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Curves in $\mathbb{R}^3$

1.1 Basic Definitions

For an open subset $U \subset \mathbb{R}^m$, and a function $f : U \to \mathbb{R}^n$, $f = (f_1, ..., f_m)$, represented as a column vector, we have that

$$D_j \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} D_j f(x) = \frac{\partial f}{\partial x^j} f(x) = \lim_{t \to 0} \frac{f(x + t \cdot e_j) - f(x)}{t} = \frac{d}{dt} f(x + t \cdot e_j)\Big|_{t=0} = \begin{pmatrix} \frac{\partial f_1}{\partial x^j} \\ \vdots \\ \frac{\partial f_m}{\partial x^j} \end{pmatrix}$$

where

$$e_j = (0_1, 0_2, ..., 1_j, ..., 0_m);$$

and $x \in \mathbb{R}^n$ We call this function f **continuously differentiable**, denoted $f \in \mathcal{C}^1$, if

$$\frac{\partial f}{\partial x^1}, ..., \frac{\partial f}{\partial x^m}$$

exist and are continuous. We say that $f \in \mathcal{C}^k$ if all partial derivatives up to the k^{th} derivative exist and are continuous. We say that f is **smooth** if f is infinitely differentiable (denoted $f \in \mathcal{C}^\infty$), i.e. if $f \in \mathcal{C}^k$ for all $k = 1, 2, ...$.

The **total derivative**, or **Jacobian** of $f : \mathbb{R}^m \to \mathbb{R}^n$, $\in \mathcal{C}^1$ is the matrix

$$Df(x) = (D_1 f(x), ..., D_n(fx)) = \begin{bmatrix} \frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x^1} & \cdots & \frac{\partial f_n}{\partial x^m} \end{bmatrix} \in \mathbb{R}^{n,m*}$$

The function f is **differentiable** if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{v \to 0} \frac{f(a + v) - (f(a) + A \cdot v)}{||v||} = 0$$

Fact. If f is continuously differentiable, this implies that f is differentiable with $A = Df(x)$, which implies that $\frac{\partial f}{\partial x^j}$ exist for all j.

For all $f : U \to \mathbb{R}^m$, where U is open in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, we have the **directional derivative:**

$$D_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \frac{d}{dt} f(x + tv)\Big|_{t=0}$$

*This notation indicates n rows, and m columns

In particular, for $f : U \to \mathbb{R}$ where U is open in $\mathbb{R}^n$ and $f \in \mathcal{C}^1$, we have that

$$Df(x) = \left(\frac{\partial f}{\partial x^1}(x), ..., \frac{\partial f}{\partial x^n}(x)\right) = grad(f)$$

and for $v = (v^1, ..., v^n)$ that

$$D_v f(x) = v^1 \frac{\partial f}{\partial x^1} + ... + v^n \frac{\partial f}{\partial x^n} = \langle v, grad(f)\rangle$$

For $a, b \in \mathbb{R}^n$, we say that a is **orthogonal** to b if $\langle a, b\rangle = 0$, denoted $a \perp b$. The **norm** of a vector $v \in \mathbb{R}^n$ is defined as $||v|| = \sqrt{\langle v, v\rangle} = \sqrt{\sum_{i=1}^n v_i^2}$, and the **metric** in $\mathbb{R}^n$ between vectors is defined as $d(v, w) := ||v - w||$.

1.2 Curves

Definition. A **curve** is a map $\alpha : I \to \mathbb{R}^n$ where $I = (a, b) \subset \mathbb{R}$ and $\alpha \in \mathcal{C}^1$. We write

$$\alpha' = D_\alpha(t) = \lim_{h \to 0} \frac{\alpha(t + h) - \alpha(t)}{h} \in \mathbb{R}^n.$$

A curve is called **regular** if $\alpha'(t) \neq 0$ for all $t \in I$ (in other words, the parameterization of the curve 'never stops moving'). We say that α has **unit speed** if $||\alpha'(t)|| = 1$ for all $t \in I$.

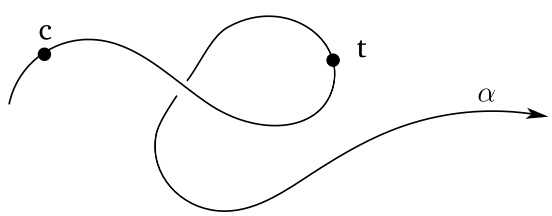
The **arc length** of α from a to b is defined as

$$\langle(\alpha) := \int_a^b ||\alpha'(t)||dt = \int_a^b \sqrt{\langle\alpha'(t), \alpha'(t)\rangle}dt$$

Let $r : J \to I$, where J and I are open intervals in $\mathbb{R}$, and call $\beta = \alpha \circ r : I \to J \to \mathbb{R}^n$ a **reparameterization** of α.

1.3 Regular Curves and unit speed

Lemma 1. *If $\alpha : I \to \mathbb{R}^n$ is a regular curve, then there exists a reparameterization β of α such that β has unit speed.*



Proof. Let

$$s(t) = \int_c^t ||\alpha'(u)||du$$

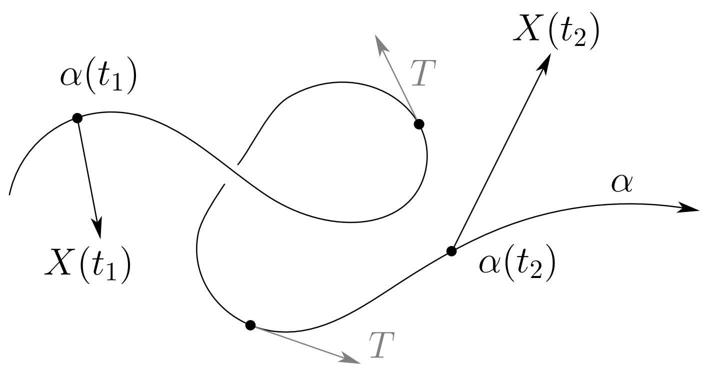
Since α is assumed to be regular, this implies that $0 < ||\alpha'(t)|| = s'(t)$. By the inverse function theorem, we get that s is invertible, i.e., $r := s^{-1}$ is a function $r : J \to I$. Define $\beta := \alpha \circ r$. Since $\alpha = \beta \circ r^{-1} = \beta \circ s$, we have

$$\alpha'(t) = \beta'(s(t))s'(t) \Rightarrow ||\beta'(s(t))|| = \left|\left|\frac{\alpha'(t)}{s'(t)}\right|\right| = \left|\left|\frac{\alpha'(t)}{\alpha'(t)}\right|\right| = 1$$

$\square$

1.4 Vector Fields

Definition. A **vector field** on the curve $\alpha : I \to \mathbb{R}^n$ is a function $X : I \to \mathbb{R}^n$ that assigns a vector to each point $\alpha(t)$.



For a curve α, we have the **tangent vector field**, $T := \alpha'(t) = \frac{\partial \alpha}{\partial t}(t)$ (depicted with gray arrows on the diagram above). A **frame field** on α consists of vector fields $E_1, ..., E_n$ of α such that

$$\langle E_i, E_j \rangle = \delta_{i,j} := \begin{cases} 0, i \neq j \\ 1, i = k \end{cases}$$

for all $t \in I$. In this case, any vector $v \in \mathbb{R}^n$ at $\alpha(t)$ can be written as

$$v = \langle v, E(t) \rangle E_1(t) + ... + \langle v, E_n(t) \rangle E_n(t)$$

1.5 Curvature

For this section, we restrict our discussion to $\mathbb{R}^3$.

Definition. Let $\alpha : I \to \mathbb{R}^3$ be a curve with both unit speed, and infinitely differentiable. Let $T := \alpha'$ be the tangent vector field. Note that

$$\langle T(t), T(t) \rangle = ||T(t)||^2 = ||\alpha'(t)||^2 = 1 \tag{1.5.1}$$

Remark. You can show that the following is true with the product rule:

$$\frac{\partial}{\partial t} \langle v(t), w(t) \rangle = \left\langle \frac{\partial}{\partial t} v(t), w(t) \right\rangle + \left\langle v(t), \frac{\partial}{\partial t} w(t) \right\rangle$$

Differentiating (1.5.1) gives us that

$$0 = \frac{d}{dt} \langle T(t), T(t) \rangle = \langle T'(t), T(t) \rangle + \langle T, T'(t) \rangle = 2\langle T'(t), T(t) \rangle \Rightarrow \langle T', T \rangle = 0 \tag{1.5.2}$$

The **curvature** of α is then defined as $\kappa(t) := ||T'(t)||$. We assume that $\kappa(t) \neq 0 \forall t$, and then define the **principal normal vector**, $N(t) := \frac{T'(t)}{\kappa(t)}$. Finally, we define the **binormal vector field** to be $B(t) := T(t) \times N(t)$, where '$\times$' is the cross-product in $\mathbb{R}^3$. Recall,

$$v \times w := det \begin{pmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \in \mathbb{R}^3$$

1.6 The Frenet Formulas

Proposition 2. *For a unit-speed curve α with $\kappa > 0$, T, N, B is a frame field. Additionally, there exists a function $\tau : I \to \mathbb{R}$ called the **torsion** of α, such that the following formulas are true:*

$$\begin{array}{lllll} T' & = & & \kappa \cdot N & \\ N' & = & -\kappa \cdot T & & +\tau \cdot B \\ B' & = & & -\tau \cdot N & \end{array} \tag{1.6.1}$$

Proof. We'd first like to show that T, N, B is a frame field. Well, it immediately follows that $\langle T, T\rangle = 1$, from the unit speed of α, and $\langle B, T\rangle = \langle B, N\rangle = 0$, because $B = T \times N$. We then have that $\langle N, T\rangle = \langle \frac{T'}{\kappa}, T\rangle = \frac{1}{\kappa}\langle T', T\rangle = 0$, as we saw in (1.5.2). Then,

$$||N|| = \left|\left|\frac{T'}{\kappa}\right|\right| = \frac{1}{\kappa}||T'|| = \frac{1}{||T'||} \cdot ||T'|| = 1, \rightsquigarrow \langle N, N\rangle = ||N||^2 = 1$$

From the definition of the cross-product, we have that

$$||B|| = ||T \times N|| = ||T|| \cdot ||N|| \cdot sin(\zeta)$$

where ζ is the angle between T and N. However, this angle is 90^o, so $||B|| = 1 \rightsquigarrow \langle B, B\rangle = 1$. We conclude that T, N, B is a frame field, and we now aim to show equality in the the three equations from (1.6.1).

