isomorphism. We recall the relevant details in Section 3.

Following Mazur [Maz97] we study ordinary deformations of ρ_0. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbf{Q}_p$. An $\mathcal{O}$-deformation of ρ_0 is a local complete Noetherian $\mathcal{O}$-algebra A with residue field $\mathbf{F}$ and maximal ideal $\mathfrak{m}_A$ together with a strict equivalence class of continuous representations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ satisfying $\rho_0 = \rho \mod \mathfrak{m}_A$. An *ordinary* deformation is a deformation that satisfies

$$\rho|_{D_\mathfrak{q}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

for $\mathfrak{q} \mid p$, where $\chi_i|_{I_\mathfrak{q}} = \epsilon^{k_i}$ with integers $k_1 \geq k_2$ depending on $\mathfrak{q}$ and ϵ is the p-adic cyclotomic character. Here $D_\mathfrak{q}$ and $I_\mathfrak{q}$ denote the decomposition group and the inertia group of $\mathfrak{q} \mid p$.

A part of our approach rests on studying possible reducible deformations of ρ_0 and showing that there are indeed not "too many" of those. As we have already shown in [BK09] the uniqueness of ρ_0 itself implies that there are no non-trivial reducible infinitesimal deformations of ρ_0. In the current paper we go further and carefully study reducible deformations of ρ_0 to other Artinian local rings.

To exhibit modular irreducible deformations we apply the cohomological congruences of [Ber09] and the Galois representations constructed by Taylor *et al.* using the strengthening of Taylor's result in [BH07]. We also make use of a result of Urban [Urb05] who proves that $\rho_\pi|_{D_\mathfrak{q}}$ is ordinary at $\mathfrak{q} \mid p$ if π is ordinary at $\mathfrak{q}$. This together with the non-existence of a non-trivial reducible infinitesimal deformation implies the existence of an $\mathcal{O}$-algebra surjection

$$R \twoheadrightarrow \mathbf{T}, \tag{1.2}$$

where R is the universal Σ-minimal deformation ring (cf. Definition 4.8) and $\mathbf{T}$ is a Hecke algebra acting on cuspidal automorphic forms of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight 2 and fixed level.

Remark 1.1. As we remarked already in [BK09] the approach of [SW97] (where an analogous problem is studied for representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$) breaks down in the imaginary quadratic case because of the non-existence of an ordinary reducible characteristic 0 deformation (cf. [BK09], Corollary 5.22). In [BK09] we assumed that the p-valuation of a Hecke L-value was small such that by the Main Conjecture there were even no reducible deformations to any Artinian rings larger than the residue field, which implied that the ideal of reducibility I_{re} of the universal deformation ring R was maximal. This simplified the proof of "$R = T$", because it follows from the results of Bellaiche-Chenevier [BC06] and Calegari [Cal06] that if R surjects onto a characteristic 0 ring (in our case $\mathbf{T}$) and the ideal I_{re} is maximal, then R is a discrete valuation ring.

In this paper we exploit that in all cases there is, in fact, a surjection $R/I_{\mathrm{re}} \twoheadrightarrow \mathbf{T}/J$ by the definitions of the Eisenstein ideal J and the ideal of reducibility I_{re}. By finding bounds on the orders of both sides we show that this map must be an isomorphism. Indeed, if we write n for the ϖ-valuation[1] of a normalised Hecke L-value (for a Hecke character that gives rise to χ_0 in (1.1)), then the order of the congruence module $\mathbf{T}/J$ is bounded from below by $\#\mathcal{O}/\varpi^n$ by the full result of [Ber09]. On the deformation side we first bound the dimension of R in Section 5.1 by using the filtration of $\mathrm{ad}\rho_0$ by 1-dimensional pieces, similar to a calculation in [SW97]. Strengthening an argument of [BK09] and using a result of Urban we show in Section 5.2 that the Main Conjecture of Iwasawa Theory implies that there are no reducible ordinary deformations to $\mathcal{O}/\varpi^m$ for $m > n$. Together with the non-existence of non-trivial reducible infinitesimal deformations this implies that $R/I_{\mathrm{re}} = \mathcal{O}/\varpi^m$ for $m \leq n$ (see Proposition 6.5). We therefore have that the surjection $R/I_{\mathrm{re}} \twoheadrightarrow \mathbf{T}/J$ is an isomorphism, from which we deduce (using our commutative algebra criterion - Proposition 6.9) that $R \cong \mathbf{T}$ in Theorem 6.10. using the principality of I_{re}.

[1] Here ϖ denotes a uniformizer of $\mathcal{O}$.

Remark 1.2. As suggested to us by Frank Calegari our commutative algebra criterion allows one to replace the condition $p \| B_{2,\omega^{k-2}}$ from [Cal06] Theorem 2.3 by $p \mid B_{2,\omega^{k-2}}$, which proves an $R = T$ result for certain residually reducible representations of $G_{\mathbf{Q}}$. More precisely, if $\mathbf{T}$ denotes the (classical) cuspidal Hecke algebra of weight 2 and level $\Gamma_1(p)$ generated by T_ℓ for $\ell \neq p$, write J the maximal ideal of $\mathbf{T}$ containing $T_\ell - 1 - \ell^{k-1}$ for all $\ell \neq p$. Under the assumption that the χ^{1-k}-eigenspace (where $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ denotes the p-adic cyclotomic character) of the class group of $\mathbf{Q}(\zeta_p)$ is cyclic, Calegari considers deformations of the unique non-split representation

$$\overline{\rho} = \begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix} \mod p$$

with $\overline{\rho}$ unramified over $\mathbf{Q}(\zeta_p)$. He requires that the deformations ρ be unramified outside p, of determinant $\chi\omega^{k-2}$, where ω is the Teichmüller lift of χ mod p, and such that the restriction of ρ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$ arises as a generic fiber of a finite flat group scheme over the ring of integers of L. Here L stands for the completion at p of the maximal real subfield of $\mathbf{Q}(\zeta_p)$. If one denotes by R the corresponding universal deformation ring, the arguments of our Sections 5.2 and 6.1 can be adapted, using [Cal06] Lemmas 4.2 and 4.5 and the Main conjecture proven by Mazur and Wiles [MW84], to prove the cyclicity and an upper bound for R/I_{re} in terms of the p-valuation of the relevant Bernoulli number. The lower bound by the same number on the size of the quotient of $\mathbf{T}_J$ by the Eisenstein ideal generated by $T_\ell - 1 - \ell^{k-1}$ for all $\ell \neq p$ is stated, for example, as Proposition 5.1 in [SW97]. By the commutative algebra criterion (Proposition 6.9) we can therefore conclude $R = \mathbf{T}_J$.

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2. Notation and terminology

2.1. **Galois groups.** Let F be an imaginary quadratic extension of $\mathbf{Q}$ of discriminant $d_F \neq 3, 4$ and $p > 3$ a rational prime which splits in F. We fix once and for all a prime $\mathfrak{p}$ of F lying over p and write $\overline{\mathfrak{p}}$ for the prime of F such that $(p) = p\mathcal{O}_F = \mathfrak{p}\overline{\mathfrak{p}}$. Let Cl_F denote the class group of F. We assume that $p \nmid \# \mathrm{Cl}_F$ and that any prime $q \mid d_F$ satisfies $q \not\equiv \pm 1 \pmod{p}$.

For a field K write G_K for the Galois group $\mathrm{Gal}(\overline{K}/K)$. If K is a finite extension of $\mathbf{Q}_\ell$ for some rational prime ℓ, we write $\mathcal{O}_K$ (respectively ϖ_K, and $\mathbf{F}_K$) for the ring of integers of K (respectively for a uniformizer of K, and $\mathcal{O}_K/\varpi_K\mathcal{O}_K$).

If $K \supset F$ is a number field, $\mathcal{O}_K$ will denote its ring of integers. If $\mathfrak{q}$ is a place of K, we write $K_\mathfrak{q}$ for the completion of K with respect to the absolute value $|\cdot|_\mathfrak{q}$ determined by $\mathfrak{q}$ and set $\mathcal{O}_{K,\mathfrak{q}} = \mathcal{O}_{K_\mathfrak{q}}$ (if $\mathfrak{q}$ is archimedean, we set $\mathcal{O}_{K,\mathfrak{q}} = K_\mathfrak{q}$). We also write $\varpi_\mathfrak{q}$ for a uniformizer of $K_\mathfrak{q}$, $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_{K,\mathfrak{q}}$, and $k_\mathfrak{q}$ for its residue field.

Fix once and for all compatible embeddings $i_\mathfrak{q} : \overline{F} \hookrightarrow \overline{F}_\mathfrak{q}$ and $\overline{F}_\mathfrak{q} \hookrightarrow \mathbf{C}$, for every prime $\mathfrak{q}$ of F, so we will often regard elements of $\overline{F}_\mathfrak{q}$ as complex numbers without explicitly mentioning it. If w is a place of $K \subset \overline{F}$, it determines a place v of F, and we always regard K_w as a subfield of $\overline{F}_v$ as determined by the embedding i_v. This also allows us to identify G_{K_w} with the decomposition group $D_{\overline{v}} \subset G_K$ of a place $\overline{v}$ of $\overline{F}$. We will denote that decomposition group by $D_\mathfrak{q}$. Abusing notation somewhat we will denote the image of $D_{\mathfrak{q}}$ in any quotient of G_K also by $D_{\mathfrak{q}}$. We write $I_{\mathfrak{q}} \subset D_{\mathfrak{q}}$ for the inertia group.

Let Σ be a finite set of places of K. Then K_Σ will denote the maximal Galois extension of K unramified outside the primes in Σ. We also write G_Σ for G_{F_Σ}. Moreover, for a positive integer n, denote by μ_n the group of nth roots of unity.

2.2. **Hecke characters.** For a number field K, denote by $\mathbf{A}_K$ the ring of adeles of K and set $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$. By a *Hecke character* of K we mean a continuous homomorphism

$$\lambda : K^\times \setminus \mathbf{A}_K^\times \to \mathbf{C}^\times.$$

For a place $\mathfrak{q}$ of K write $\lambda^{(\mathfrak{q})}$ for the restriction of λ to $K_{\mathfrak{q}}$ and $\lambda^{(\infty)}$ for the restriction of λ to $\prod_{\mathfrak{q}|\infty} K_{\mathfrak{q}}$. The latter will be called the *infinity type* of λ. We also usually write $\lambda(\varpi_{\mathfrak{q}})$ to mean $\lambda^{(\mathfrak{q})}(\varpi_{\mathfrak{q}})$. Given λ there exists a unique ideal $\mathfrak{f}_\lambda$ of K largest with respect to the following property: $\lambda^{(\mathfrak{q})}(x) = 1$ for every finite place $\mathfrak{q}$ of K and $x \in \mathcal{O}_{K,\mathfrak{q}}^\times$ such that $x - 1 \in \mathfrak{f}_\lambda \mathcal{O}_{K,\mathfrak{q}}$. The ideal $\mathfrak{f}_\lambda$ is called the *conductor* of λ. If $K = F$, there is only one archimedean place, which we will simply denote by ∞. For a Hecke character λ of F, one has $\lambda^{(\infty)}(z) = z^m \overline{z}^n$ with $m, n \in \mathbf{R}$. If $m, n \in \mathbf{Z}$, we say that λ is of type (A_0). We always assume that our Hecke characters are of type (A_0). Write $L(s,\lambda)$ for the Hecke L-function of λ.

Let λ be a Hecke character of infinity type $z^a \left(\frac{z}{\overline{z}}\right)^b$ with conductor prime to p. Assume $a, b \in \mathbf{Z}$ and $a > 0$ and $b \geq 0$. Put

$$L^{\mathrm{alg}}(0,\lambda) := \Omega^{-a-2b} \left(\frac{2\pi}{\sqrt{d_F}}\right)^b \Gamma(a+b) \cdot L(0,\lambda),$$

where Ω is a complex period. In most cases, this normalization is integral, i.e., lies in the integer ring of a finite extension of $F_{\mathfrak{p}}$. See [Ber08] Theorem 3 for the exact statement. Put

$$L^{\mathrm{int}}(0,\lambda) = \begin{cases} L^{\mathrm{alg}}(0,\lambda) & \text{if } \mathrm{val}_p(L^{\mathrm{alg}}(0,\lambda)) \geq 0 \\ 1 & \text{otherwise.} \end{cases}$$

For $z \in \mathbf{C}$ we write $\overline{z}$ for the complex conjugate of z. The action of complex conjugation extends to an automorphism of $\mathbf{A}_F^\times$ and we will write $\overline{x}$ for the image of $x \in \mathbf{A}_F^\times$ under that automorphism.