We immediately have the first identity in 1.6.1, because $N = \frac{T'}{\kappa}$ by definition. Next, we'll show that B' is colinear to N, in order to show the third identity in (1.6.1). In fact, we'll show that $B' \perp T$, and that $B' \perp B$. We have

$$\langle B, T\rangle = 0 \xrightarrow{d/dt} 0 = \langle B', T\rangle + \langle B, T'\rangle = \langle B', T\rangle + \underbrace{\langle B, \kappa N\rangle}_{0} \Rightarrow B' \perp T$$

Also,

$$\langle B, B\rangle = 1 \xrightarrow{d/dt} 0 = 2\langle B', B\rangle \Rightarrow B' \perp B$$

As B' is orthogonal to both T and B, and since $B, N,$ and T make up a frame field, there exists some τ such that $B' = -\tau N$. For the second equation in (1.6.1), remember that we can write any vector v as a linear combination of the vector in our frame field, $v = \langle v, T\rangle T + \langle v, N\rangle N + \langle v, B\rangle B$. Applying this to $v = N'$, we'll show

$$N' = \underbrace{\langle N', T\rangle}_{-\kappa} T + \underbrace{\langle N', N\rangle}_{=0} N + \underbrace{\langle N', B\rangle}_{=\tau} B$$

We have that

$$\langle N, T\rangle = 0 \xrightarrow{d/dt} 0 = \langle N', T\rangle + \langle N, T'\rangle \Rightarrow \langle N', T\rangle = -\langle N, \kappa N\rangle = -\kappa\langle N, N\rangle = -\kappa$$

and additionally,

$$\langle N, N\rangle = 1 \xrightarrow{d/dt} 0 = 2\langle N', N\rangle$$

$$\langle N, B\rangle = 0 \xrightarrow{d/dt} 0 = \langle N', B\rangle + \langle N, B'\rangle \Rightarrow \langle N', B\rangle = -\langle N, -\tau N\rangle = -(-\tau)\langle N, N\rangle = \tau$$

$\square$

1.7 Frenet Frame for Reparameterizations

If α is a regular curve, let $\beta(t) = \alpha(v(t))$ be a reparameterization of α with unit speed. Then, the Frenet frame for α is defined to be the Frenet frame for β at the corresponding points: if $\alpha(s) = \beta(r^{-1}(s))$, then $T_\alpha(s) := T_\beta(r^{-1}(s)),\ N_\alpha(s) := N_\beta(r^{-1}(s)),\ B_\alpha(s) = B_\beta(r^{-1}(s))$.

1.8 Isometries

Definition. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry of** $\mathbb{R}^n$ if it preserves the distance function, $d(F(x), F(y)) = d(x, y)$.

1.9 Examples

1. Let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ be the translation map by $a \in \mathbb{R}^n$, $T_a(x) = x + a$. Of course, note that $T_a^{-1} = T_{-a}$, and that translation maps preserve distances (and i.e., are isometries).
2. Let A be an invertible matrix with $A^{-1} = A^t$ (an orthogonal matrix) written $A \in O(n) :=$ the set of all orthogonal matrices in $\mathbb{R}^{n,n}$. Interpret A as a linear map $\mathbb{R}^n \to \mathbb{R}^n$, $A(x) = Ax$. Then,
$$\langle Ax, Ay\rangle = \langle x, A^t Ay\rangle = \langle x, y\rangle$$
Which implies that $||Ax|| = ||x|| \Rightarrow d(Ax, Ay) = d(x, y)$.
3. If f and g are isometries, so is their composition, because
$$d(f \circ g(x), f \circ g(y)) = d(g(x), g(y)) = d(x, y)$$

1.10 Orthogonal Matrices and Isometries

Claim. Let F be an isometry. Then, there exists a unique $a \in \mathbb{R}^n$ and $A \in O(n)$ such that $F = T_a \circ A$.

Proof. Take $a := F(0)$, and define the isometry A to be $A := T_a^{-1} \circ F$. We now must show that A is an orthogonal matrix, i.e., that:

1. $\langle Ax, Ay\rangle = \langle x, y\rangle (\iff A^{-1} = A^t)$
2. A is a linear map

We show this as follows:

(1) We have that $A(0) = T_a^{-1}(F(0)) = F(0) - a = F(0) - F(0) = 0$, and that $d(Ax, Ay) = d(x, y)$ because A is an isometry. Furthering this idea,
$$||A(x)|| = d(A(x), 0) = d(A(x), A(0)) = d(x, 0) = ||x||$$
$$\Rightarrow d(Ax, Ay) = d(x, y)$$
using this,
$$||Ax - Ay|| = ||x - y|| \Rightarrow \langle Ax - Ay, Ax - Ay\rangle = \langle x - y, x - y\rangle$$
the bilinearity of the inner product then implies,
$$\langle Ax, Ax\rangle - \langle Ax, Ay\rangle - \langle Ay, Ax\rangle + \langle Ay, Ay\rangle = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$
and so,
$$||Ax||^2 - 2\langle Ax, Ay\rangle + ||Ay||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2 \rightsquigarrow \langle Ax, Ay\rangle = \langle x, y\rangle$$
and we have (1).

(2) For (2), we want to show that A is linear, e.g. that $A(x + y) = A(x) + A(y)$, and $A(cx) = cA(x)$ for $c \in \mathbb{R}$. Writing $x = \sum_{j=1}^n x^j e_j$ and noticing that $\langle Ae_j, Ae_k\rangle = \langle e_j, e_k\rangle = \delta_{jk}$, it follows that $\{Ae_j\}$ form an orthonormal basis of $\mathbb{R}^n$. Therefore, we can expand $A(x)$ in this basis:
$$A(x) = \sum_{j=1}^n \langle A(x), A(e_j)\rangle A(e_j) = \sum_{j=1}^n \langle x, e_j\rangle A(e_j) = \sum_{j=1}^n x^j A(e_j)$$
and the rest of this proof is simply following through the sums when considering either cx or $x + y$.

The uniqueness of a follows from seeing that if we assume $F = T_a \circ A = T_b \circ B$, then

$$T_a \circ A(0) = T_b \circ B(0) \Rightarrow T_a(0) = T_b(0) \Rightarrow a = b$$

and then

$$T_a \circ A = T_a \circ B \rightsquigarrow T_a^{-1} \circ T_a \circ A = T_a^{-1} \circ T_a \circ B \Rightarrow A = B$$

□

1.11 Congruent Curves

Definition. Let α, β be two curves in $\mathbb{R}^n$, $\alpha : I \to \mathbb{R}^n, \beta : I \to \mathbb{R}^n$. Then, α and β are called **congruent** if there exists an isometry F of $\mathbb{R}^n$ such that $\beta = F \circ \alpha$.

Theorem 3. *Let α, β be curves of unit speed. Then, α and β are congruent if and only if $\kappa_\alpha = \kappa_\beta$, and $\tau_\alpha = \pm\tau_\beta$.*

Proof. ($\Rightarrow$) Let $F(x) = T_a \circ A(x) = Ax + a$ for $a \in \mathbb{R}^3, A \in O(3)$ where $\beta = F \circ \alpha$. First note that the Jacobian $DF(x) = A$, and this implies that

$$T_\beta(t) = \beta'(t) = (F \circ \alpha)'(t) = DF(\alpha(t)) \cdot \alpha'(t) = A \cdot T_\alpha(t).$$

This implies that

$$\kappa_\beta = ||T'_\beta|| = ||A \cdot T'_\alpha(t)|| = ||T'_\alpha(t)|| = \kappa_\alpha$$

and so,

$$N_\beta = \frac{T'_\beta}{\kappa_\beta} = \frac{A \cdot T'_\alpha}{\kappa_\alpha} = A \cdot N_\alpha$$

$$\Rightarrow B_\beta = T_\beta \times N_\beta = (AT_\alpha \times AN_\alpha) = \pm A(T_\alpha \times N_\alpha)$$

Where the last equality comes from the cross product of orthogonal matrices, and the $\pm$ is determined by the determinant of A, which is ± 1. This gives us that $B_\beta = \pm A \cdot B_\alpha$, implying (from the Frenet formulas):

$$B'_\beta = \pm A \cdot B'_\alpha \Rightarrow -\tau_\beta N_\beta = \mp A \tau_\alpha N_\alpha = \mp \tau_\alpha N_\beta$$

$$\Rightarrow \tau_\beta = \pm\tau_\alpha$$

($\Leftarrow$) Assume that $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \pm\tau_\beta$. We need to show that there exists an isometry F such that $\beta = F \circ \alpha$. Fix $t_0 \in I$. Then, there exists an isometry F such that F maps the Frenet frame of α at $\alpha(t_0)$ to the Frenet frame of β at $\beta(t_0)$. Indeed, $F = T_{\beta(t_0)} \circ A \circ T_{-a}(t_0)$ where $A \in O(3)$ is the unique orthogonal matrix that maps the Frenet fields (at 0) to each other. Let $\overline{\alpha} = F \circ \alpha$. We claim that $\overline{\alpha} = \beta$, and we know that at $t_0 : \overline{\alpha}(t_0) = \beta(t_0)$ and

1. $T_{\overline{\alpha}}(t_0) = T_\beta(t_0)$
2. $N_{\overline{\alpha}}(t_0) = N_\beta(t_0)$
3. $B_{\overline{\alpha}}(t_0) = B_\beta(t_0)$

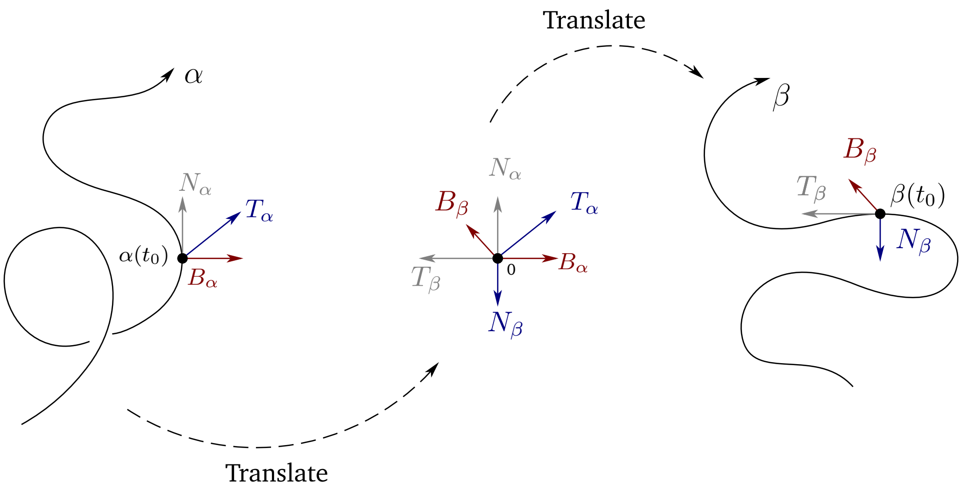


Figure 1.1: The idea is to translate the Frenet field to 0, to perform some kind of linear map that maps the translated Frenet field of α to what the Frenet field of β looks like at t_0, and then to translate the result to $\beta(t_0)$.

We have that $\kappa_{\overline{\alpha}} = \kappa_\alpha = \kappa_\beta$, and we first assume that $\tau_\alpha = \tau_\alpha = \tau_\beta$ at all $t \in I$. We want to show that $\overline{\alpha} = \beta : I \to \mathbb{R}^3$. It's enough to show that $T_{\overline{\alpha}} = T_\beta$ for all t, and that $\overline{\alpha}(t_0) = \beta(t_0) \Rightarrow \overline{\alpha} = \beta$ (from calculus. This discussion is in O'Neil's Book). To this end, define the following function $f : I \to \mathbb{R}$ as follows, $f(t) := \langle T_{\overline{\alpha}}(t), T_\beta(t)\rangle + \langle N_{\overline{\alpha}}(t), N_\beta(t)\rangle + \langle B_{\overline{\alpha}}(t), B_\beta(t)\rangle \in \mathbb{R}$. So,

$$f'(t) = \langle T'_{\overline{\alpha}}, T_\beta\rangle + \langle T_{\overline{\alpha}}, T'_\beta\rangle + \langle N'_{\overline{\alpha}}, N_\beta\rangle + \langle N_{\overline{\alpha}}, N'_\beta\rangle + \langle B'_{\overline{\alpha}}, B_\beta\rangle + \langle B_{\overline{\alpha}}, B'_\beta\rangle$$

Using the Frenet formulas,

$$= \langle \kappa_{\overline{\alpha}} N_{\overline{\alpha}}, T_\beta\rangle + \langle T_{\overline{\alpha}}, \kappa_\beta N_\beta\rangle + \langle -\kappa_{\overline{\alpha}} T_{\overline{\alpha}} + \tau_{\overline{\alpha}} B_{\overline{\alpha}}, N_\beta\rangle + \langle N_{\overline{\alpha}} - \kappa_\beta T_\beta + \tau_\beta B_\beta\rangle + \langle -\tau_{\overline{\alpha}} N_{\overline{\alpha}}, B_\beta\rangle + \langle B_{\overline{\alpha}}, -\tau_\beta N_\beta\rangle = 0$$

where the equality to 0 follows from the cancellation of all the previous terms. Also,

$$f(t_0) = \langle T_{\overline{\alpha}}(t_0), T_\beta(t_0)\rangle + \langle N_{\overline{\alpha}}(t_0), N_\beta(t_0)\rangle + \langle B_{\overline{\alpha}}(t_0), B_\beta(t_0)\rangle = 1 + 1 + 1 = 3$$

Recall that $\langle v, w\rangle = ||v|| \cdot ||w|| \cdot cos(\zeta)$, so $\langle T_{\overline{\alpha}}(t), T_\beta(t)\rangle \leq 1, \langle N_{\overline{\alpha}}(t), N_\beta(t)\rangle \leq 1, \langle B_{\overline{\alpha}}(t), B_\beta(t)\rangle \leq 1$, so $f(t) \leq 3$ with equality only when the angle between each component in the inner products is 0; when coliniarity is present. Since $\langle T_{\overline{\alpha}}(t), T_\beta(t)\rangle 1$, this tells us that $T_{\overline{\alpha}}(t) = T_\beta(t)$ for all t, and we have case (1). Now, when $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = -\tau_\beta$, we first reflect α to obtain $-\alpha : I \to \mathbb{R}^3$ (note here that $-\alpha = -1 \circ \alpha$, where -1 is the isometry that reflects the image of α). We calculate $T_{-\alpha} = -T_\alpha \Rightarrow \kappa_{-\alpha} = \kappa_\alpha$, and that $N_{-\alpha} = -N_\alpha$. Therefore, $B_{-\alpha} = T_{-\alpha} \times N_{-\alpha} = B_\alpha$, and so $B'_{-\alpha} = B'_\alpha$ and by Frenet, $-\tau_{-\alpha} \cdot N_{-\alpha} = -\tau_\alpha N_\alpha$ implies that $\tau_{-\alpha} = -\tau_\alpha$. Since $-\alpha$ and α are isometric, we can apply the previous case to $-\alpha$ instead of α. Hence, α is isometric to $-\alpha$, which is isometric to β. $\square$

1.12 Personal Addendum

I'm not surprised that we never formally introduced the chain rule, but I thought it might be best to formally state anyway: **The chain rule in higher dimensions:** Fix differentiable functions $f : \mathbb{R}^m \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, and a point $a \in \mathbb{R}^n$. Let $D_a g$ denote the total derivative at g at a and $D_{g(a)} f$ denote the total derivative of f at $g(a)$. These two derivatives are linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{R}^m \to \mathbb{R}^k$ respectively, so we can compose them - the chain rule for total derivatives says that their composite is the total derivative of $f \circ g$ at a:

$$D_a(f \circ g) = D_{g(a)} f \circ D_a g$$

2

Submanifolds of $\mathbb{R}^n$

2.1 The Inverse Function Theorem

Theorem 4. *Let U be an open subset of $\mathbb{R}^n$, $F : U \to \mathbb{R}^n$ be a map of class $\mathcal{C}^k$. If the Jacobian $DF(a)$ is invertible for some fixed $a \in U$, then there exist open sets $U_0, V_0 \subset \mathbb{R}^n$ such that $a \in U_0 \subset U$ and $F\big|_{U_0} : U_0 \to V_0$ is a $\mathcal{C}^k$-diffeomorphism, i.e., $F\big|_{U_o}$ is bijective and both $F\big|_{U_0}$ and $(F\big|_{U_0})^{-1}$ are $\mathcal{C}^k$ maps.*

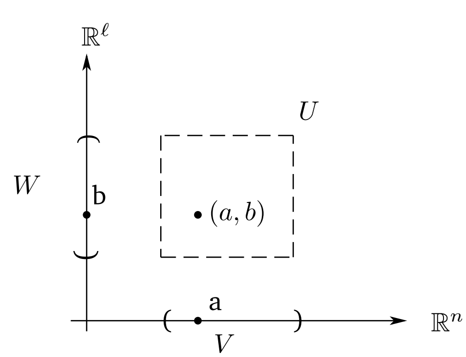
2.2 The Global Inverse Function Theorem

Corollary 5. *Let $U \subset \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^n$ be a $\mathcal{C}^k$ map. If F is injective and $DF(x)$ is invertible for all $x \in U$, then $F(U) \subset \mathbb{R}^n$ is open, and $F : U \to F(U)$ is a $\mathcal{C}^k$ diffeomorphism.*

Proof. $F : U \to F(U)$ is certainly bijective. For $x \in U$, there exist U_0, V_0 open subsets of $\mathbb{R}^n$ with $x \in U_0, F(x) \in V_0 \subset F(U) \subset \mathbb{R}^n$. This tells us that $F(U)$ is open. We also have that $(F\big|_{U_0})^{-1}$ is a $\mathcal{C}^k$ map. Since this is true for all $x \in U$, this gives us that $F^{-1} \in \mathcal{C}^k$. $\square$

2.3 The Implicit Function Theorem

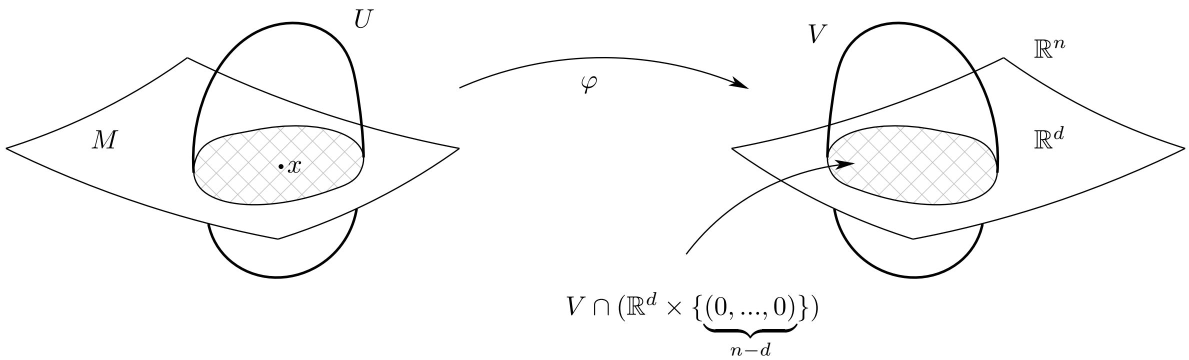
Theorem 6. *Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^\ell$ with coordinates $(x, y) = (x^1, ..., x^n, y^1, ..., y^\ell) \in \mathbb{R}^n \times \mathbb{R}^\ell$. Let $F : U \to \mathbb{R}^\ell$ be a map with $F \in \mathcal{C}^k$. Let $(a, b) \in U$, call $c = F(a, b)$, and assume that $(\frac{\partial F}{\partial y}(a, b)) \in \mathbb{R}^{n+\ell,\ell}$ is invertible. Then, there exist an open set $V \subset \mathbb{R}^n, W \subset \mathbb{R}^\ell$ with $a \in V, b \in W$ and there exists $(g : V \to W) \in \mathcal{C}^k$ such that $g(x) = y \iff F(x, y) = c$ for all $(x, y) \in V \times W$.*