For a Hecke character λ of F, we denote by λ^c the Hecke character of F defined by $\lambda^c(x) = \lambda(\overline{x})$.

2.3. **Galois representations.** For a field K and a topological ring R, by a *Galois representation* we mean a continuous homomorphism $\rho : G_K \to \mathrm{GL}_n(R)$. If $n = 1$ we usually refer to ρ as a *Galois character*. We write $K(\rho)$ for the fixed field of $\ker \rho$ and call it the *splitting field of* ρ. If ρ is a Galois character and M is an R-module, we denote by $M(\rho)$ the $R[G_K]$-module M with the G_K-action given by ρ. If K is a number field and $\mathfrak{q}$ is a finite prime of K with inertia group $I_{\mathfrak{q}}$ we say that ρ is unramified at $\mathfrak{q}$ if $\rho|_{I_{\mathfrak{q}}} = 1$. If Σ is a finite set of places of K such that ρ is unramified at all $v \notin \Sigma$, then ρ can be regarded as a representation of G_{K_Σ}.

Let E be a finite extension of $\mathbf{Q}_p$. Every Galois representation $\rho : G_K \to \mathrm{GL}_n(E)$ can be conjugated (by an element $M \in \mathrm{GL}_n(E)$) to a representation $\rho_M : G_K \to \mathrm{GL}_n(\mathcal{O}_E)$. We denote by $\overline{\rho}_M : G_K \to \mathrm{GL}_n(\mathbf{F}_E)$ its reduction modulo $\varpi_E \mathcal{O}_E$. It is sometimes called a *residual representation* of ρ. The isomorphism class of its semisimplification $\overline{\rho}_M^{\mathrm{ss}}$ is independent of the choice of M and we simply write $\overline{\rho}^{\mathrm{ss}}$.

Let $\epsilon : G_F \to \mathbf{Z}_p^\times$ denote the p-adic cyclotomic character. For any subgroup $G \subset G_F$ we will also write ϵ for $\epsilon|_G$. Our convention is that the Hodge-Tate weight of ϵ at $\mathfrak{p}$ is 1.

Let λ be a Hecke character of F of type (A_0) and $\Sigma_\lambda = \{\mathfrak{q} \mid p\mathfrak{f}_\lambda\}$. We define (following Weil) a $\mathfrak{p}$-adic Galois character

$$\lambda_{\mathfrak{p}} : G_{\Sigma_\lambda} \to \overline{F}_{\mathfrak{p}}^\times$$

associated to λ by the following rule: For a finite place $\mathfrak{q} \nmid p\mathfrak{f}_\lambda$ of F, put $\lambda_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{q}}) = i_{\mathfrak{p}}(i_\infty^{-1}(\lambda(\varpi_{\mathfrak{q}})))$ where $\mathrm{Frob}_{\mathfrak{q}}$ denotes the *arithmetic* Frobenius at $\mathfrak{q}$. It takes values in the integer ring of a finite extension of $F_{\mathfrak{p}}$.

Definition 2.1. For a topological ring R we call a Galois representation $\rho : G_\Sigma \to \mathrm{GL}_2(R)$ *ordinary* if

$$\rho|_{D_{\mathfrak{q}}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

for $\mathfrak{q} \mid p$, where $\chi_i|_{I_{\mathfrak{q}}} = \epsilon^{k_i}$ with integers $k_1 \geq k_2$ depending on $\mathfrak{q}$.

2.4. **Automorphic representations of $\mathbf{A}_F$ and their Galois representations.** Set $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$. For K_f an open compact subgroup of $G(\mathbf{A}_f)$, denote by $S_2(K_f)$ the space of cuspidal automorphic forms of $G(\mathbf{A})$ of weight 2, right-invariant under K_f (for more details see Section 3.1 of [Urb95]). For ψ a finite order Hecke character write $S_2(K_f, \psi)$ for the forms with central character ψ. This is isomorphic as a $G(\mathbf{A}_f)$-module to $\bigoplus \pi_f^{K_f}$ for automorphic representations π of a certain infinity type (see Theorem 2.2 below) with central character ψ. Here π_f denotes the restriction of π to $\mathrm{GL}_2(\mathbf{A}_f)$ and $\pi_f^{K_f}$ stands for the K_f-invariants.

For $g \in G(\mathbf{A}_f)$ we have the usual Hecke action of $[K_f g K_f]$ on $S_2(K_f)$ and $S_2(K_f, \psi)$. For primes $\mathfrak{q}$ such that the vth component of K_f is $\mathrm{GL}_2(\mathcal{O}_{F,v})$ we define $T_{\mathfrak{q}} = [K_f \begin{bmatrix} \varpi_{\mathfrak{q}} & \\ & 1 \end{bmatrix} K_f]$.

Combining the work of Taylor, Harris, and Soudry with results of Friedberg-Hoffstein and Laumon/Weissauer, one can show the following (see [BH07] for general case of cuspforms of weight k):

Theorem 2.2 ([BH07] Theorem 1.1)**.** *Given a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{bmatrix} t_1 & * \\ & t_2 \end{bmatrix} \mapsto \left(\frac{t_1}{|t_1|}\right) \left(\frac{|t_2|}{t_2}\right)$$

and cyclotomic central character ψ (i.e., $\psi^c = \psi$), let Σ_π denote the set consisting of the places of F lying above p, the primes where π or π^c is ramified, and the primes ramified in $F/\mathbf{Q}$.

Then there exists a finite extension E of $F_{\mathfrak{p}}$ and a Galois representation

$$\rho_\pi : G_{\Sigma_\pi} \to \mathrm{GL}_2(E)$$

such that if $\mathfrak{q} \notin \Sigma_\pi$, then ρ_π is unramified at $\mathfrak{q}$ and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})$ is $x^2 - a_{\mathfrak{q}}(\pi)x + \psi(\varpi_{\mathfrak{q}})(\# k_{\mathfrak{q}})$, where $a_{\mathfrak{q}}(\pi)$ is the Hecke eigenvalue corresponding to $T_{\mathfrak{q}}$. Moreover, ρ_π is absolutely irreducible.

Urban studied in [Urb98] the case of ordinary automorphic representations π, and together with results in [Urb05] on the Galois representations attached to ordinary Siegel modular forms showed:

Theorem 2.3 (Corollary 2 of [Urb05])**.** *Let $\mathfrak{q}$ be a prime of F lying over p. If π is unramified at $\mathfrak{q}$ and ordinary at $\mathfrak{q}$, i.e., $|a_{\mathfrak{q}}(\pi)|_{\mathfrak{q}} = 1$, then the Galois representation ρ_π is ordinary at $\mathfrak{q}$. Moreover*

$$\rho_\pi|_{D_{\mathfrak{q}}} \cong \begin{bmatrix} \Psi_1 & * \\ & \Psi_2 \end{bmatrix},$$

where $\Psi_2|_{I_{\mathfrak{q}}} = 1$ and $\Psi_1|_{I_{\mathfrak{q}}} = \det \rho_\pi|_{I_{\mathfrak{q}}} = \epsilon$.

We conjecture that the assumption on the central character, inherent to the method of proof of Theorem 2.2, is not necessary (see also [CD06] Conjecture 3.2).

Conjecture 2.4. *Let π be a cuspidal automorphic representation as in Theorem 2.2 but without the assumption on the central character. Then the representation ρ_π exists as above and the conclusion of Theorem 2.3 remains valid for it.*

Definition 2.5. Let E be a finite extension of $F_{\mathfrak{p}}$ and $\rho : G_\Sigma \to \mathrm{GL}_2(E)$ a Galois representation for a finite set of places Σ. We say that ρ is *modular* if there exists a cuspidal automorphic representation π as in Theorem 2.2, such that $\rho \cong \rho_\pi$ (possibly after enlarging E).

From now on we fix a finite extension E of $F_{\mathfrak{p}}$ which we assume to be sufficiently large (see Section 4.2 and Remark 4.5, where this condition is made more precise). To simplify notation we put $\mathcal{O} := \mathcal{O}_E$, $\mathbf{F} = \mathbf{F}_E$ and $\varpi = \varpi_E$.

3. Uniqueness of a residual Galois representation

Let $p = \mathfrak{p}\overline{\mathfrak{p}}$ as before be a split prime. Let Σ be a finite set of finite primes of F containing $\mathfrak{p}$. In this paper we will be studying deformations of a certain class of residual Galois representations which we now describe.

From now on let

$$\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$$

be a Galois representation satisfying both of the following conditions:

(Red): $\rho_0 = \begin{bmatrix} 1 & * \\ & \chi_0 \end{bmatrix}$ for a Galois character $\chi_0 : G_\Sigma \to \mathbf{F}^\times$;

(Sc): ρ_0 has scalar centralizer.

Definition 3.1. We will say that a Galois character $\chi_0 : G_\Sigma \to \mathbf{F}^\times$ is Σ-*admissible* if

1. χ_0 is ramified at $\mathfrak{p}$ and at $\overline{\mathfrak{p}}$;
2. if $\mathfrak{q} \in \Sigma$, then either χ_0 is ramified at $\mathfrak{q}$ or $\chi_0(\mathrm{Frob}_{\mathfrak{q}}) \neq (\#k_v)^{\pm 1}$ (as elements of $\mathbf{F}$);
3. $\dim_{\mathbf{F}}(H^1(G_\Sigma, \mathbf{F}(\chi_0))) = \dim_{\mathbf{F}}(H^1(G_\Sigma, \mathbf{F}(\chi_0^{-1}))) = 1$.

Remark 3.2. It follows from the global Euler characteristic formula ([Mil06] Theorem 5.1), that the cohomology spaces in condition (3) of Definition 3.1 are at least one-dimensional.

An important consequence of Σ-admissibility of χ_0 is the uniqueness of ρ_0 up to isomorphism. We record this fact as a proposition.

Proposition 3.3. *Suppose $\rho' : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ is a Galois representation satisfying condition* (Red) *and* (Sc) *for a Σ-admissible character χ_0. Then $\rho' \cong \rho_0$.*

Proof. As ρ_0 and ρ' correspond to non-zero elements in $\mathrm{Ext}_{\overline{\mathbf{F}}_p[G_\Sigma]}(\mathbf{1}, \chi_0) \cong H^1(G_\Sigma, \overline{\mathbf{F}}_p(\chi_0^{-1}))$, the proposition follows from condition (3) in Definition 3.1. □

We will be interested in studying deformations of a representation ρ_0 satisfying conditions (Red) and (Sc) for a Σ-admissible character χ_0.

Let us shortly explain the significance of these assumptions to the deformation problem we study in the coming sections. Condition (Sc) is necessary to ensure that the deformation functor studied in the following sections is representable. The assumption that χ_0 is Σ-admissible is crucial for our method. On the one hand Proposition 3.3 guarantees that modular Galois representations which we construct (and which are defined only up to semisimplification) are deformations of ρ_0. On the other hand, Σ-admissibility of χ_0 is essential for ensuring that the ideal of reducibility of the universal deformation is principal, which is crucial for our commutative algebra criterion (Proposition 6.9) that we use to prove modularity of Σ-minimal deformations of ρ_0 (Theorem 6.10).

We will now formulate some conditions (see Theorem 3.5 below) that ensure Σ-admissibility of a Galois character (for details see [BK09], Section 3). At the same time we want to emphasize that there is no reason why every Σ-admissible character should satisfy all of these conditions, however, it is in the context of Theorem 3.5 that we were able to construct examples of Σ-admissible characters (see Remark 3.7 below). Also, it is only under condition (3) of Theorem 3.5 that one can relate (by twisting, see [Ber09] Lemma 8) to cusp forms with cyclotomic character for which one knows how to attach Galois representations to automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ by Theorem 2.2. Conjecturally, however, this condition should be unnecessary (see Conjecture 2.4).