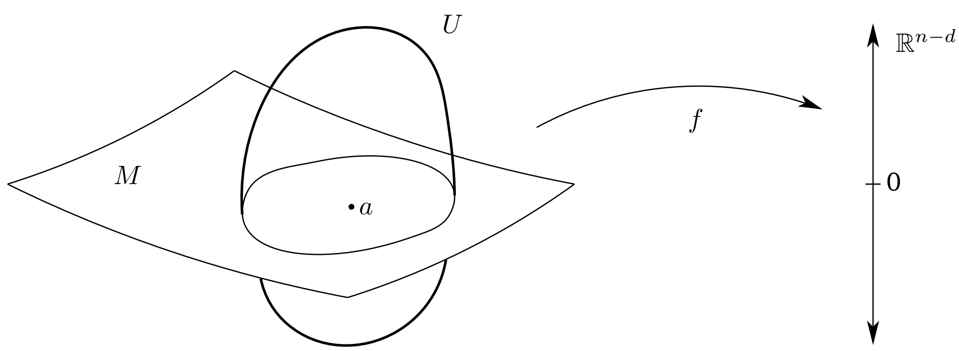
2.4 Equivalent Conditions for being a Submanifold

Claim. Let $M \subset \mathbb{R}^n$ and let d be a natural number, $d \leq n$. The following are equivalent:

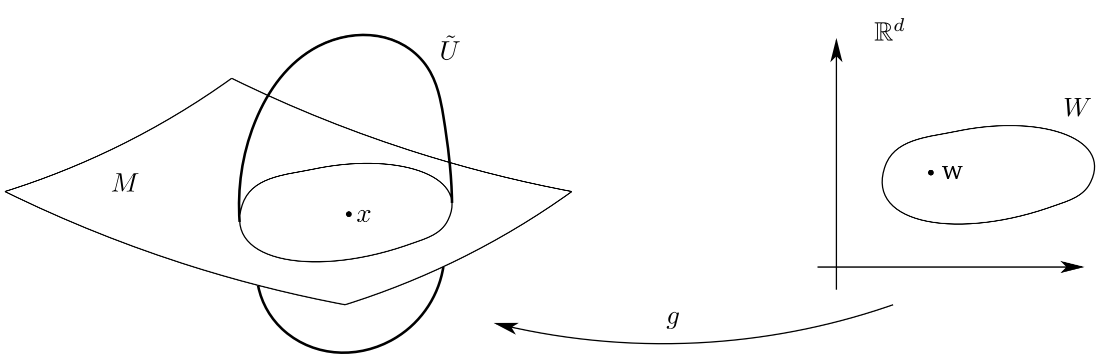
(a) For all $x \in M$, there exist open subsets U, V in $\mathbb{R}^n$ with $x \in U$ and there exists a map $\varphi : U \to V$ such that φ is a $\mathcal{C}^k$-diffeomorphism, and $\varphi(M \cap U) = V \cap (\mathbb{R}^d \times \{\underbrace{(0, ..., 0)}_{n-d}\})$.



(b) For all $x \in M$ there exists $U \subset \mathbb{R}^n$ with $x \in U$, and there exists a map $f : U \to \mathbb{R}^{n-d}$ such that $f \in \mathcal{C}^k$ and for all $x \in U$, $rank(Df(x)) = n - d$ and $M \cap U = \{x \in U | f(x) = 0\}$.



(c) For all $x \in M$, there exist $\tilde{U} \subset \mathbb{R}^n$ with $x \in \tilde{U}$ such that there exists an open set $W \subset \mathbb{R}^d$ and a map $g : W \to \tilde{U}$, $g \in \mathcal{C}^k$, such that for all $w \in W$, $rank(Dg(w)) = d$ and $g : W \to M \cap \tilde{U}$ is a homeomorphism.



If M satisfies any of the above conditions, we call M a $\mathcal{C}^k$**-submanifold of** $\mathbb{R}^n$ **of dimension** d. For $k = \infty$, we simply refer to M as a **smooth** manifold. The maps g in condition (c) are called the local parameterizations of M at x.

Proof. $(a \Rightarrow b)$: Denote $\varphi(x) = (\varphi^1(x), ..., \varphi^n(x))$, and let $f(x) := (\varphi^{d+1}, ..., \varphi^n(x))$. Since $\varphi \in \mathcal{C}^k$, it follows that $f \in \mathcal{C}^k$. Also, $f(x) = 0$ if and only if $\varphi(x) \in \mathbb{R}^d \times \{(0_1, ..., 0_{n-d})\}$, which by assuming condition (a) happens if and only if $x \in M \cap U$. Since φ is a $\mathcal{C}^k$ diffeomorphism, this tells us that $rank(D\varphi(x)) = n$ for all x, which implies that $rank(f(x)) = n - d$ for all $x \in U$.

$(b \Rightarrow c)$: We have that $rank(Df(x)) = n - d$, so without loss of generality, we may assume that

$$rank\left(\frac{\partial f^i}{\partial x^j}(x_0)\right)_{\substack{i=1...n-d \\ j=d+1...n}} = n - d$$

(after possibly rearranging the basis of $\mathbb{R}^n$). By the implicit function theorem, there exists an open set $W \subset \mathbb{R}^d$, and there exists an open subset $\tilde{W} \subset \mathbb{R}^{n-d}$ such that $(x_0^1, ..., x_0^d) \in W$, $(x_0^{d+1}, ..., x_0^n) \in \tilde{W}$, $w \in \tilde{W} \subset U$ and there exists a map $(\tilde{g} : W \to \tilde{W}) \in \mathcal{C}^k$ such that

$$\tilde{g}(x^1, ..., x^d) = (x^{d+1}, ..., x^n) \iff f(x^1, ..., x^n) = 0 \iff x \in M \cap (W \times \tilde{W}) \tag{2.4.1}$$

Where the last statement follows from assuming (b). Now, define

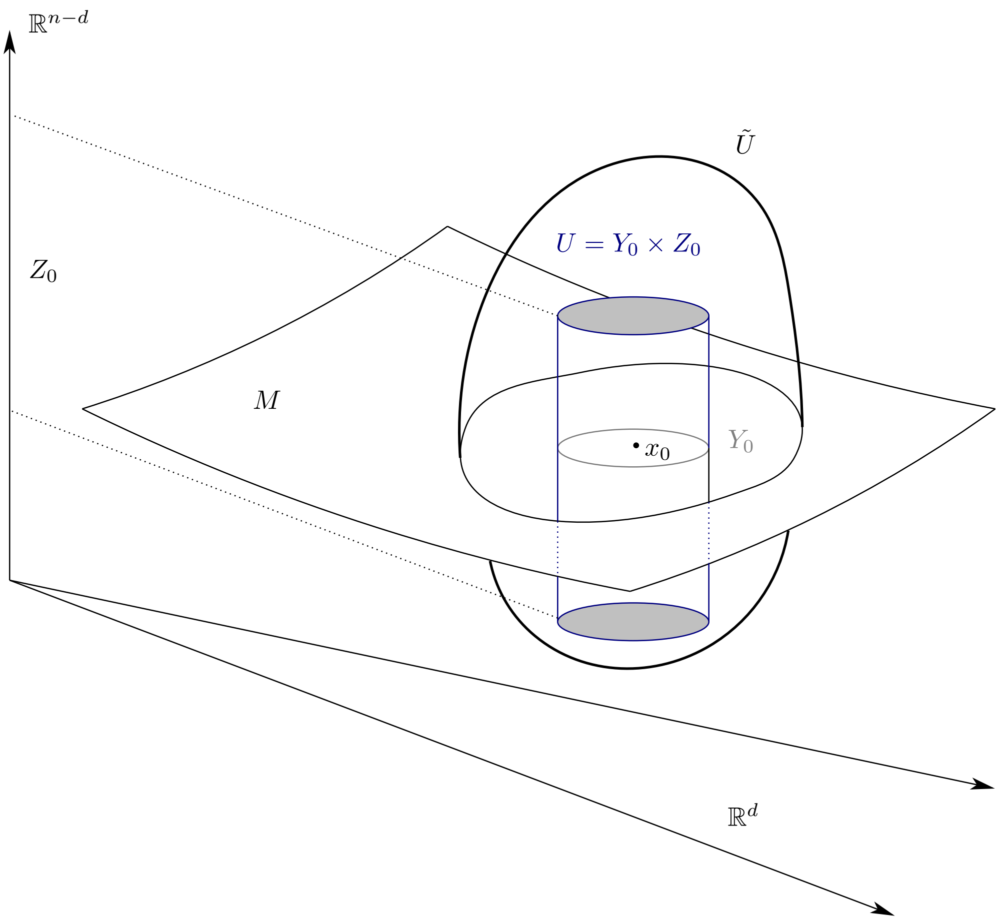
$$g : W \to \mathbb{R}^n; \qquad g(x^1, ..., x^d) := (x^1, ..., x^d, \tilde{g}(x^1, ..., x^d)).$$

Let $\tilde{U} = W \times \tilde{W}$, so $g : W \to \tilde{U}$, $g \in \mathcal{C}^k$ is 1-1, and

$$rank(Dg) = rank(D(id_{\mathbb{R}^d} \times \tilde{g})) = rank\left(\begin{array}{cc} \underbrace{\left[\begin{array}{cccc} 1 & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \\ \hline \frac{\partial \tilde{g}^1}{\partial x^1} & \cdots & \cdots & \frac{\partial \tilde{g}^d}{\partial x^1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tilde{g}^1}{\partial x^{n-d}} & \cdots & \cdots & \frac{\partial \tilde{g}^d}{\partial x^{n-d}} \end{array}\right]}_{\text{d}} & \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\1\end{array}}\right\} d \\ \left.\vphantom{\begin{array}{c}\frac{\partial}{\partial}\\ \vdots \\ \frac{\partial}{\partial}\end{array}}\right\} n-d \end{array} \end{array}\right) = d$$

Finally, $g(W) = M \cap \tilde{U}$ because of (2.4.1), and is onto. Thus, g is a continuous bijection, and $g^{-1}(x^1, ..., x^n) = (x^1, ..., x^d)$ is continuous, which implies that $g : W \to M \cap \tilde{U}$ is a homeomorphism.

$(c \Rightarrow a)$: Just to review, we are given that for any $x \in M$ there exists $\tilde{U} \subset \mathbb{R}^n$, $x \in \tilde{U}$ such that there exists an open set $W \subset \mathbb{R}^d$ and a $\mathcal{C}^k$ differentiable map $g : W \to \tilde{U}$, where $g : W \to M \cap \tilde{U}$ is a homeomorphism, and for all $w \in W$, $rank(Dg) = d$. We need to show that for all $x_o \in M$, there exist open sets $U, V \subset \mathbb{R}^n$ and a map $\varphi : U \to V, \varphi \in \mathcal{C}^k, \varphi(M \cap U) = V \cap (\mathbb{R}^d \times \{0\})$.



Let $x \in M \cap \tilde{U}$, and let $w_0 = g^{-1}(x_0)$. Without loss of generality, we can rearrange things such that

$$rank\left(\frac{\partial g^i}{\partial x^j}\right)_{\substack{i=1...d \\ j=1...d}} = d$$

Let $g = (g^1, ..., g^n)$, define $h = (g^1, ..., g^d)$, $k = (g^{d+1}, ..., g^n)$ so that $g = (h, k)$. Note that $rank(Dh) = d$, and also that $h, k \in \mathcal{C}^k$. By the inverse function theorem, there exists an open set $W_o \subset \mathbb{R}^d$, and an open set $Y_0 \subset h(W)$, and an open subset $Z_0 \subset \mathbb{R}^{n-d}$ such that $x_0 \in Y_0 \times Z_0 \subset \tilde{U}$, and $h|_{w_0} : W_0 \to h(W_0)$ is a $\mathcal{C}^k$ diffeomorphism. Let $U = Y_0 \times Z_0$. Define $\varphi : U \to \mathbb{R}^n$ as follows:

$$\varphi(x^1, ..., x^d, x^{d+1}, ..., x^n) := (h^{-1}(x^1, ..., x^d), (x^{d+1}, ..., x^n) - \underbrace{k(h^{-1}(x^1, ..., x^d)))}_{\in \tilde{U}}$$

Note that $\varphi \in \mathcal{C}^k$, because $h^{-1}, k \in \mathcal{C}^k$ and φ is also injective. Define $V := Image(\varphi)$. It isn't hard to check that $rank(D\varphi) = n$ for all $x \in Y_0 \times Z_0$. By the Global inverse function theorem, $\varphi(U) = V$ is open, and $\varphi : U \to \varphi(U)$ is a $\mathcal{C}^k$ diffeomorphism - we have shown the first part of what we'd like to show.

It remains to show that $\varphi(M \cap U) = \varphi(U) \cap (\mathbb{R}^d \times \{0\})$.

($\subset$) Assume that $y \in \varphi(M \cap U)$. This gives us that $y = \varphi(x)$, where $x \in M \cap U \subset M \cap \tilde{U}$. As such, $x = g(w)$ for some $w \in W$, so:

$$\begin{aligned} y = \varphi(x) &= \varphi(g(w)) = \varphi(h(w), k(w)) \\ &= (h^{-1}(h(w)), k(w) - k(h^{-1}(h(w)))) = (w, k(w) - k(w)) = (w, 0) \in \mathbb{R}^d \times \{0\}. \end{aligned} \quad (2.4.2)$$

We conclude that $\varphi(M \cap U) \subset \varphi(U) \cap (\mathbb{R}^d \times \{0\})$.

($\supset$) Now, take $y \in \varphi(U) \cap (\mathbb{R}^d \times \{0\})$. This tells us that $y = \varphi(x) \in \mathbb{R}^d \times \{0\}$, for some $x \in U$. This means that the second component of φ has to be 0;

$$\begin{aligned}(x^{d+1}, ..., x^n) - k(h^{-1}(x^1, ..., x^d)) = 0 \Rightarrow x = (x^1, ..., x^n) = (x^1, ..., x^d, k(h^{-1}(x^1, ..., x^d))) \\ = (h(h^{-1}(x^1, ..., x^d)), k(h^{-1}(x^1, ..., x^d))) = (h, k)(h^{-1}(x^1, ..., x^d)) = g(\underbrace{h^{-1}(x^1, ..., x^d)}_{\in W_0 \subset W})\end{aligned} \tag{2.4.3}$$

Which tells us that $x \in g(W) = M \cap U \Rightarrow y = \varphi(x)$ for $x \in M \cap U$.

$\square$

2.5 Examples

(a) $\mathbb{R}^d \times \{0\}$ is a submanifold of $\mathbb{R}^n$. More generally, every linear subspace V of $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$. More generally, every affine subspace $V + a$ (a linear subspace with the adjoinment of a point) is a submanifold of $\mathbb{R}^n$ of dimension $dim(V)$, and of class $\mathcal{C}^\infty$.

(b) Any open subset of a submanifold of $\mathbb{R}^n$ is again a submanifold of $\mathbb{R}^n$.

(c) Let W be an open subset of $\mathbb{R}^d$, and let $f : W \to \mathbb{R}^{n-d}$ with $f \in \mathcal{C}^k$. Then, $graph(f) := \{(x, f(x)) \in \mathbb{R}^n | x \in W\}$ is a d-dimensional submanifold of $\mathbb{R}^n$. This is true, because we can take g in condition (c) to be $g : W \to W \times \mathbb{R}^{n-d}$, $g(x) := (x, f(x))$.

(d) Let S^n be the n-sphere. Then, S^n is a submanifold of $\mathbb{R}^{n+1}$ of dimensional n. In showing this, use condition (b) with $f : U \to \mathbb{R}$, $U := \mathbb{R}^{n+1} - \{0\}$ is an open subset of $\mathbb{R}^{n+1}$, and $f(x) = ||x||^2 - 1 = (x^1)^2 + ... + (x^{n+1})^2 - 1$. Then, $Df(x) = (2x^1, ..., 2x^{n+1}) = 2x$. This tells us that $rank(Df) = 1 = (n+1) - n$. Clearly, $S^n \cap U = \{x \in U | f(x) = 0\}$.

(e) Define the cylinder $c = \{x \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1\}$, and use condition (b) where $f(x) = (x^1)^2 + (x^2)^2 - 1$.

(f) If $M \subset \mathbb{R}^n$, $M' \subset \mathbb{R}^n$ are submanifolds of $\mathbb{R}^n$, then $M \times M'$ is a submanifold of $\mathbb{R}^{2n}$ of where $dim(M \times M') = dim(M) + dim(M')$ (as a note, $M \cap M'$ doesn't work in general). To see this, use condition (c), where

$$g : W \to (M \cap \tilde{U}) \quad g' : W' \to (M' \cap \tilde{U}')\} \Rightarrow g \times g' : W \times W' \to (M \cap \tilde{U}) \times (M' \times \tilde{U}')$$

(g) $S^1 \subset \mathbb{R}^2 \Rightarrow S^1 \times \mathbb{R}^2$ is a submanifold of $\mathbb{R}^4$ of dimension 3. We call $S^1 \times S^1 \subset \mathbb{R}^4$ the **torus**, more specifically,

$$\underbrace{S^1 \times .... \times S^1}_{n}$$

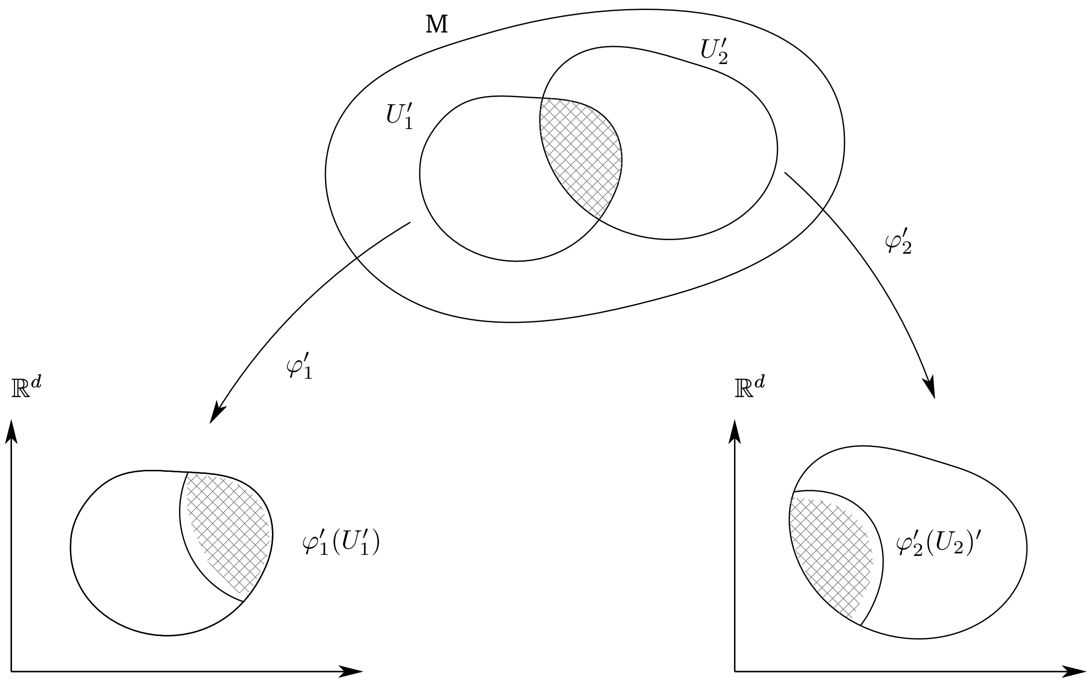
is called the n-**torus**.

As an exercise, give a condition on the map $f : M \cap U \to \mathbb{R}^{n-d}$ (for all $x \in M$) such that this guarantees that M is a product, $M = M_1 \times M_2$.

2.6 Lemma

Lemma 7. *Let M be a d-dimensional submanifold of $\mathbb{R}^n$. Let $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$ be as in section 2.4(a). Denote $U_1' = M \cap U_1, V_1'$ is the projection of V_1 to $\mathbb{R}^d$, $U_2' = M \cap U_2$, V_2' is the projection of V_2 to $\mathbb{R}^d$. Also denote by $\varphi_1' : U_1' \to V_1'$, $\varphi_2' : U_2' \to V_2'$. Note that φ_1', φ_2' do not have a notion of $\mathcal{C}^k$ differentiability, since U_1' and U_2' are not (necessarily) open in $\mathbb{R}^n$. However, we do have that* *(a)* $\varphi_i'(U_1' \cap U_2') \subset \mathbb{R}^d$ *is open for* $i = 1, 2$

(b) $\varphi_2' \circ (\varphi_1')^{-1}$ *is a* $\mathcal{C}^k$*-diffeomorphism.*



Proof.