Definition 3.4. Let R be a commutative ring, $J \subset R$ an ideal, M a free R-module and N a submodule of M. We will say that N is *saturated with respect to J* if

$$N = \{m \in M \mid mJ \subset N\}.$$

Let $\chi_0 : G_\Sigma \to \mathbf{F}^\times$ be a Galois character. Let S_p be the set of primes of $F(\chi_0)$ lying over p. Write M_{χ_0} for $\prod_{\mathfrak{q} \in S_p}(1 + \mathfrak{P}_v)$ and T_{χ_0} for its torsion submodule. The quotient M_{χ_0}/T_{χ_0} is a free $\mathbf{Z}_p$-module of finite rank. Let $\overline{\mathcal{E}}_{\chi_0}$ be the closure in M_{χ_0}/T_{χ_0} of the image of $\mathcal{E}_{\chi_0}$, the group of units of the ring of integers of $F(\chi_0)$ which are congruent to 1 modulo every prime in S_p.

In [BK09] the authors proved the following theorem (see Theorem 3.5 and the proof of Corollary 3.6 in [loc. cit.]).

Theorem 3.5. *Let $\chi_0 : G_\Sigma \to \mathbf{F}^\times$ be a Galois character. Assume that χ_0 satisfies all of the following conditions:*

1. *χ_0 is ramified at $\mathfrak{p}$;*
2. *if $\mathfrak{q} \in \Sigma$, then either χ_0 is ramified at $\mathfrak{q}$ or $\chi_0(\mathrm{Frob}_\mathfrak{q}) \neq (\#k_\mathfrak{q})^{\pm 1}$ (as elements of $\mathbf{F}$);*
3. *χ_0 is anticyclotomic, i.e., $\chi_0(c\sigma c) = \chi_0(\sigma)^{-1}$ for every $\sigma \in G_\Sigma$ and c the generator of $\mathrm{Gal}(F/\mathbf{Q})$;*
4. *the $\mathbf{Z}_p$-submodule $\overline{\mathcal{E}}_{\chi_0} \subset M_{\chi_0}/T_{\chi_0}$ is saturated with respect to the ideal $p\mathbf{Z}_p$,*
5. *The χ_0^{-1}-eigenspace of the p-part of $\mathrm{Cl}_{F(\chi_0)}$ is trivial.*

Then χ_0 *is* Σ*-admissible.*

Remark 3.6. Note that conditions (1) and (3) of Theorem 3.5 imply that χ_0 is also ramified at $\overline{\mathfrak{p}}$. Observe that χ_0 satisfies conditions (1)-(5) of Theorem 3.5 if and only if so does χ_0^{-1}. Indeed, conditions (1)-(4) in Theorem 3.5 are invariant under taking the inverse. Moreover, since χ_0 is anticyclotomic, the extension $F(\chi_0)/\mathbf{Q}$ is Galois, hence the χ_0^{-1}-eigenspace of the p-part of $\mathrm{Cl}_{F(\chi_0)}$ vanishes if and only if the χ_0-eigenspace does.

Remark 3.7. Condition (4) in Theorem 3.5 implies that every $\epsilon \in \overline{\mathcal{E}}_{\chi_0}$ which is a pth power of an element of M_{χ_0}/T_{χ_0} must be a pth power of an element of $\overline{\mathcal{E}}_{\chi_0}$. In particular the map

$$\overline{\mathcal{E}}_{\chi_0} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \to (M_{\chi_0}/T_{\chi_0}) \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$$

is injective. One way to check condition (4) in practice is to compute the p-part C of the class group of $F(\chi_0)(\mu_p)$ as a $\mathrm{Gal}(F(\chi_0)(\mu_p)/F(\chi_0))$-module. In particular, if ω denotes the character of $\mathrm{Gal}(F(\chi_0)(\mu_p)/F(\chi_0))$ giving the action on μ_p, then Kummer theory implies that Condition (4) is satisfied if the ω-part of C is trivial. In [BK09], Section 6, the authors exhibited an example of a Galois representation whose mod ϖ-reduction satisfied conditions (Red) and (Sc) for a Galois character satisfying conditions (1)-(5) of Theorem 3.5.

4. Deformations of ρ_0

In this section we will fix a residual representation ρ_0 arising as the reduction of modular Galois representations $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O})$ and define a deformation problem.

4.1. **Eisenstein congruences.** Let ϕ_1, ϕ_2 be two Hecke characters with infinity types $\phi_1^{(\infty)}(z) = z$ and $\phi_2^{(\infty)}(z) = z^{-1}$. Put $\gamma = \phi_1\phi_2$.

Denote by $\mathfrak{S}$ the finite set of places where both ϕ_i are ramified, but ϕ is unramified. Write $\mathfrak{M}_i$ for the conductor of ϕ_i. For an ideal $\mathfrak{N}$ in $\mathcal{O}_F$ and a finite place $\mathfrak{q}$ of F put $\mathfrak{N}_\mathfrak{q} = \mathfrak{N}\mathcal{O}_{F,\mathfrak{q}}$. We define

$$K^1(\mathfrak{N}_\mathfrak{q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid a-1, c \equiv 0 \mod \mathfrak{N}_\mathfrak{q} \right\},$$

and

$$U^1(\mathfrak{N}_\mathfrak{q}) = \{k \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid \det(k) \equiv 1 \mod \mathfrak{N}_\mathfrak{q}\}.$$

Now put

$$K_f := \prod_{\mathfrak{q}\in\mathfrak{S}} U^1(\mathfrak{M}_{1,\mathfrak{q}}) \prod_{\mathfrak{q}\notin\mathfrak{S}} K^1((\mathfrak{M}_1\mathfrak{M}_2)_\mathfrak{q}) \subset G(\mathbf{A}_f). \tag{4.1}$$

From now on, let Σ be a finite set of places of F containing

$$S_\phi := \{\mathfrak{q} \mid \mathfrak{M}_1\mathfrak{M}_1^c\mathfrak{M}_2\mathfrak{M}_2^c\} \cup \{\mathfrak{q} \mid pd_F\}.$$

We denote by $\mathbf{T}(\Sigma)$ the $\mathcal{O}$-subalgebra of $\mathrm{End}_\mathcal{O}(S_2(K_f,\gamma))$ generated by the Hecke operators $T_\mathfrak{q}$ for all places $\mathfrak{q} \notin \Sigma$. Following [Tay88] (p. 107) we define idempotents $e_\mathfrak{p}$ and $e_{\overline{\mathfrak{p}}}$, commuting with each other and with $\mathbf{T}(\Sigma)$ acting on $S_2(K_f,\gamma)$. They are characterized by the property that any element $h \in X := e_\mathfrak{p}e_{\overline{\mathfrak{p}}}\, S_2(K_f,\gamma)$ which is an eigenvector for $T_\mathfrak{p}$ and $T_{\overline{\mathfrak{p}}}$ satisfies $|a_\mathfrak{p}(h)|_p = |a_{\overline{\mathfrak{p}}}(h)|_p = 1$, where $a_\mathfrak{p}(h)$ (resp. $a_{\overline{\mathfrak{p}}}(h)$) is the $T_\mathfrak{p}$-eigenvalue (resp. $T_{\overline{\mathfrak{p}}}$-eigenvalue) corresponding to h. Let

$\mathbf{T}^{\mathrm{ord}}(\Sigma)$ denote the quotient algebra of $\mathbf{T}(\Sigma)$ obtained by restricting the Hecke operators to X.

Let $J(\Sigma) \subset \mathbf{T}(\Sigma)$ be the ideal generated by

$$\{T_{\mathfrak{q}} - \phi_1(\varpi_{\mathfrak{q}}) \cdot \#k_{\mathfrak{q}} - \phi_2(\varpi_{\mathfrak{q}}) \mid \mathfrak{q} \not\in \Sigma\}.$$

Definition 4.1. Denote by $\mathfrak{m}(\Sigma)$ a maximal ideal of $\mathbf{T}^{\mathrm{ord}}(\Sigma)$ containing the image of $J(\Sigma)$. We set $\mathbf{T}_\Sigma := \mathbf{T}^{\mathrm{ord}}(\Sigma)_{\mathfrak{m}(\Sigma)}$. Moreover, set $J_\Sigma := J(\Sigma)\mathbf{T}_\Sigma$. We refer to J_Σ as the *Eisenstein ideal of* $\mathbf{T}_\Sigma$.

Theorem 4.2 ([Ber05], Theorem 6.3, [Ber09] Theorem 14)**.** *Let ϕ be an unramified Hecke character of infinity type $\phi^{(\infty)}(z) = z^2$. There exist Hecke characters ϕ_1, ϕ_2 with $\phi_1/\phi_2 = \phi$ such that their conductors are divisible only by ramified primes or inert primes not congruent to $\pm 1 \mod p$ and such that*

$$\#(\mathbf{T}_\Sigma/J_\Sigma) \geq \#(\mathcal{O}/(L^{\mathrm{int}}(0, \phi))).$$

Proof. [Ber09] Theorem 14 states this inequality for the Hecke algebra $\mathbf{T}(\Sigma)$. However, the Eisenstein cohomology class used in the proof of [Ber09] Theorem 14 is ordinary because by [Ber08] Lemma 9 its $T_{\mathfrak{p}}$-eigenvalue (resp. $T_{\overline{\mathfrak{p}}}$-eigenvalue) is the p-adic unit $p\phi_1(\mathfrak{p}) + \phi_2(\mathfrak{p})$ (resp. $p\phi_1(\overline{\mathfrak{p}}) + \phi_2(\overline{\mathfrak{p}})$). Therefore one can prove the statement for the ordinary cuspidal Hecke algebra. □

Remark 4.3. If ϕ is unramified then $\overline{\phi_{\mathfrak{p}}\epsilon}$ is anticyclotomic (see [Ber09] Lemma 1). The condition on the conductor of the auxiliary character ϕ_1 together with our assumption on the discriminant of F therefore ensure that for $\chi_0 = \overline{\phi_{\mathfrak{p}}\epsilon}$ conditions (1)-(3) of Theorem 3.5 are automatically satisfied for $\Sigma = S_\phi$.

The assumption on the ramification of ϕ can be relaxed. For example, Proposition 16 and Theorem 28 of [Ber08] and Proposition 9 and Lemma 11 of [Ber09] imply the following:

Theorem 4.4. *Let ϕ_1, ϕ_2 be as at the start of this section with $p \nmid \#(\mathcal{O}_F/\mathfrak{M}_1\mathfrak{M}_2)^\times$. Assume both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are coprime to (p) and the conductor of $\phi := \phi_1/\phi_2$ and divisible only by primes split in $F/\mathbf{Q}$. Suppose $\frac{L(0,\overline{\phi})}{L(0,\phi)} \in \mathcal{O}$. If the torsion part of $H^2_c(S_{K_f}, \mathbf{Z}_p)$ is trivial, where*

$$S_{K_f} = G(\mathbf{Q})\backslash G(\mathbf{A})/K_f U(2)\mathbf{C}^\times$$

then

$$\#(\mathbf{T}_\Sigma/J_\Sigma) \geq \#(\mathcal{O}/(L^{\mathrm{int}}(0, \phi_1/\phi_2))).$$

Remark 4.5. In fact, by replacing $\mathbf{Z}_p$ by the appropriate coefficient system, the result is true for characters ϕ_1, ϕ_2 of infinity type $z\overline{z}^{-m}$ and z^{-m-1}, respectively, for $m \geq 0$. For Theorems 4.2 and 4.4, the field E needs to contain the values of the finite parts of ϕ_1 and ϕ_2 as well as $L^{\mathrm{int}}(0, \phi_1/\phi_2)$.

We will from now on assume that we are either in the situation of Theorem 4.2 or 4.4 and fix the characters ϕ_1, ϕ_2 and $\phi = \phi_1/\phi_2$, with corresponding conditions on the set Σ and definitions of K_f, $\mathbf{T}_\Sigma$, and J_Σ. We also assume from now on that $\mathrm{val}_p(L^{\mathrm{int}}(0, \phi)) > 0$, however our main result (Theorem 6.13) remains vacuously true when $L^{\mathrm{int}}(0, \phi)$ is a p-adic unit (see Remark 5.10). Put $\chi_0 = \overline{\phi_{\mathfrak{p}}\epsilon}$ and assume that χ_0 is Σ-admissible. If we are in the situation of Theorem 4.4 then suppose also that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not divisible by any primes $\mathfrak{q}$ such that $\#k_{\mathfrak{q}} \equiv 1 \mod p$. (This last assumption is only used in the proof of Theorem 4.9.)

4.2. **Definition of ρ_0.** Write

$$S_2(K_f,\gamma)_{\mathfrak{m}(\Sigma)} = \bigoplus_{\pi\in\Pi_\Sigma} \pi_f^{K_f}$$

for a finite set Π_Σ of ordinary cuspidal automorphic representations with central character γ, such that $\pi_f^{K_f} \neq 0$. The set Π_Σ is non-empty by Theorem 4.2 under our assumption that $\mathrm{val}_p(L^{\mathrm{int}}(0,\phi)) > 0$.

Let $\pi \in \Pi_\Sigma$. Let $\rho_\pi : G_\Sigma \to \mathrm{GL}_2(E)$ be the Galois representation attached to π by Conjecture 2.4 (This is another point where we assume that E is large enough). If we assume that $\phi^c = \overline{\phi}$ then the condition on the central character in Theorem 2.2 can be satisfied (after possibly twisting with a finite character), see [Ber09] Lemma 8, so we do not need to refer to Conjecture 2.4 in this case.

The representation ρ_π is unramified at all $\mathfrak{q} \notin S_\phi$, and satisfies

$$\mathrm{tr}\,\rho_\pi(\mathrm{Frob}_\mathfrak{q}) = a_\mathfrak{q}(\pi)$$

and

$$\det \rho_\pi(\mathrm{Frob}_\mathfrak{q}) = \gamma(\varpi_\mathfrak{q}) \cdot \#(k_\mathfrak{q}).$$

By definition, $\mathbf{T}_\Sigma$ injects into $\prod_{\pi\in\Pi_\Sigma} \mathrm{End}_\mathcal{O}(\pi^{K_f})$. Since $T_\mathfrak{q}$ acts on π by multiplication by $a_\mathfrak{q}(\pi) \in \mathcal{O}$ the Hecke algebra $\mathbf{T}_\Sigma$ embeds, in fact, into $B = \prod_{\pi\in\Pi_\Sigma} \mathcal{O}$ (in particular $\mathbf{T}_\Sigma$ has no $\mathbf{Z}$-torsion).

Observe that $(\mathrm{tr}\,\rho_\pi(\sigma))_{\pi\in\Pi_\Sigma} \in \mathbf{T}_\Sigma \subset B$ for all $\sigma \in G_\Sigma$. This follows from the Chebotarev Density Theorem and the continuity of ρ_π (note that $\mathbf{T}_\Sigma$ is a finite $\mathcal{O}$-algebra).

Fix $\pi \in \Pi_\Sigma$ for the rest of this subsection. Define $\rho'_\pi := \rho_\pi \otimes \phi_{2,\mathfrak{p}}^{-1}$. Then ρ'_π satisfies

$$\mathrm{tr}\,\rho'_\pi(\mathrm{Frob}\,\mathfrak{q}) \equiv 1 + (\phi_\mathfrak{p}\epsilon)(\mathrm{Frob}\,\mathfrak{q}) \pmod{\varpi} \quad \text{for } \mathfrak{q} \notin S_\phi,$$

and

$$\det \rho'_\pi = \gamma \cdot \phi_{2,\mathfrak{p}}^{-2} \cdot \epsilon = \phi_\mathfrak{p}\epsilon.$$

By choosing a suitable lattice Λ one can ensure that ρ'_π has image inside $\mathrm{GL}_2(\mathcal{O})$. The Chebotarev Density Theorem and the Brauer-Nesbitt Theorem imply that

$$(\overline{\rho}'_\pi)^{\mathrm{ss}} \cong 1 \oplus \overline{\phi}_\mathfrak{p}\overline{\epsilon}.$$

By Theorem 2.2 ρ'_π is irreducible, and by Theorem 2.3 ρ'_π is ordinary so a standard argument (see e.g. Proposition 2.1 in [Rib76]) shows the lattice Λ may be chosen in such a way that $\overline{\rho}'_\pi$ is not semi-simple and

$$\overline{\rho}'_\pi = \begin{bmatrix} 1 & * \\ & \overline{\phi}_\mathfrak{p}\overline{\epsilon} \end{bmatrix} \tag{4.2}$$

and is ordinary. We put

$$\rho_0 := \overline{\rho}'_\pi. \tag{4.3}$$

Note that the isomorphism class of ρ_0 is in fact independent of the choice of $\pi \in \Pi_\Sigma$ by Proposition 3.3.

Remark 4.6. The representation ρ_0 satisfies conditions (Red) and (Sc) of Section 3. The fact that ρ'_π is ordinary combined with (4.2) implies that

$$\overline{\rho}'_\pi|_{D_{\overline{\mathfrak{p}}}} \cong \begin{bmatrix} 1 & \\ & (\overline{\phi}_\mathfrak{p}\overline{\epsilon})|_{D_{\overline{\mathfrak{p}}}} \end{bmatrix}. \tag{4.4}$$

Note that in fact (4.4) together with condition (2) in Definition 3.1 implies that the extension $F(\rho_0)/F(\chi_0)$ is unramified away from $\mathfrak{p}$.

Remark 4.7. In Remark 4.6 of [BK09] we pointed out that the extension $F(\rho_0)/F(\chi_0)$ under the conditions of Theorem 3.5 is totally ramified at $\mathfrak{p}$, so, in particular, there exists $\tau \in I_{\mathfrak{p}}$ such that

$$\rho_0(\tau) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

This implies that no twist of ρ_0 by a character is invariant under $c \in \mathrm{Gal}(F/\mathbf{Q})$ and so no character twists of ρ_0 arise from base change. However, we would like to withdraw our additional statement in Remark 4.6 of [BK09] (and corresponding remark in Section 4.6 of [Ber09]) that this implies that (twists of) deformations of ρ_0 cannot arise from base change. The G_Σ-invariant lattice in E^2 that we consider need not be invariant under complex conjugation. In fact, Hida proved in [Hid82] the existence of deformations of a twist of ρ_0 for anticyclotomic characters χ_0 starting from congruences of CM forms over $\mathbf{Q}$.

4.3. **Deformation problem.** Recall that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ is split. Let Σ, ϕ, χ_0 and ρ_0 be as above. Recall that we have assumed that χ_0 is Σ-admissible and have shown in Section 4.2 that ρ_0 satisfies conditions (Red) and (Sc) of Section 3. Hence by Proposition 3.3, ρ_0 is unique up to isomorphism. By (4.4) the extension $F(\rho_0)/F(\chi_0)$ splits at $\overline{\mathfrak{p}}$. In this section we study deformations of ρ_0.

Denote the category of local complete Noetherian $\mathcal{O}$-algebras with residue field $\mathbf{F}$ by $\mathrm{LCN}(E)$. An $\mathcal{O}$-deformation of ρ_0 is a pair consisting of $A \in \mathrm{LCN}(E)$ and a strict equivalence class of continuous representations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ such that $\rho_0 = \rho \pmod{\mathfrak{m}_A}$, where $\mathfrak{m}_A$ is the maximal ideal of A. As is customary we will denote a deformation by a single member of its strict equivalence class. Note that the Hodge-Tate weights of $\phi_{\mathfrak{p}}\epsilon$ are -1 at $\mathfrak{p}$ and $+1$ at $\overline{\mathfrak{p}}$.

Following [SW97] we make the following definition:

Definition 4.8. We say that an $\mathcal{O}$-deformation $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0 is Σ-*minimal* if ρ is ordinary,

$$\det \rho = \phi_{\mathfrak{p}}\epsilon,$$

and for all primes $\mathfrak{q} \in \Sigma$ such that $\# k_{\mathfrak{q}} \equiv 1 \pmod{p}$ one has

$$\rho|_{I_{\mathfrak{q}}} \cong \begin{bmatrix} 1 & \\ & \phi_{\mathfrak{p}}|_{I_{\mathfrak{q}}} \end{bmatrix}.$$

Note that by our assumption on the conductor of ϕ, we in fact have $\phi_{\mathfrak{p}}|_{I_{\mathfrak{q}}} = 1$ for $\mathfrak{q}$ as above. Also the ordinarity condition means in this case that

$$\rho|_{I_{\mathfrak{p}}} \cong \begin{bmatrix} 1 & * \\ & \epsilon^{-1} \end{bmatrix}$$

and

$$\rho|_{I_{\overline{\mathfrak{p}}}} \cong \begin{bmatrix} \epsilon & * \\ & 1 \end{bmatrix}.$$

Note that ρ_0 satisfies all the conditions in Definition 4.8. Since ρ_0 has a scalar centralizer and Σ-minimality is a deformation condition in the sense of [Maz97], there exists a universal deformation ring which we will denote by $R_\Sigma \in \mathrm{LCN}(E)$, and a universal Σ-minimal $\mathcal{O}$-deformation $\rho_{\Sigma,\mathcal{O}} : G_\Sigma \to \mathrm{GL}_2(R_\Sigma)$ such that for every $A \in \mathrm{LCN}(E)$ there is a one-to-one correspondence between the set of $\mathcal{O}$-algebra maps $R_\Sigma \to A$ (inducing identity on $\mathbf{F}$) and the set of Σ-minimal deformations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0.

The arguments from Section 4.2 together with the uniqueness of ρ_0 (Proposition 3.3 - recall that we are all the time assuming that χ_0 is Σ-admissible) can now be reinterpreted as:

Theorem 4.9. *For any $\pi \in \Pi_\Sigma$ there is an $\mathcal{O}$-algebra homomorphism $r_\pi : R_\Sigma \twoheadrightarrow \mathcal{O}$ inducing ρ'_π.*

Proof. The only property left to be checked is Σ-minimality. This is clear since ρ_π is unramified away from S_ϕ, and no $\mathfrak{q} \in S_\phi$ satisfies $\# k_\mathfrak{q} \equiv 1 \pmod p$ by construction (if we are in the case of Theorem 4.2) or assumption (in the case of Theorem 4.4). □

The next three propositions were proved in [BK09] (see Propositions 5.4, 5.5 and Lemma 5.6 in [loc. cit.]).

Proposition 4.10. *There does not exist any non-trivial upper-triangular Σ-minimal deformation of ρ_0 to $\mathrm{GL}_2(\mathbf{F}[x]/x^2)$.*

Proposition 4.11. *The universal deformation ring R_Σ is generated as an $\mathcal{O}$-algebra by traces.*

Proposition 4.12. *The image of the map $R_\Sigma \to \prod_{\pi \in \Pi_\Sigma} \mathcal{O}$ given by $x \mapsto (r_\pi(x))_\pi$ is $\mathbf{T}_\Sigma$.*

Corollary 4.13. *There exists a surjective $\mathcal{O}$-algebra homomorphism $r : R_\Sigma \twoheadrightarrow \mathbf{T}_\Sigma$.*

Proof. This is a direct consequence of Proposition 4.12. □

5. Galois cohomology calculations

In this section we bound the Krull dimension of the universal deformation ring R_Σ via its tangent space. We conclude that R_Σ is a quotient of $\mathcal{O}[[X]]$, which will later allow us to prove that R_Σ is a complete intersection (see Corollary 6.14). We also study reducible deformations of ρ_0 to $\mathcal{O}/\varpi^n$.