(a) We have that

$$\varphi_i'(U_1' \cap U_2') = \varphi_i(U_1 \cap U_2 \cap M) = \varphi_i(U_1 \cap U_2) \cap (\mathbb{R}^d \times \{0\})$$

and as φ_i is a diffeomorphism, we have that $\varphi_i(U_1 \cap U_2)$ is open in $\mathbb{R}^n$, and as such the whole term on the right hand side of the equation is open in $\mathbb{R}^d$.

(b) We have that $\varphi_2' \circ (\varphi_1')^{-1}$ is a composition of $\mathcal{C}^k$ maps as follows:

$$\underbrace{\varphi_1'(U_1' \cap U_2')}_{\subset \mathbb{R}^d,\ open} \xrightarrow{include} \underbrace{\varphi_1(U_1 \cap U_2)}_{\subset \mathbb{R}^n,\ open} \xrightarrow{\varphi_1^{-1}} \underbrace{U_1 \cap U_2}_{\subset \mathbb{R}^n,\ open} \xrightarrow{\varphi_2} \underbrace{\varphi_2(U_1 \cap U_2)}_{\subset \mathbb{R}^n,\ open} \xrightarrow{project} \underbrace{\varphi_2'(U_1' \cap U_2' \cap M)}_{\subset \mathbb{R}^d.\ open}$$

Which implies that $\varphi_2' \circ (\varphi_1')^{-1} \in \mathcal{C}^k$, and $(\varphi_2' \circ (\varphi_1')^{-1})^{-1} = \varphi_1' \circ (\varphi_2')^{-1} \in \mathcal{C}^k$. We then have our claim. $\square$

2.7 $\mathcal{C}^k$ -curves

Definition. Let M be a d-dimensional submanifold of $\mathbb{R}^n$. Then, a $\mathcal{C}^k$ **curve on** M is a map $\alpha : I \to M$, where I is an open interval of $\mathbb{R}$, such that α is of class $\mathcal{C}^k$, as a map $\alpha : I \to \mathbb{R}^n$. A **tangent vector** of M at some point $a \in M$ is a vector $v \in \mathbb{R}^n$ such that there exists $\alpha : I \to M$, a $\mathcal{C}^k$-curve such that $\alpha(0) = a, \alpha'(0) = v$. The **tangent space** of M at a (denoted T_aM) is the set of all tangent vectors at a.

2.8 Submanifolds and Tangent Spaces

Proposition 8. *Let M be a submanifold of $\mathbb{R}^n$ of dimension d, and let $a \in M$.*

(a) If $f : U \to \mathbb{R}^{n-d}$ as in definition 2.4b, with $a \in U$, then $T_aM = Ker(Df(a))$.

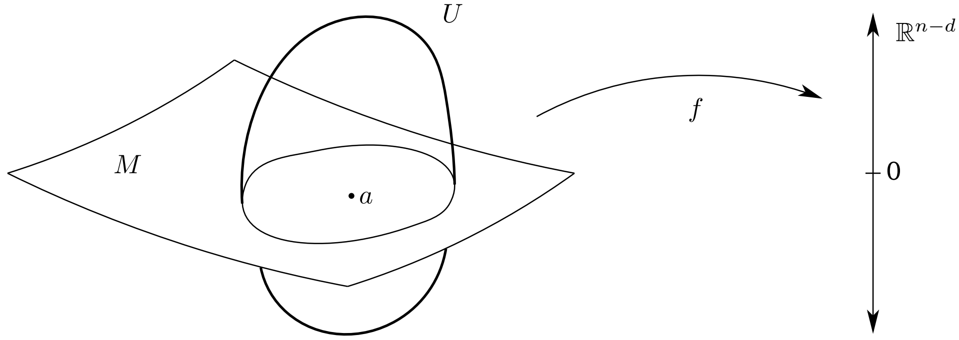


Figure 2.1: For all $x \in M$ there exists $U \subset \mathbb{R}^n$ with $x \in U$, and there exists a map $f : U \to \mathbb{R}^{n-d}$ such that $f \in \mathcal{C}^k$ and for all $x \in U$, $rank(Df(x)) = n - d$ and $M \cap U = \{x \in U | f(x) = 0\}$.

(b) If $g : W \to \tilde{W}$ as in definition 2.4c with $a \in \tilde{U}$, then $T_aM = Im(Dg(b))$ where $b = g^{-1}(a)$.

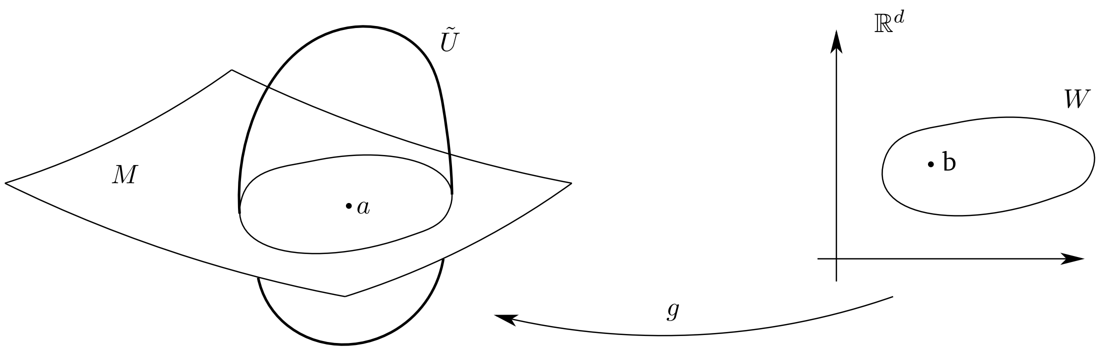


Figure 2.2: For all $x \in M$, there exist $\tilde{U} \subset \mathbb{R}^n$ with $x \in \tilde{U}$ such that there exists an open set $W \subset \mathbb{R}^d$ and a map $g : W \to \tilde{U}$, $g \in \mathcal{C}^k$, such that for all $w \in W$, $rank(Dg(w)) = d$ and $g : W \to M \cap \tilde{U}$ is a homeomorphism.

Proof. We have that $Df : \mathbb{R}^n \to \mathbb{R}^{n-d}$ is a linear map with $rank(Df(a)) = n - d$. This implies that $dim(Ker(Df(a))) = d$. Hence, it's enough to show that

$$Im(Dg(b)) \subset_{(1)} T_aM \subset_{(2)} Ker(Df(a))$$

1. Let $v = Dg(b) \cdot w$, for some $w \in \mathbb{R}^d$. Let $\epsilon > 0$ be such that $b + t \cdot w \in W$, $\forall t \in (-\epsilon, \epsilon)$. Then, define the curve $\alpha : (-\epsilon, \epsilon) \to M$ by, $\alpha(t) := g(b + t \cdot w)$. Then $\alpha \in \mathcal{C}^k, \alpha(0) = g(b) = a$, and by the chain rule,
$$v'(0) = Dg(b + 0 \cdot w) \cdot \frac{\partial(b + t \cdot w)}{\partial t},$$
but since $Dg(b + 0 \cdot w) \cdot \frac{\partial(b+t\cdot w)}{\partial t} = Dg(b) \cdot w = v$, this implies that $v \in T_aM$.

2. Let $v \in T_aM$, let $v = \alpha'(0)$, $\alpha : I \to M$ with $\alpha(0) = a$. WLOG, assumed that $\alpha(I) \subset U$. Since $\alpha(I) \subset M$ and $f(M) \cap U = 0$, this implies that $f(\alpha(t)) = 0$ for all $t \in I$. Taking derivatives, we have that
$$0 = Df(\alpha(0)) \cdot \alpha'(0) = Df(a) \cdot v^* \Rightarrow v \in Ker(Df(a))$$

□

2.9 Functions of class $\mathcal{C}^k$

Definition. Let M be a $\mathcal{C}^k$-submanifold of $\mathbb{R}^n$. Then, a function $f : M \to \mathbb{R}$ is **of class** $\mathcal{C}^k$ if, for all $x \in M$, there exists an open set $U \subset \mathbb{R}^n$ where $x \in U$ such that there also exists $\tilde{f} : U \to \mathbb{R} \in \mathcal{C}^k$ such that $\tilde{f}|_{U\cap M} = f|_{U\cap M}$. We denote the set of all such $\mathcal{C}^k$-functions by $\mathcal{C}^k(M, \mathbb{R})$.

2.10 A map $\mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$

Let $a \in M$, and let $v \in T_aM$. We define $D_v : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ by $D_v(f) := \frac{d}{dt} f \circ \alpha|_{t=0}$, where $f \circ \alpha$ is a function from $I \to \mathbb{R}$, for any $\mathcal{C}^k$ curve α with $\alpha(0) = a, \alpha'(0) = v$, and any $f \in \mathcal{C}^k(M, \mathbb{R})$. Then,

(a) D_v is well-defined

(b) D_v is linear, $D_v(f + g) = D_v(f) + D_v(g)$, $D_v(cf) = c \cdot D_v(f)$ for $f, g \in \mathcal{C}^k(M, \mathbb{R})$ and $\in \mathbb{R}$.

Proof.

(a) We need to show that D_v is independent of our choice of α. Chose an extension $\tilde{f} : U \to \mathbb{R}$, $a \in U$. Then,
$$D_v f = \frac{d}{dt} f \circ \alpha|_{t=0} = \frac{d}{dt} \tilde{f} \circ \alpha|_{t=0} = D\tilde{f}(\alpha(0)) \circ \alpha'(0) = D\tilde{f}(a) \cdot v$$
and for other choices of $\tilde{f}$ (eg, $\tilde{\tilde{f}}$), $f \circ \alpha = \tilde{f} \circ \alpha = \tilde{\tilde{f}} \circ \alpha$ is always the same function.
(b)
$$\frac{d}{dt}(cf + g) \circ \alpha|_{t=0} = c\frac{d}{dt} f \circ \alpha|_{t=0} + \frac{d}{dt} g \circ \alpha|_{t=0}$$
and
$$\frac{d}{dt}(fg) \circ \alpha = \frac{d}{dt}(f \circ g) \cdot (g \circ \alpha) = \frac{d}{dt}(f \circ \alpha) \cdot (g \circ \alpha) + (f \circ \alpha) \cdot \frac{d}{dt}(g \circ \alpha)$$
Setting $t = 0$, we have our claim. □

Remark. The properties in Lemma 2.6 and 2.10 will be used below to define general concepts of manifolds and their tangent spaces without referring to the ambient space $\mathbb{R}^n$.