5.1. **The Krull dimension of R_Σ is less than 3.** Let ρ_0 be as in Section 4 and let $\mathcal{U}$ be the representation space for ρ_0. Then $\mathcal{U}$ is a two-dimensional $\mathbf{F}$-vector space with a G_Σ-stable filtration

$$0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U},$$

where $\mathcal{U}_1$ is the 1-dimensional $\mathbf{F}$-vector space on which G_Σ acts trivially. The quotient $\mathcal{U}_2 = \mathcal{U}/\mathcal{U}_1$ is a 1-dimensional $\mathbf{F}$-vector space on which G_Σ acts via χ_0. Let $\mathcal{W} = \mathrm{ad}\rho_0 = \mathrm{Hom}_\mathbf{F}(\mathcal{U}, \mathcal{U})$ be the adjoint representation, and let

$$\mathcal{W}^{\mathrm{Sel}} = \{f \in \mathcal{W} : f(\mathcal{U}) \subseteq \mathcal{U}_1, f(\mathcal{U}_1) = 0\}.$$

Recall that ρ_0 splits when restricted to $I_{\overline{\mathfrak{p}}}$ (cf. (4.4)). Define the $I_{\overline{\mathfrak{p}}}$-submodule

$$\mathcal{W}^{\mathrm{Sel,t}} = \{f \in \mathcal{W} : f(\mathcal{U}) \subseteq \mathcal{U}_2, f(\mathcal{U}_2) = 0\}.$$

For $\mathfrak{q} \in \Sigma$ let

$$H_\mathfrak{q} = \begin{cases} H^1(I_\mathfrak{p}, \mathcal{W}/\mathcal{W}^{\mathrm{Sel}}) & \text{if } \mathfrak{q} = \mathfrak{p}, \\ H^1(I_{\overline{\mathfrak{p}}}, \mathcal{W}/\mathcal{W}^{\mathrm{Sel,t}}) & \text{if } \mathfrak{q} = \overline{\mathfrak{p}}, \\ H^1(I_\mathfrak{q}, \mathcal{W}) & \text{if } \# k_\mathfrak{q} \equiv 1 \mod p, \\ 0 & \text{otherwise.} \end{cases}$$

Define the following Selmer group:

$$H^1_\Sigma(F,\mathcal{W}) = \ker(H^1(G_\Sigma,\mathcal{W}) \to \bigoplus_{\mathfrak{q}\in\Sigma} H_\mathfrak{q}).$$

Proposition 5.1.

$$\dim_\mathbf{F} H^1_\Sigma(F,\mathcal{W}) \leq 1.$$

Proof. We proceed in a similar manner to the proof of Proposition 3.1 in [SW97] Section 3. Let $\Sigma_1 \subset \Sigma$ comprise those primes in Σ such that $\# k_\mathfrak{q} \equiv 1 \mod p$, together with $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Let $\mathcal{W}_1 = \mathrm{Hom}_\mathbf{F}(\mathcal{U}_2,\mathcal{U})$, and let $\mathcal{W}_2 = \mathrm{Hom}_\mathbf{F}(\mathcal{U}_1,\mathcal{U})$. There is a commutative diagram of G_Σ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{W}_1 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{F} & \longrightarrow & \mathcal{W}/\mathcal{W}^{\mathrm{Sel}} & \longrightarrow & \mathcal{W}_2 & \longrightarrow & 0 \end{array}$$

having exact rows. Similarly, we have a commutative diagram of $I_{\overline{\mathfrak{p}}}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{W}_1 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{W}_1 & \longrightarrow & \mathcal{W}/\mathcal{W}^{\mathrm{Sel,t}} & \longrightarrow & \mathbf{F} & \longrightarrow & 0 \end{array}$$

having exact rows. (The third vertical arrow is induced by a splitting of the exact sequence for $\mathcal{W}_2$ as $I_{\overline{\mathfrak{p}}}$-modules below.) Together these induce the following commutative diagram of cohomology groups:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G_\Sigma,\mathcal{W}_1) & \longrightarrow & H^1(G_\Sigma,\mathcal{W}) & \longrightarrow & H^1(G_\Sigma,\mathcal{W}_2)\ , \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & X & \longrightarrow & \bigoplus_{\mathfrak{q}\in\Sigma_1} H_\mathfrak{q} & \longrightarrow & Y \end{array}$$

where

$$X = H^1(I_\mathfrak{p},\mathbf{F}) \oplus \bigoplus_{\mathfrak{q}\in\Sigma_1\setminus\{\mathfrak{p}\}} H^1(I_\mathfrak{q},\mathcal{W}_1)$$

and

$$Y = H^1(I_{\overline{\mathfrak{p}}},\mathbf{F}) \oplus \bigoplus_{\mathfrak{q}\in\Sigma_1\setminus\{\overline{\mathfrak{p}}\}} H^1(I_\mathfrak{q},\mathcal{W}_2).$$

Note that the rows in the diagram are exact. Indeed, the only thing to check is the exactness on the left, which can be verified easily by observing that the H^0-terms form a short exact sequence in each of the cases. Hence by the snake lemma there is an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) = H^1_\Sigma(F,\mathcal{W}) \to \ker(\gamma).$$

We claim now that $\ker(\alpha) = 0$ so that $\ker(\beta) \hookrightarrow \ker(\gamma)$. First observe that $\mathcal{W}_1$ fits into the short exact sequence

$$0 \to \mathcal{W}^{\mathrm{Sel}} \to \mathcal{W}_1 \xrightarrow{f} \mathbf{F} \to 0.$$

Since $\mathcal{W}_1 \cong \rho_0 \otimes \chi_0^{-1}$, it is clear that $\mathcal{W}_1^{G_\Sigma} = 0$ and hence the associated long exact cohomology sequence yields the exact sequence

$$0 \to \mathbf{F} \to H^1(G_\Sigma, \mathcal{W}^{\mathrm{Sel}}) \to H^1(G_\Sigma, \mathcal{W}_1) \xrightarrow{H^1(f)} H^1(G_\Sigma, \mathbf{F}).$$

Since $\mathcal{W}^{\mathrm{Sel}} \cong \mathbf{F}(\chi_0^{-1})$, it follows from the Σ-admissibility of χ_0 that the second arrow is surjective. Hence $H^1(f)$ is injective.

Lemma 5.2.

$$\ker(\alpha) \hookrightarrow \ker(H^1(G_\Sigma, \mathbf{F}) \to \bigoplus_{\mathfrak{q}\in\Sigma_1} H^1(I_{\mathfrak{q}}, \mathbf{F}))) = 0.$$

Proof. Consider the following diagram

$$\begin{array}{ccc} H^1(G_\Sigma, \mathcal{W}_1) & \xrightarrow{H^1(f)} & H^1(G_\Sigma, \mathbf{F}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi:=\oplus_{q\in\Sigma_1}\mathrm{res}_q} \\ X & \xrightarrow{f'} & \bigoplus_{q\in\Sigma_1} H^1(I_q, \mathbf{F}) \end{array}$$

where f' is the identity on $H^1(I_p, \mathbf{F})$ and is induced by f on the rest of the components. By definitions of the maps involved it is clear that the diagram is commutative. Since $H^1(f)$ is injective as observed above, if $c \in \ker\alpha$, its image must lie in $\ker\phi$, hence the inclusion of the kernels in the statement of the lemma follows. Now consider $c \in \ker\phi$, i.e., c is unramified at all $q \in \Sigma_1$. But since $\mathbf{F}$ is a p-power order abelian group, we see that c is also unramified at all $q \notin \Sigma_1$. Hence it is unramified everywhere. Since $H^1(G_\Sigma, \mathbf{F}) = \mathrm{Hom}(G_\Sigma, \mathbf{F})$ and by our assumption $p \nmid \#\mathrm{Cl}_F$ it is clear that $c = 0$. □

Similarly, $\mathcal{W}_2$ fits into the short exact sequence

$$0 \to \mathbf{F} \to \mathcal{W}_2 \to \mathrm{Hom}(U_1, U_2) \cong \mathbf{F}(\chi_0) \to 0.$$

The associated long exact cohomology sequence yields the commutative diagram

$$\begin{array}{ccccc} H^1(G_\Sigma, \mathbf{F}) & \hookrightarrow & H^1(G_\Sigma, \mathcal{W}_2) & \longrightarrow & H^1(G_\Sigma, \mathbf{F}(\chi_0)) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle f_2} \\ \bigoplus_{\mathfrak{q}\in\Sigma_1} H^1(I_{\mathfrak{q}}, \mathbf{F}) & \hookrightarrow & Y & \longrightarrow & \bigoplus_{\mathfrak{q}\in\Sigma_1\setminus\{\overline{\mathfrak{p}}\}} H^1(I_{\mathfrak{q}}, \mathbf{F}(\chi_0)) \end{array}$$

having exact rows. As above we see that $\ker(f_1) = 0$ and therefore

$$\ker(\gamma) \hookrightarrow \ker(f_2) \subset H^1(G_\Sigma, \mathbf{F}(\chi_0)).$$

Since the character χ_0 is Σ-admissible we have that $\dim_{\mathbf{F}} H^1(G_\Sigma, \mathbf{F}(\chi_0)) = 1$. This concludes the proof of Proposition 5.1. □

Proposition 5.3. *We have*

$$H^1_\Sigma(F, \mathrm{ad}^0\rho_0) = \mathrm{Hom}(\mathfrak{m}_{R_\Sigma}/(\mathfrak{m}_{R_\Sigma}^2 + \varpi R_\Sigma), \mathbf{F}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, \mathbf{F}[x]/x^2).$$

Proof. See for example [dS97], Theorem 15 which together with Lemma 5.4 yields the Proposition. □

Lemma 5.4. $H_{\overline{\mathfrak{p}}}$ *is the cohomological condition corresponding to ordinarity at* $\overline{\mathfrak{p}}$ *of an infinitesimal deformation, i.e., a deformation to* $\mathrm{GL}_2(\mathbf{F}[X]/X^2)$.

Proof. Let ρ be an infinitesimal deformation of ρ_0 with $\det(\rho) = \Psi = \chi_0$. Write

$$\rho(g)\rho_0(g)^{-1} = 1 + \epsilon c_\rho(g).$$

Then c defines a 1-cocycle with values in $\mathrm{ad}^0\rho_0$. For $g \in I_{\overline{\mathfrak{p}}}$, we have for a suitable basis (with respect to which ρ_0 splits and ρ is lowertriangular)

$$c_\rho(g) \in \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} = \mathcal{W}^{\mathrm{Sel,t}},$$

so the image of $[c_\rho]$ in $H_{\overline{\mathfrak{p}}}$ is zero, as desired. Conversely, if this condition is satisfied then $\rho|_{I_{\overline{\mathfrak{p}}}}$ is lower-triangular , hence so is $\rho|_{D_{\overline{\mathfrak{p}}}}$ and ρ is ordinary at $\overline{\mathfrak{p}}$. □

Corollary 5.5. *We have $R_\Sigma = \mathcal{O}[[X]]/I$ for an ideal I.*

Proof. This is an application of Nakayama, see e.g. [dS97] Theorem 15 or [Til96], Lemma 5.1. Note that we have $H^1_\Sigma(F, \mathrm{ad}\,\rho_0) = H^1_\Sigma(F, \mathrm{ad}^0\,\rho_0)$ since $p \nmid \#\,\mathrm{Cl}_F$ by our assumption. □

5.2. **No reducible deformation beyond p-valuation of L-value.** Set $\Psi := \phi_{\mathfrak{p}}\epsilon$ and write Ψ_r for Ψ mod ϖ^r with $r > 0$. Note that $\mathrm{Gal}(F(\Psi)/F) = \Gamma \times \Delta$ with $\Gamma \cong \mathbf{Z}_p$ and Δ a finite abelian group. Assume $p \nmid \#\Delta$. Set $\tilde{\chi}_0 := \Psi|_\Delta$.

For a finite abelian extension K of F write $M(K)$ for the maximal abelian pro-p-extension of K unramified away from the primes lying over $\mathfrak{p}$. For a character $\varphi : \mathrm{Gal}(K/F) \to \mathcal{O}^\times$ write $M(K)_\varphi$ for the maximal abelian pro-p-extension of K unramified away from the primes lying over $\mathfrak{p}$ such that $\mathrm{Gal}(K/F)$ acts on $\mathrm{Gal}(M(K)_\varphi/K)$ via φ^{-1}.

Lemma 5.6. *Any Σ-minimal uppertriangular deformation ρ_r of ρ_0 to $\mathcal{O}/\varpi^r$ must have the form*

$$\rho_r = \begin{bmatrix} 1 & * \\ & \Psi_r \end{bmatrix}.$$

Proof. We will prove this by induction on r. It is true for $r = 1$, so assume this holds for $r = k$. So, we can write

$$\rho_{k+1} = \begin{bmatrix} 1 + \alpha\varpi^k & b \\ & d \end{bmatrix},$$

where $\alpha : G_\Sigma \to \mathbf{F}$ is a group homomorphism (the group operation on $\mathbf{F}$ being addition). Arguing as in the proof of Proposition 5.4 of [BK09] we see that α can only be ramified at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Ordinarity at $\mathfrak{p}$ (respectively at $\overline{\mathfrak{p}}$) forces $\rho_{k+1}|_{I_{\mathfrak{p}}}$ (respectively $\rho_{k+1}|_{I_{\overline{\mathfrak{p}}}}$) to have a free $\mathcal{O}/\varpi^{k+1}$-submodule (respectively quotient) on which $I_{\mathfrak{p}}$ (respectively $I_{\overline{\mathfrak{p}}}$) acts trivially. This together with the fact that χ_0 is ramified at $\mathfrak{p}$ and at $\overline{\mathfrak{p}}$ easily imply that $\alpha|_{I_{\mathfrak{p}}} = \alpha|_{I_{\overline{\mathfrak{p}}}} = 0$. Finally, using the assumption that $p \nmid \#\,\mathrm{Cl}_F$ we conclude that $\alpha = 0$. Since $\det \rho_{k+1} = \Psi_{k+1}$, we get $d = \Psi_{k+1}$. □

Write e for the ramification index of $\mathcal{O}$ over $\mathbf{Z}_p$. Note that the exponent of $\mathcal{O}/\varpi^r$ is $p^{\lceil r/e \rceil}$, where for a real number α, $\lceil \alpha \rceil$ denotes the smallest integer $n \geq \alpha$. By a slight abuse of terminology we will say that a deformation ρ of ρ_0 is *upper-triangular* if there exists a member of the strict equivalence of ρ that is upper-triangular.