*Note that this isn't 'multiplication', as someone pointed out in class; rather, it's the linear transform $Df(a)$ acting on v

3 Manifolds

3.1 Introductory Definitions

Definition. Let M be a set (we have no topology on this set right now).

(a) A ($\mathcal{C}^k$-) chart on M is a map $\varphi : U \to \mathbb{R}^d$ where $U \subset M$ such that

(i) φ is injective

(ii) $\varphi(U) \subset \mathbb{R}^d$ is open

(b) Two charts $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ are called **($\mathcal{C}^k$-) compatible** if:

(i) $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^d$

(ii) $\psi \circ \varphi^{-1} : \varphi(U \cap V) \xrightarrow{\varphi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V)$ is a $\mathcal{C}^k$-diffeomorphism.

(c) A ($\mathcal{C}^k$-) atlas $\mathscr{A}$ is a collection of charts $\mathscr{A} = \{\varphi_i : U_i \to \mathbb{R}^d | i \in I\}$ such that

(i) $\cup_{i \in I} U_i = M$

(ii) Any two charts $\varphi_i, \varphi_j \in \mathscr{A}$ are $\mathcal{C}^k$-compatible

(d) An atlas is called **maximal** if it is not properly contained in any other larger atlas (i.e., there does not exist any chart $\psi : V \to \mathbb{R}^n$ not in $\mathscr{A}$ such that ψ is compatible with all other charts of $\mathscr{A}$).

(e) A **($\mathcal{C}^k-$) manifold of dimension d** is a tuple $(M, \mathscr{A})$ where M is a set an $\mathscr{A}$ is an atlas on M. We often simply write M instead of $(M, \mathscr{A})$ with the understanding that $\mathscr{A}$ is also given.

3.2 Compatibility with an Atlas

Lemma 9. *Let M be a set and let $\mathscr{A} = \{\varphi_i : U_i \to \mathbb{R}^d\}$ be an atlas on M. Let $\varphi : U \to \mathbb{R}^d$ and $\psi : V \to \mathbb{R}^d$ be any two charts such that φ is compatible with any $\varphi_i \in \mathscr{A}$, and ψ is compatible with any $\varphi_i \in \mathscr{A}$. Then, ψ and φ are also compatible.*

Proof. * We have to show the two notions of compatibility: let $x \in U \cap V$. Since $\mathscr{A}$ is an atlas, this implies the existence of $\varphi_i : U_i \to \mathbb{R}^d \in \mathscr{A}$ such that $x \in U_i$. Then,

$$\varphi(x) \in \varphi(U_i \cap (U \cap V)) = \varphi \circ \varphi_i^{-1}(\varphi_i(U_i \cap U \cap V)) = \underbrace{(\varphi \circ \varphi_i^{-1})}_{\mathcal{C}^k-diffeomorphism} (\underbrace{\varphi_i(U_i \cap U)}_{\text{open in } \mathbb{R}^d} \cap \underbrace{\varphi_i(U_i \cap V)}_{\text{open in } \mathbb{R}^d}) \overset{open}{\subset} \mathbb{R}^d$$

*Before really jumping into this proof, note that $\mathcal{C}^k$-compatibility is *not* an equivalence relation (it isn't transitive).

This implies that $\varphi(x) \in \underbrace{\varphi(U_i \cap (U \cap V))}_{\text{open in } \mathbb{R}^d} \subset \varphi(U \cap V)$, which implies that $\varphi(U \cap V)$ is open in $\mathbb{R}^d$. Similarly, we find that $\psi(U \cap V)$ is open in $\mathbb{R}^d$.

Working on the second condition for compatibility, we have that

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to U \cap V \to \psi(U \cap V)$$

is bijective. Locally, there is a $\varphi_i : U_i \to \mathbb{R}^d, x \in U_i$, $\varphi_i \in \mathscr{A}$ so that

$$\begin{tikzcd}
\varphi(U_i \cap U \cap V) \arrow[r, "\varphi^{-1}"] \arrow[rr, bend left, "\in \mathcal{C}^k"] & U_i \cap U \cap V \arrow[r, "\varphi_i"] & \varphi_i(U_i \cap U \cap V) \arrow[r, "\varphi_i^{-1}"] \arrow[rr, bend right, "\in \mathcal{C}^k"'] & U_i \cap U \cap V \arrow[r, "\psi"] & \psi(U_i \cap U \cap V)
\end{tikzcd}$$

Which implies that $\psi \circ \varphi^{-1}|_{\varphi(U_i \circ (U \cap V))} \in \mathcal{C}^k$, so for all $x_i \in U \cap V$ there exists $\varphi_i : U_i \to \mathbb{R}^d$, and so we can remove the restriction in our domain and conclude that $\psi \circ \varphi^{-1} \in \mathcal{C}^k$. Similarly, we show that $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1} \in \mathcal{C}^k$. $\square$

3.3 Determining $\overline{\mathscr{A}}$

Corollary 10. *Let $\mathscr{A}$ be an atlas on M. Denote by $\overline{\mathscr{A}}$ the collection of all charts $\{\varphi : U \to \mathbb{R}^d | \varphi$ is compatible with all charts in $\mathscr{A}\}$. Then, $\overline{\mathscr{A}}$ is the unique maximal atlas covering $\mathscr{A}$. Thus, it is enough to specify any atlas even if it is non-maximal to determined the manifold $(M, \overline{\mathscr{A}})$.*

3.4 Examples

(a) Every d-dimensional submanifold of $\mathbb{R}^n$ is also a manifold in the sense of definition 3.1, because we can take the atlas $\mathscr{A} = \{\varphi_i' : U_i' \to V_i'\}$ as defined in Lemma 2.6. We proved that each φ_i' is a chart and the first qualification for compatibility follows from lemma 2.6 (a), and the second follows from lemma 2.6 (b).

(b) We know that $(\mathbb{R}, \mathscr{A} = \{id_{\mathbb{R}}\})$ is the standard $\mathcal{C}^\infty$ (smooth) structure on $\mathbb{R}$. But we also have non-standard smooth structures as follows: let $M = \mathbb{R}$, $\overline{\mathscr{A}} = \{\varphi\}$ where $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^3$. Note, $\mathscr{A}$ and $\overline{\mathscr{A}}$ are non-equivalent smooth structures because $\varphi \circ id_{\mathbb{R}} = x^3$ has inverse $x \mapsto x^{\frac{1}{3}}$ which is not smooth (in fact at 0, it isn't $\mathcal{C}^1$).

(c) The set of $m \times n$ matrices, $Mat(m \times n, \mathbb{R}) = \mathbb{R}^{m,n}$ can be identified with $\mathbb{R}^{m \cdot n}$, call this $\rho : \mathbb{R}^{m,n} \to \mathbb{R}^{m \cdot n}$ (which is a bijection), that looks like

$$\begin{bmatrix} a_{1,1} & & \\ & \ddots & \\ & & a_{m,n} \end{bmatrix} \mapsto (a_{1,1}, ..., a_{m,n})$$

This defines an atlas for the matrices $(\mathbb{R}^{m,n}, \mathscr{A} = \{\rho\})$.

(d) The general linear group $GL(n, \mathbb{R}) \subset \mathbb{R}^{n,m}$ is the set of matrices A with determinant $det(A) \neq 0$. Mapping $GL(n, \mathbb{R}) \xrightarrow{\rho} \mathbb{R}^{n^2}$, we see that $\rho(GL(n, \mathbb{R})) \subset_{open} \mathbb{R}^{n^2}$, because $det(A) : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous, implying that $det^{-1}(\mathbb{R} - \{0\})$ is an open subset of $\mathbb{R}^{n^2}$. This defines the manifold $GL(n, \mathbb{R}), \mathscr{A} = \{\rho|_{GL(n,\mathbb{R})}\}$.

(e) If $(M_1, \mathscr{A}_1)$ and $(M_2, \mathscr{A}_2)$ are two manifolds of dimension d_1, d_2 repectively, then

$$(M_1 \times M_2, \mathscr{A}_1 \times \mathscr{A}_2 = \{\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{d_1+d_2}\})$$

is a manifold of dimension $d_1 + d_2$. To see this, note that

$$(\varphi_1 \times \varphi_2)(U_1 \times U_2) = \underbrace{\varphi_1(U_1)}_{\text{open in } \mathbb{R}^{d_1}} \times \underbrace{\varphi_2(U_2)}_{\text{open in } \mathbb{R}^{d_2}} \subset_{open} \mathbb{R}^{d_1+d_2}$$

$$(\psi_1 \times \psi_2) \circ (\varphi_1 \times \varphi_2)^{-1} = (\psi_1 \circ \varphi_1^{-1}) \times (\psi_2 \circ \varphi_2^{-1})$$

(f) The real projective plane, denoted $\mathbb{R}P^n$ is defined as the set of lines in $\mathbb{R}^{n+1}$;

$$\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\}/\sim, \quad \text{where } x \sim y \iff \exists\, c \in \mathbb{R} | x = c \cdot y.$$

We define $\mathscr{A} = \{\varphi_j : U_j \to \mathbb{R}^n | j = 1, ..., n+1\}$ where

$$U_j = \{[x] \in \mathbb{R}P^n | x \in \mathbb{R}^{n+1} - \{0\}\},\ x^j \neq 0\} \subset \mathbb{R}P^n,$$

and point out that $\cup_{j=1}^n U_j = \mathbb{R}P^n$. For $x = (x^1, ..., x^{n+1})$,

$$\varphi_j([x]) = \left(\frac{x^1}{x^j}, \ldots, \frac{\hat{x^j}}{x^j}, \ldots, \frac{x^{n+1}}{x^j} \right) \in \mathbb{R}^n$$

where the $\hat{x^j}$ indicates the removal of the j^{th} coordinate. This map is well defined, because if we take $c \cdot x = (cx^1, ..., cx^{n+1})$, then

$$\varphi_j([cx]) = \left(\frac{cx^1}{cx^j}, \ldots, \frac{\hat{cx^j}}{cx^j}, \ldots, \frac{cx^{n+1}}{cd^j} \right) = \left(\frac{x^1}{x^j}, \ldots, \frac{\hat{x^j}}{x^j}, \ldots, \frac{x^{n+1}}{x^j} \right) = \varphi_j([x]) \in \mathbb{R}^n$$

Considering the requirements for being a chart, φ_j is clearly injective, and $\varphi_j(U_j)$ clearly open by the definition of φ_j.