Lemma 5.7. *Suppose there exists a Σ-minimal uppertriangular deformation ρ_r of ρ_0 to $\mathrm{GL}_2(\mathcal{O}/\varpi^r)$. Then there exists a surjective map of groups*

$$\mathrm{Gal}(M(F(\Psi_r))_{\Psi_r}/F(\Psi_r)) \twoheadrightarrow \mathbf{Z}/p^{\lceil r/e \rceil}.$$

Proof. Consider the b-entry of ρ_r as a function $b : G_\Sigma \to \mathcal{O}/\varpi^r$. Suppose that the restriction $b|_{\mathrm{Gal}(F(\rho_r)/F(\Psi_r))} \in \varpi\mathcal{O}/\varpi^r\mathcal{O}$. This means that the mod ϖ reduction of b is zero after we split Ψ_r, so that $F(\rho_0) \subset F(\Psi_r)$, but this is impossible as $F(\rho_0)/F$ is non-abelian. So, we get that $b|_{\mathrm{Gal}(F(\rho_r)/F(\Psi_r))} \notin \varpi\mathcal{O}/\varpi^r\mathcal{O}$. Hence $\mathrm{Gal}(F(\rho_r)/F(\Psi_r))$ has a cyclic quotient of order $p^{\lceil r/e \rceil}$. Since $F(\rho_r) \subset M(F(\Psi_r)_{\Psi_r})$, we are done. □

Write $L^{\mathrm{int}}(0,\phi) = u\varpi^n$, where u is a unit. One has $\mathrm{val}_p(\varpi) = 1/e$. We also have for any $x \in \mathcal{O}$, that $\mathrm{val}_p(x) = \frac{1}{e}\mathrm{val}_\varpi(x)$.

Remark 5.8. Note that we have the following sequence of equalities:

$$\begin{aligned}(5.1)\quad \#\mathcal{O}/L^{\mathrm{int}}(0,\phi) = \#\mathcal{O}/\varpi^n = p^{\frac{n[\mathcal{O}:\mathbf{Z}_p]}{e}} = p^{n[\mathbf{F}:\mathbf{F}_p]} = \\ = p^{\mathrm{val}_\varpi(L^{\mathrm{int}}(0,\phi))[\mathbf{F}:\mathbf{F}_p]} = p^{\mathrm{val}_p(L^{\mathrm{int}}(0,\phi))[\mathcal{O}:\mathbf{Z}_p]}.\end{aligned}$$

Proposition 5.9. *Let n be as above. There is no upper-triangular Σ-minimal deformation $\rho_{n+1} : G_\Sigma \to \mathrm{GL}_2(\mathcal{O}/\varpi^{n+1})$.*

Proof. Suppose ρ_{n+1} exists. By Lemma 5.7 there exists an element

$$x \in \mathrm{Gal}(M(F(\Psi_{n+1}))_{\Psi_{n+1}}/F(\Psi_{n+1}))$$

of order $p^{\lceil (n+1)/e \rceil}$. Note that since $p \nmid \#\Delta$, the character $\tilde{\chi}_0$ is the Teichmüller lift of a character with values in $\mathbf{F}^\times$. Let $\mathbf{F}_0$ be the smallest subfield $\mathbf{F}_p \subset \mathbf{F}_0 \subset \mathbf{F}$ such that the image of (the reduction of) $\tilde{\chi}_0$ is contained in $\mathbf{F}_0^\times$. Let $\mathcal{O}_0$ be the corresponding extension of $\mathbf{Z}_p$, i.e., $\mathbf{Z}_p \subset \mathcal{O}_0 \subset \mathcal{O}$. Note that we can assume without loss of generality that $\mathcal{O}_0$ is unramified over $\mathbf{Z}_p$. By Lemma 5.6, $\rho_{n+1} : G_\Sigma \to \mathrm{GL}_2(\mathcal{O}/\varpi^{n+1})$ has the form

$$\rho_{n+1}(\sigma) = \begin{bmatrix} 1 & b(\sigma) \\ & \Psi_{n+1}(\sigma) \end{bmatrix}.$$

Let $y \in \Delta$ be a generator (note that Δ is cyclic as a group isomorphic to a subgroup of $\mathbf{F}_0^\times$). Then it is easy to see that $\tilde{\chi}_0^{-1}(y^k)b(x)$ are all elements of order $p^{\lceil (n+1)/e \rceil}$, and they generate a subgroup H of $b(\mathrm{Gal}(F(\rho_{n+1})/F(\Psi_{n+1}))$ of order $(p^{\lceil (n+1)/e \rceil})^{[\mathbf{F}_0:\mathbf{F}_p]} = (p^{\lceil (n+1)/e \rceil})^{[\mathcal{O}_0:\mathbf{Z}_p]}$ - the last equality is true because $\mathcal{O}_0/\mathbf{Z}_p$ is unramified. (This subgroup still has exponent only $p^{\lceil (n+1)/e \rceil}$.)

Now we have

$$(H \otimes_{\mathbf{Z}_p} \mathcal{O})^{\tilde{\chi}_0^{-1}} = (H \otimes_{\mathbf{Z}_p} \mathcal{O}_0 \otimes_{\mathcal{O}_0} \mathcal{O})^{\tilde{\chi}_0^{-1}} = (H \otimes_{\mathbf{Z}_p} \mathcal{O}_0)^{\tilde{\chi}_0^{-1}} \otimes_{\mathcal{O}_0} \mathcal{O},$$

and hence

(5.2)

$$\begin{aligned}\#(H \otimes_{\mathbf{Z}_p} \mathcal{O})^{\tilde{\chi}_0^{-1}} = (\#(H \otimes_{\mathbf{Z}_p} \mathcal{O}_0)^{\tilde{\chi}_0^{-1}})^{[\mathcal{O}:\mathcal{O}_0]} \geq (\#H)^{[\mathcal{O}:\mathcal{O}_0]} = p^{\lceil (n+1)/e \rceil [\mathcal{O}:\mathbf{Z}_p]} = \\ p^{\lceil (n+1)/e \rceil e [\mathbf{F}:\mathbf{F}_p]} \geq p^{(n+1)[\mathbf{F}:\mathbf{F}_p]} = \#\mathcal{O}/\varpi^{n+1}.\end{aligned}$$

But the group H can be regarded as a subgroup of $\mathrm{Gal}(F(\rho_{n+1})/F(\Psi_{n+1}))$. We have $F(\rho_{n+1}) \subset M(F(\Psi_{n+1}))_\Psi$, hence we arrive at a contradiction by Theorem 5.11 below. □

Remark 5.10. Note that the proof of Proposition 5.9 is valid also for $n=0$. Hence it shows that in the case when $L^{\mathrm{int}}(0,\phi)$ is a p-adic unit, ρ_0 as in Section 3 is not Σ-minimal.

This implies that when $n = 0$ Theorem 6.13 remains vacuously true, since condition (5) in that theorem can never be satisfied.

Theorem 5.11. *One has*

$$\#(\mathrm{Gal}(M(F(\Psi_{n+1}))_\Psi/F(\Psi_{n+1}))\otimes\mathcal{O})^{\tilde{\chi}_0^{-1}} \leq \#\mathcal{O}/L^{\mathrm{int}}(0,\phi) = \#(\mathcal{O}/\varpi^n).$$

Proof. One estimates the order of the left hand side by relating it to the Selmer group of a Hecke character. Using standard methods one can bound the order of the latter from above by the L-value applying the main conjecture proven by Rubin. For details see Lemma 5.14 and 5.15 of [BK09], where this is done for $n = 1$. The proofs of both of these lemmas generalize easily to arbitrary n. □

6. $R_\Sigma = \mathbf{T}_\Sigma$

In this section we will prove the main result which asserts that the surjection in Corollary 4.13 is an isomorphism. As a consequence we obtain a result on modularity of Galois representations and deduce some properties of the Hecke algebra and cyclicity of a certain Galois group. As in Section 5.2 in what follows we assume that $p \nmid \#\Delta$. As before, throughout this section we assume that χ_0 is Σ-admissible.

6.1. **The ideal of reducibility.** We briefly recall some general facts about Eisenstein representations from Section 3 of [Cal06] and Section 2 of [BC06]: Let $(R, \mathfrak{m}_R, \mathbf{F})$ be a local p-adically complete ring. Let G be a topological group and consider a continuous representation $\rho : G \to \mathrm{GL}_2(R)$ such that $\mathrm{tr}\,(\rho) \bmod \mathfrak{m}_R$ is the sum of two distinct characters $\tau_i : G \to \mathbf{F}^\times, i = 1, 2$. Moreover, assume that

$$\dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathrm{cts},\mathbf{F}[G]}(\tau_1,\tau_2) = \dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathrm{cts},\mathbf{F}[G]}(\tau_2,\tau_1) = 1.$$

Definition 6.1. The *ideal of reducibility* of R is the smallest ideal I of R such that $\mathrm{tr}\,(\rho) \bmod I$ is the sum of two characters. We will denote the ideal of reducibility of R_Σ by I_{re}.

Proposition 6.2. *The ideal I_{re} is principal.*

Proof. See [Cal06], Proof of Lemma 3.4. This uses in a crucial way condition (3) of Σ-admissibility of χ_0 (see Definition 3.1). □

Theorem 6.3 (Urban)**.** *Let $(R, \mathfrak{m}_R, \mathbf{F})$ be a local Artinian ring. Let ρ_1, ρ_2, and ρ be three representations of a topological group G with coefficients in R (with ρ having image in $\mathrm{GL}_m(R)$). Assume the following are true:*

- *ρ and $\rho_1 \oplus \rho_2$ have the same characteristic polynomials;*
- *The mod $\mathfrak{m}_R$-reductions $\overline{\rho}_1$ and $\overline{\rho}_2$ of ρ_1 and ρ_2 respectively are absolutely irreducible and non-isomorphic;*
- *The mod $\mathfrak{m}_R$-reduction $\overline{\rho}$ of ρ is indecomposable and the subrepresentation of $\overline{\rho}$ is isomorphic to $\overline{\rho}_1$.*

Then there exists $g \in \mathrm{GL}_m(R)$ such that

$$\rho(h) = g \begin{bmatrix} \rho_1(h) & * \\ & \rho_2(h) \end{bmatrix} g^{-1}$$

for all $h \in G$.

Proof. This is Theorem 1 in [Urb99]. □

Corollary 6.4. *Let $R \in \mathrm{LCN}(E)$ and suppose $\rho : G_\Sigma \to \mathrm{GL}_2(R)$ is a Σ-minimal deformation of ρ_0. Let $I \subset R$ be an ideal such that $R/I \in \mathrm{LCN}(E)$ and is an Artin ring. Then I contains the ideal of reducibility of R if and only if ρ mod I is an upper-triangular deformation of ρ_0 to $\mathrm{GL}_2(R/I)$.*

Proof. If $\rho \bmod I$ is isomorphic to an upper-triangular deformation of ρ_0 to $\mathrm{GL}_2(R/I)$, then clearly $\operatorname{tr}\rho \bmod I$ is a sum of two characters (which are distinct, since they must reduce to χ_0 and 1 modulo $\mathfrak{m}_R$), so I contains the ideal of reducibility. The converse is an easy consequence of Theorem 6.3. □

Proposition 6.5. *Assume $n = \mathrm{val}_\varpi(L^{\mathrm{int}}(0,\phi)) > 0$. Then one has $R_\Sigma/I_{\mathrm{re}} = \mathcal{O}/\varpi^r$ where $0 < r \leq n$.*

Proof. Write S for R_Σ/I_{re}. Then S is a local complete ring. Moreover, by Corollary 5.5 we have that S is a quotient of $\mathcal{O}[[X]]$, and hence $R_\Sigma/\varpi R_\Sigma$ (and thus $S/\varpi S$) is a quotient of $\mathbf{F}[[X]]$. But $\mathbf{F}[[X]]$ is a dvr, so $S/\varpi S = \mathbf{F}[[X]]/X^m$ for some $m \in \mathbf{Z}_+ \cup \{\infty\}$. (By $\mathbf{F}[[X]]/X^\infty$ we mean $\mathbf{F}[[X]]$.) We will first show that $m = 1$. Suppose $m \neq 1$. Then $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, S/\varpi S)$ contains at least two elements - the map $R_\Sigma \twoheadrightarrow \mathbf{F} \hookrightarrow S/\varpi S$ and the surjection $R_\Sigma \twoheadrightarrow S/\varpi S$. These two elements give rise to two distinct elements in $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, \mathbf{F}[X]/X^2)$, the trivial one and the surjection $R_\Sigma \twoheadrightarrow S/\varpi S \twoheadrightarrow \mathbf{F}[X]/X^2$. By the definition of R_Σ there is a one-to-one correspondence between the deformations to $\mathbf{F}[X]/X^2$ and elements of $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, \mathbf{F}[X]/X^2)$. The trivial element corresponds to the trivial deformation to $\mathbf{F}[X]/X^2$, i.e., with image contained in $\mathrm{GL}_2(\mathbf{F})$, which is clearly upper-triangular. However, the deformation corresponding to the surjection must also be upper-triangular by Corollary 6.4 since $\ker(R_\Sigma \twoheadrightarrow S/\varpi S \twoheadrightarrow \mathbf{F}[X]/X^2)$ contains I_{re} and $\mathbf{F}[X]/X^2$ is Artinian. But we know by Proposition 4.10 that ρ_0 does not admit any non-trivial Σ-minimal upper-triangular deformations to $\mathbf{F}[X]/X^2$. Hence we must have $m = 1$.

Thus by the complete version of Nakayama's Lemma ([Eis95], Exercise 7.2) we know that S is generated (as a $\mathcal{O}$-module) by one element. So $S = \mathcal{O}/\varpi^r$ with $r \in \mathbf{Z}_+ \cup \{\infty\}$. Finally we must have $0 < r \leq n$, $r \neq \infty$ since by Corollary 6.4 if $r > n$ or $r = \infty$, then there would be an upper-triangular Σ-minimal deformation of ρ_0 to $\mathcal{O}/\varpi^{n+1}$, which is impossible by Proposition 5.9. □

Remark 6.6. Note that the fact that S is a quotient of $\mathcal{O}[[X]]$ is actually not necessary for the proof of Proposition 6.5. Indeed, it suffices to know that S (and hence $S/\varpi S$) is topologically finitely generated as an $\mathcal{O}$-algebra. It is an easy fact that then $S/\varpi S = \mathbf{F}$ if and only if $S/\varpi S$ has no quotient isomorphic to $\mathbf{F}[X]/X^2$.

Corollary 6.7. *Let $n = \mathrm{val}_\varpi(L^{\mathrm{int}}(0,\phi)) > 0$. Suppose there exists $\pi \in \Pi_\Sigma$ such that $\operatorname{tr}\rho'_\pi(\mathrm{Frob}_\mathfrak{q}) = 1 + (\phi_\mathfrak{p}\epsilon)(\mathrm{Frob}_\mathfrak{q}) \bmod \varpi^n$ for all $\mathfrak{q} \notin \Sigma$. Write $r_{\pi,n}$ for the composite $R_\Sigma \twoheadrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}/\varpi^n$ (the first arrow being r_π). Then $I_{\mathrm{re}} = \ker r_{\pi,n}$.*

Proof. By the definition of the ideal of reducibility we have $I_{\mathrm{re}} \subset \ker r_{\pi,n}$. But, by Proposition 6.5, $R_\Sigma/I_{\mathrm{re}} = \mathcal{O}/\varpi^r$ for $0 < r \leq n$, so we have $\mathcal{O}/\varpi^r = R_\Sigma/I_{\mathrm{re}} \twoheadrightarrow R_\Sigma/\ker r_{\pi,n} = \mathcal{O}/\varpi^n$ and the corollary follows. □

6.2. **A commutative algebra criterion and R=T theorem.** Let R and S denote complete local Noetherian $\mathcal{O}$-algebras with residue field $\mathbf{F}$. Suppose that S is finitely generated as a module over $\mathbf{Z}_p$ and has no $\mathbf{Z}$-torsion. Write $\mathfrak{m}_R$ and $\mathfrak{m}_S$ for the maximal ideals of R and S respectively.

Lemma 6.8. *Let $x \in S$ be such that $\#(S/xS) < \infty$. Then multiplication by x is injective.*

Proof. Consider the exact sequence

$$0 \to \ker x \to S \xrightarrow{\cdot x} S \to S/xS \to 0.$$

Since S is finitely generated as a module over $\mathbf{Z}_p$ we can tensor the exact sequence with $\mathbf{Q}_p$ and conclude that since $S/xS \otimes \mathbf{Q}_p = 0$, we must also have $\ker x \otimes \mathbf{Q}_p = 0$. Since $\ker x$ is a finitely generated $\mathbf{Z}_p$-module it must be a finite group. Hence if $a \in \ker x$, $p^m a = 0$ for some $m \in \mathbf{Z}_{\geq 0}$. Thus a is a torsion element, which implies $a = 0$ by our assumption on S. □

Proposition 6.9. *Suppose there exists a surjective $\mathcal{O}$-algebra map $\phi : R \twoheadrightarrow S$ inducing identity on the residue fields and an element $\pi \in R$ such that the bottom map in the following commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow & & \downarrow \\ R/\pi R & \longrightarrow & S/\phi(\pi)S \end{array}$$

is an isomorphism. If $R/\pi R = \mathcal{O}/\varpi^r$ for some positive integer r, then ϕ is an isomorphism.

Proof. Write $x = \phi(\pi)$. We have $\phi(\pi^k) = x^k$. First, we are going to show that $R/\pi^k R = S/x^k S$ for every positive integer k. Indeed, the map $R/\pi^k R \to S/\phi(\pi^k)S$ is surjective because $R \to S$ is. So, it remains to prove injectivity. The map $\pi^{k-1}R/\pi^k R \to \pi^k R/\pi^{k+1}R$ given by multiplication by π is clearly surjective, so we get

$$\#(R/\pi R) \geq \#(\pi R/\pi^2 R) \geq \dots.$$

Moreover, we have the short exact sequence

$$0 \to \pi^{k-1}R/\pi^k R \to R/\pi^k R \to R/\pi^{k-1}R \to 0,$$

so $\#(R/\pi^k R) = \#(R/\pi^{k-1}R)\#(\pi^{k-1}R/\pi^k R)$. Combining this equation with the previous sequence of inequalities, we get by induction on k that $\#(R/\pi^k R) \leq (\#(R/\pi R))^k$. On the other hand, the maps

$$S/xS \overset{\cdot x}{\twoheadrightarrow} xS/x^2S \overset{\cdot x}{\twoheadrightarrow} x^2S/x^3S \overset{\cdot x}{\twoheadrightarrow} \dots \overset{\cdot x}{\twoheadrightarrow} x^{k-1}S/x^kS$$

are all injective because multiplication by x on S is injective by Lemma 6.8. Thus we get

$$\#(S/x^kS) = \#(S/xS)^k = \#(R/\pi R)^k \geq \#(R/\pi^k R).$$

This proves injectivity of the map $R/\pi^k R \twoheadrightarrow S/x^k S$.

Hence

$$\varprojlim_k R/\pi^k R = \varprojlim_k S/x^k S.$$

So, it suffices to show that $R = \varprojlim_k R/\pi^k R$ and $S = \varprojlim_k S/x^k S$. The first follows from Lemma 7.14 on page 197 in [Eis95] since for every power of $\mathfrak{m}_R$, say $\mathfrak{m}_R^k$ there is a power π^s of π such that $(\pi)^s \subset \mathfrak{m}_R^k$ (indeed, take $s = k$) and for every power of (π), say $(\pi)^k$ there is a power of $\mathfrak{m}_R^s$ of $\mathfrak{m}_R$ such that $\mathfrak{m}_R^s \subset (\pi)^k$ (indeed, since $\mathfrak{m}_R = (\pi, \varpi)$, we get $\mathfrak{m}_R^r \subset (\pi)$, so $s = kr$ works). The situation for S is analogous. □

We will now state some consequences of Proposition 6.9.

Theorem 6.10. *The map $r : R_\Sigma \twoheadrightarrow \mathbf{T}_\Sigma$ in Corollary 4.13 is an isomorphism.*

Proof. Write ρ_J for the G_Σ-representation induced by $R_\Sigma \to \mathbf{T}_\Sigma/J_\Sigma$. By definition of J_Σ and the Chebotarev density theorem we see that $\operatorname{tr}\rho_J = 1 + \phi_{\mathfrak{p}}\epsilon$, hence $r^{-1}(J_\Sigma) \supset I_{\mathrm{re}}$ and hence $J_\Sigma \supset r(I_{\mathrm{re}})$. By Proposition 6.2 the ideal I_{re} is principal and we fix a generator π of I_{re}. Thus the surjection

$$R_\Sigma/\pi R_\Sigma \twoheadrightarrow \mathbf{T}_\Sigma/r(\pi)\mathbf{T}_\Sigma \twoheadrightarrow \mathbf{T}_\Sigma/J_\Sigma,$$

must be an isomorphism since $R_\Sigma/\pi R_\Sigma = \mathcal{O}/\varpi^r$ with $r \leq n$ by Proposition 6.5 and $\#(\mathbf{T}_\Sigma/J_\Sigma) \geq \#\mathcal{O}/\varpi^n$ by Theorems 4.2 and 4.4. Here $n = \operatorname{val}_\varpi(L^{\mathrm{int}}(0,\phi))$. Hence r induces the commutative diagram in Proposition 6.9 with $R = R_\Sigma$, $S = \mathbf{T}_\Sigma$ (note that $\mathbf{T}_\Sigma$ is $\mathbf{Z}$-torsion-free and finitely generated as a module over $\mathbf{Z}_p$ since by definition it acts faithfully on a finite dimensional vector space of characteristic zero). □

Corollary 6.11. *Let $n = \operatorname{val}_\varpi(L^{\mathrm{int}}(0,\phi))$. Suppose there exists $\pi \in \Pi_\Sigma$ such that $\operatorname{tr}\rho'_\pi(\mathrm{Frob}_{\mathfrak{q}}) = 1 + (\phi_{\mathfrak{p}}\epsilon)(\mathrm{Frob}_{\mathfrak{q}}) \bmod \varpi^n$ for all $\mathfrak{q} \notin \Sigma$. Then $R_\Sigma = \mathbf{T}_\Sigma = \mathcal{O}$, i.e., $R_\Sigma = \mathbf{T}_\Sigma$ is a discrete valuation ring.*

Proof. By Corollary 6.7 we have $r_\pi^{-1}(\varpi^n\mathcal{O}) = I_{\mathrm{re}}$. So, we see that r_π induces the commutative diagram in Proposition 6.9 with $R = R_\Sigma$, $S = \mathcal{O}$, π a generator of I_{re}. □

Remark 6.12. Note that Corollary 6.11 implies that if $n = \operatorname{val}_\varpi(L^{\mathrm{int}}(0,\phi)) > 0$ and there exists an automorphic representation $\pi \in \Pi_\Sigma$ such that π has Hecke eigenvalues congruent to the Hecke eigenvalues of an Eisenstein series mod ϖ^n, then $\Pi_\Sigma = \{\pi\}$.

6.3. **Modularity theorem.** In this section we state a modularity theorem which is a consequence of the results of the previous sections. To make its statement self-contained, we explicitly include all the assumptions we have made so far.

Theorem 6.13. *Let F be an imaginary quadratic field and $p > 3$ a rational prime which splits in F. Fix a prime $\mathfrak{p}$ of F over p. Assume that $p \nmid \#\,\mathrm{Cl}_F$ and that any prime $q \mid \mathrm{disc}_F$ satisfies $q \not\equiv \pm 1 \pmod{p}$. Let ϕ_1, ϕ_2 be Hecke characters of F with split conductors and of infinity type z and z^{-1} respectively such that $\phi := \phi_1/\phi_2$ is unramified. Assume that the conductor $\mathfrak{M}_1$ of ϕ_1 is coprime to (p) and that $p \nmid \#(\mathcal{O}_F/\mathfrak{M}_1)^\times$. Write $\mathrm{Gal}(F(\phi_{\mathfrak{p}}\epsilon)/F) = \Gamma \times \Delta$ with $\Gamma \cong \mathbf{Z}_p$ and Δ a finite group. Assume $p \nmid \#\Delta$.*

Let $\rho : G_\Sigma \to \mathrm{GL}_2(E)$ be a continuous irreducible representation that is ordinary at all places $\mathfrak{q} \mid p$. Suppose $\overline{\rho}^{\mathrm{ss}} \cong \chi_1 \oplus \chi_2$ with $\chi_1 = \overline{\phi_{1,\mathfrak{p}}\epsilon}$, $\chi_2 = \overline{\phi_{2,\mathfrak{p}}}$. Set $\chi_0 := \chi_1\chi_2^{-1}$. If all of the following conditions are satisfied:

1. $\Sigma \supset \{\mathfrak{q} \mid pd_F\mathfrak{M}_1\mathfrak{M}_1^c\}$,
2. $p \nmid \#\Delta$
3. χ_0 *is Σ-admissible (cf. Definition 3.1),*
4. $\det(\rho) = \phi_1\phi_2\epsilon$,
5. $\rho \otimes \phi_{2,\mathfrak{p}}^{-1}$ *is Σ-minimal,*

then ρ is modular in the sense of Definition 2.5.

6.4. **More consequences of our result.** We will now state some corollaries of the $R = T$ theorem proved in the previous section. As before, we assume that χ_0 is Σ-admissible. We will also discuss how our approach can be used to partially recover the results of Skinner and Wiles [SW99].

Corollary 6.14. *The Eisenstein ideal J_Σ is a principal ideal and the Hecke algebra $\mathbf{T}_\Sigma$ is a complete intersection (and hence Gorenstein).*

Proof. The first statement follows immediately from Theorem 6.10 and the fact that the ideal of reducibility $I_{\rm re}$ (which is principal - see Proposition 6.2) is mapped exactly onto the Eisenstein ideal J_Σ (see proof of Theorem 6.10). The fact that $\mathbf{T}_\Sigma$ is Gorenstein follows from Proposition 6.4 in [Bas63]. Note that the proposition is applicable since the maximal ideal of $\mathbf{T}_\Sigma$ is generated by ϖ and a generator of the ideal of reducibility (see Proof of Proposition 6.9) and p is clearly a non-zero divisor. Finally since $R_\Sigma = T_\Sigma = \mathcal{O}[[X]]/I$ with the ideal I of codimension 1, the Gorenstein condition is equivalent with principality of I ([Eis95], Corollary 21.20). Hence $\mathbf{T}_\Sigma$ is a complete intersection. □

Remark 6.15. The properties of the Hecke algebra and the Eisenstein ideal stated in Corollary 6.14 were proved for their counterparts over $\mathbf{Q}$ by Mazur ([Maz77], Theorem 11, and Chapter 2, Section 14). See also a discussion in [CE05] (page 99 and Corollary 3.17) where these properties, like here, are derived as a consequence of an $R = T$ theorem.

Corollary 6.16. *Let the notation be as in Section 5. There exists a Σ-minimal upper-triangular deformation of ρ_0 to $\mathrm{GL}_2(\mathcal{O}/\varpi^n)$. Moreover, the $\mathcal{O}$-module*

$$\mathrm{Gal}(M(F(\Psi_n))_\Psi/F(\Psi_n)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}$$

is isomorphic to $\mathcal{O}/\varpi^n$.

Proof. The first statement follows from the isomorphisms $R_\Sigma/I_{\rm re} \cong \mathbf{T}_\Sigma/J_\Sigma \cong \mathcal{O}/\varpi^n$ combined with Theorem 6.3. The existence of the upper-triangular deformation together with Lemma 5.7 provides an element of $\mathrm{Gal}(M(F(\Psi_n))_{\Psi_n}/F(\Psi_n))$ of order $p^{\lceil n/e \rceil}$. As in the proof of Theorem 5.11 we see that this element generates an $\mathcal{O}$-submodule of

$$\mathrm{Gal}(M(F(\Psi_n))_\Psi/F(\Psi_n)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}$$

of order $\#\mathcal{O}/\varpi^n$. Hence the second statement follows from Theorem 5.11. □

Remark 6.17. While the upper bounds on the order of the Galois group in Corollary 6.16 are predicted by the Main Conjecture of Iwasawa Theory, their exact $\mathcal{O}$-module structure is in general a mystery. Our results show that the upper bounds are in fact optimal and provide a rather definitive answer to structure question, however only in the case of a Σ-admissible character, where one assumes at the outset the cyclicity of the group of extensions (see Definition 3.1).

Finally, we will discuss to what extent our method provides an alternative to the approach of Skinner and Wiles in [SW99] for proving an $R = T$ theorem for 2-dimensional, p-adic, residually reducible, ordinary representations of $G_{\mathbf{Q}}$ with unique non-semisimple reduction which are unramified outside finitely many primes. First note that the assumptions of [SW99] are weaker than ours, because they only imply that the group of extensions of 1 by the character as in Definition 3.1 is one-dimensional, but not so if one reverses the order of 1 and the character. However, even with that weaker assumption in place our Proposition 4.10 carries over (with the same proof) showing that there are no upper-triangular Σ-minimal deformations of the residual representation ρ_0 of [SW99] to $\mathrm{GL}_2(\mathbf{F}[X]/X^2)$. Using this we proceed as in the proof of our Proposition 6.5 to show that the universal deformation ring (denoted in [SW99] by $R^{min}_{\Sigma,\mathcal{O}}$) has the property that $R^{min}_{\Sigma,\mathcal{O}}/I_{\rm re} \cong \mathcal{O}/\varpi^k$ for some $k \in \mathbf{Z}_+ \cup \{\infty\}$, where $I_{\rm re}$ denotes the ideal of reducibility. Note that by Remark 6.6 it is not necessary to know that $R^{min}_{\Sigma,\mathcal{O}}$ is a quotient of $\mathcal{O}[[X]]$. Since [SW99] in fact do exhibit a Σ-minimal upper-triangular deformation $\rho^{Eis}_{\Sigma,\mathcal{O}}$ to $\mathrm{GL}_2(\mathcal{O})$, we must have $k = \infty$, i.e., $R^{min}_{\Sigma,\mathcal{O}}/I_{\rm re} \cong \mathcal{O}$. In particular, $I_{\rm re}$ is the kernel of the map $R^{min}_{\Sigma,\mathcal{O}} \twoheadrightarrow \mathcal{O}$ correspoding to $\rho^{Eis}_{\Sigma,\mathcal{O}}$, which in [SW99] is denoted by I_Σ. It follows therefore that the surjection of $R^{min}_{\Sigma,\mathcal{O}}$ onto the (full, not just cuspidal) Hecke algebra (that in [SW99] is denoted by $T^{min}_{\Sigma,\mathcal{O}}$) descends to an isomorphism $R^{min}_{\Sigma,\mathcal{O}}/I_{\rm re} \cong T^{min}_{\Sigma,\mathcal{O}}/I^{Eis}$, where I^{Eis} denotes the Eisenstein ideal (see also Proposition 3.12 in [CE05] where an analogous statement is proved in the case when the residual representation is reducible and semi-simple). In particular our method gives an alternative proof to the statement that every *reducible* Σ-minimal deformation of ρ_0 is modular. However, we cannot apply our commutative algebra criterion (Proposition 6.9) to conclude that $R^{min}_{\Sigma,\mathcal{O}} = T^{min}_{\Sigma,\mathcal{O}}$. First, as mentioned above, the assumptions of [SW99] are weaker than our assumption of Σ-admissibility, so do not imply that the ideal of reducibility is principal, and secondly, $R^{min}_{\Sigma,\mathcal{O}}/I_{\rm re}$ is not finite. In [SW99] and [CE05] the $R = T$ statement is proved using the numerical criterion of Wiles and Lenstra which requires one to compare the size of $I_{\rm re}/I^2_{\rm re}$ with the size of the quotient of the cuspidal Hecke algebra by I^{Eis}.

References

Bas63. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

BC06. J. Bellaïche and G. Chenevier, *Lissité de la courbe de Hecke de* GL_2 *aux points Eisenstein critiques*, J. Inst. Math. Jussieu **5** (2006), no. 2, 333–349.

BCDT01. C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic).

Ber05. T. Berger, *An Eisenstein ideal for imaginary quadratic fields*, Thesis, University of Michigan, Ann Arbor, 2005.

Ber08. ———, *Denominators of Eisenstein cohomology classes for* GL_2 *over imaginary quadratic fields*, Manuscripta Math. **125** (2008), no. 4, 427–470.

Ber09. Tobias Berger, *On the Eisenstein ideal for imaginary quadratic fields*, Compos. Math. **145** (2009), no. 3, 603–632.

BH07. T. Berger and G. Harcos, *l-adic representations associated to modular forms over imaginary quadratic fields*, Int. Math. Res. Not. IMRN (2007), no. 23, Art. ID rnm113, 16.

BK09. T. Berger and K. Klosin, *A deformation problem for Galois representations over imaginary quadratic fields*, Journal de l'Institut de Math. de Jussieu **8** (2009), no. 4, 669–692.

Cal06. F. Calegari, *Eisenstein deformation rings*, Compos. Math. **142** (2006), no. 1, 63–83.

CD06. F. Calegari and N. M. Dunfield, *Automorphic forms and rational homology 3-spheres*, Geom. Topol. **10** (2006), 295–329 (electronic).

CE05. F. Calegari and M. Emerton, *On the ramification of Hecke algebras at Eisenstein primes*, Invent. Math. **160** (2005), no. 1, 97–144.

dS97. E. de Shalit, *Hecke rings and universal deformation rings*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 421–445. MR MR1638487

Eis95. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

Fuj99. K. Fujiwara, *Deformation rings and Hecke algebras in the totally real case*, Preprint (1999).

Hid82. H. Hida, *Kummer's criterion for the special values of Hecke L-functions of imaginary quadratic fields and congruences among cusp forms*, Invent. Math. **66** (1982), no. 3, 415–459.

Kis07. M. Kisin, *The Fontaine-Mazur conjecture for* GL_2, Preprint (2007).

Maz77. B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978).

Maz97. ———, *An introduction to the deformation theory of Galois representations*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 243–311.

Mil06. J. S. Milne, *Arithmetic duality theorems*, second ed., BookSurge, LLC, Charleston, SC, 2006.

MW84. B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbf{Q}$, Invent. Math. **76** (1984), no. 2, 179–330.

Rib76. K. A. Ribet, *A modular construction of unramified p-extensions of $Q(\mu_p)$*, Invent. Math. **34** (1976), no. 3, 151–162.

SW97. C. M. Skinner and A. J. Wiles, *Ordinary representations and modular forms*, Proc. Nat. Acad. Sci. U.S.A. **94** (1997), no. 20, 10520–10527.

SW99. ———, *Residually reducible representations and modular forms*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 89, 5–126 (2000).

SW01. ———, *Nearly ordinary deformations of irreducible residual representations*, Ann. Fac. Sci. Toulouse Math. (6) **10** (2001), no. 1, 185–215.

Tay88. R. Taylor, *On congruences between modular forms*, Thesis, Princeton University, Princeton, 1988.

Tay02. ———, *Remarks on a conjecture of Fontaine and Mazur*, J. Inst. Math. Jussieu **1** (2002), no. 1, 125–143.

Til96. J. Tilouine, *Deformations of Galois representations and Hecke algebras*, Published for The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1996.

TW95. R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572.

Urb95. E. Urban, *Formes automorphes cuspidales pour* GL_2 *sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. **99** (1995), no. 3, 283–324.

Urb98. ———, *Module de congruences pour* $\mathrm{GL}(2)$ *d'un corps imaginaire quadratique et théorie d'Iwasawa d'un corps CM biquadratique*, Duke Math. J. **92** (1998), no. 1, 179–220.

Urb99. ———, *On residually reducible representations on local rings*, J. Algebra **212** (1999), no. 2, 738–742.

Urb05. ———, *Sur les représentations p-adiques associées aux représentations cuspidales de* $\mathrm{GSp}_{4/\mathbb{Q}}$, Astérisque (2005), no. 302, 151–176, Formes automorphes. II. Le cas du groupe GSp(4).

Wil95. A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

[1]University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Cambridge, CB3 0WB, United Kingdom.

[2]Université Paris 13, Département de Mathématiques, Institut Galilée, 99, avenue Jean-Baptiste Clément, 93430 - Villetaneuse, France