Take $\varphi_j(U_j) = \mathbb{R}^n$ open, $\varphi_j^{-1}(x^1, \ldots, x^n) = [x^1, \ldots, 1_j, \ldots, x^n]$. Considering the first requirement for being compatible, we have that

$$\varphi_j(U_j \cap U_k) = \{x \in \mathbb{R}^n | x^k \neq 0 \text{ for } j > k \text{ or } x_{k+1} = 0 \text{ for } j < k\} \subset_{open} \mathbb{R}^n.$$

The second requirement for compatibility is as follows,

$$\varphi_k \circ \varphi_j^{-1}(x^1, .., x^n) = \varphi_k([x^1, \ldots, \underbrace{1}_{j^{th}}, \ldots, x^n]) = \left(\frac{x^1}{x^k}, \ldots, \underbrace{\frac{1}{x^k}}_{j^{th}}, \ldots, \frac{\hat{x^k}}{x^k}, \ldots, \frac{x^n}{x^k} \right)$$

This implies that $\varphi_k \circ \varphi_j^{-1}$ is a $\mathcal{C}^k$ morphism, and similarly we show that $(\varphi_k \circ \varphi_j)^{-1}$ is a $\mathcal{C}^k$ morphism.

3.5 Putting a topology on a manifold

Claim. Let M be a $\mathcal{C}^k$-manifold. We define a topology on M by calling $V \subset M$ open if and only if for each $x \in V$, there exists a chart $\varphi : U \to \mathbb{R}^d$ with $x \in U$, $\varphi \in \overline{\mathscr{A}}$ (the maximal atlas) and $U \subset V$. This defines a topology on M such that for each chart $\varphi \in \mathscr{A}$, $\varphi : U \to \varphi(U)$ is a homeomorphism.

Proof. We have to prove the three basic properties of a topology are held up by this proposal.

1. First, note that $\emptyset$ is open trivially, and M is open because $M = \cup_i U_i$, where U_i is the domain of a chart.

2. Let $V_j \subset M$ be open for all $j \in J$. We need to show that $\cup_j V_j$ is open: let $x \in \cup_{j\in J} V_j$, so $x \in V_{j_0}$. This implies that there exists $\varphi_{j_0} : U_{j_0} \to \mathbb{R}^d$, $x \in U_{j_0} \subset V_{j_0} \subset \cup_{j\in J} V_j$, implying that $\cup_{j\in J} V_j$ is open.

3. Lastly, let V_1 and V_2 be open, let $x \in V_1 \cap V_2$. This implies the existence of $\varphi_j \in \mathscr{A}, \varphi_j : U_j \to \mathbb{R}^d$, $j = 1, 2$ with $x \in U_j \subset V_j$. Define $\varphi := \varphi_1|_{U_1\cap U_2} : U_1 \cap U_2 \to \mathbb{R}^d$. It is enough to show that (i) φ is a chart, and (ii) φ is compatible with all other charts in $\mathscr{A}$ (this will prove the claim, since then we would have $x \in U_1 \cap U_2 \subset V_1 \cap V_2$, and $(\varphi|_{U_1\cap U_2} \to \mathbb{R}^d) \in \mathscr{A}$). In proving so, we do the following:

   (i) In showing that φ is a chart, we have to show that φ is injective, and $\varphi(U_1 \cap U_2)$ is open. The injectivity of φ is of course true, because φ_1 is injective (φ_1 is, after all, a chart). Also,

   $$\varphi(U_1 \cap U_2) = \varphi_1(U_1 \cap U_2),$$

   which is open since φ_1 and φ_2 are compatible. We conclude that φ is a chart.

   (ii) We have the following line of thought: let $\psi : V \to \mathbb{R}^d \in \mathscr{A}$. For the first compatibility condition,

   $$\varphi(V \cap (U_1 \cap U_2)) =^{\dagger} \underbrace{\varphi_1(V \cap U_1)}_{open,\ \varphi_1\ \&\ \psi\ compatible} \bigcap \underbrace{\varphi_1(U_1 \cap U_2)}_{open,\ \varphi_1, \varphi_2\ compatible} \subset_{open} \mathbb{R}^d$$

   and

   $$\psi(V \cap (U_1 \cap U_2)) = \underbrace{\psi(V \cap U_1)}_{open,\ \psi,\ \varphi_1\ compatible} \bigcap \underbrace{\psi(V \cap U_2)}_{open,\ \psi,\ \varphi_2\ compatible}$$

   so we have the first compatibility condition. Next, note that

   $$\psi \circ \varphi^{-1} = \underbrace{\psi \circ \varphi^{-1}|_{\varphi_1(V\cap(U_1\cap U_2))}}_{\mathcal{C}^k-diffeomorphism} : \underbrace{\varphi(V \cap (U_1 \cap U_2))}_{open} \longrightarrow \underbrace{\psi(V \cap (U_1 \cap U_2))}_{open}$$

   and as such, $\psi \circ \varphi^{-1}$ is a $\mathcal{C}^k$ diffeomorphism.

The second part of this proof is to prove that $\varphi : U \to \varphi(U)$ is a homeomorphism ‡ . Well, we have that $\varphi \in \mathscr{A}$ is injective, and so $\varphi : U \to \varphi(U)$ is bijective. We need to show that,

(1) φ is continuous

(2) φ^{-1} is continuous.

And we do so as follows:

(1) Let Y be an open subspace of $\varphi(U)$. We need to show that $\varphi^{-1}(Y)$ is open in the topology of M, so let $x \in \varphi^{-1}(Y)$. It is enough to show that $\varphi|_{\varphi^{-1}(Y)} : \varphi^{-1}(Y) \to Y$ is (i) a chart, and (ii), that φ is compatible with any $\psi \in \mathscr{A}$.

   (i) Since φ is injective, it follows that $\varphi|_{\varphi^{-1}(Y)}$ is injective. As Y is assumed open,

   $$\varphi|_{\varphi^1(Y)}(\varphi^{-1}(Y)) = Y \subset_{open} \mathbb{R}^d.$$

---

† The equality here follows from φ_1 being injective

‡ In the initial writing of this section, I was blatantly abusing terminology. It's worth mentioning that now, I think the point of this discussion was to remove ourselves from the notion of $\mathbb{R}^n$ being the ambient space of our manifold, and instead relying on the new topology we have shown to exist by part one of the proof to indicate the notion of homeomorphic spaces.

(ii) Let $\psi : V \to \mathbb{R}^d \in \mathscr{A}$. Then, for the first compatibility condition,

$$\psi(V \cap \varphi^{-1}(Y)) = \psi \circ \varphi^{-1} \circ \varphi(V \cap \varphi^{-1}(Y)) = \underbrace{\psi \circ \varphi^{-1}}_{\substack{\mathcal{C}^k \\ diffeomorphism}} \big(\underbrace{\varphi(U \cap V)}_{\substack{open; \\ \varphi \,\&\, \psi \text{ are} \\ compatible}} \cap \underbrace{Y}_{open}\big)$$

and so, we have a $\mathcal{C}^k$ map of an open set, which is then of course, open in $\mathbb{R}^d$. Similarly,

$$\varphi(V \cap \varphi^{-1}(Y)) = \underbrace{\varphi(U \cap V)}_{\substack{open; \\ \varphi \,\&\, \psi \text{ are} \\ compatible}} \cap \underbrace{Y}_{open} \subset_{open} \mathbb{R}^d.$$

For the second compatibility condition, we have that

$$\varphi|_{\varphi^{-1}(Y)} \circ \psi^{-1} = \underbrace{\varphi \circ \psi^{-1}}_{\substack{\mathcal{C}^k \\ diffeomorphism}} : \underbrace{\psi(\varphi^{-1}(Y) \cap V)}_{open} \longrightarrow \underbrace{\varphi(\psi^{-1}(Y) \cap V)}_{open}$$

which is clearly a $\mathcal{C}^k$ diffeomorphism. We do something similar when considering $\psi \circ \varphi^{-1}$.

(2) It remains to show that φ^{-1} is continuous. Let W be an open subspace of U. We need to show that $\varphi(W) \subset_{open} \mathbb{R}^d$. Let $y \in \varphi(W)$, $x = \varphi^{-1}(y)$. By the open-ness of W, there exists a chart $\psi \in \mathscr{A}$, $\psi : V \to \mathbb{R}^d$ such that $x \in V \subset W$. Since φ and ψ are compatible, $\varphi(V \cap U)$ is an open subspace of $\mathbb{R}^d$, and $y = \varphi(x) \in \underbrace{\varphi(U \cap V)}_{open} \subset \varphi(W)$, so $\varphi(W)$ is open in $\mathbb{R}^d$.

□

3.6 Restricting Charts to open subsets in M

Lemma 11. *Less formally, the points below tell us that open subspaces of a manifold are again, manifolds.*

(a) Let $\varphi : U \to \varphi(U) \in \mathscr{A}$. If $V \subset_{open} M$, then $\varphi|_V$ is a chart in $\mathscr{A}$.

(b) If $V \subset M$ open, then the collection $\mathscr{A}|_V = \{\varphi|_V | \varphi \in \mathscr{A}\}$ is an atlas for V, thus V is also a manifold.

Proof. For Homework, see the exercises at the end of the chapter. □

3.7 Examples and Definitions

(a) Let $M = \mathbb{R} \sqcup \{0^*\}$. Let $\mathscr{A} = \{\varphi_1, \varphi_2\}$ where $\varphi_1 : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, and $\varphi_2 : (\mathbb{R} - \{0\} \cup \{0^*\}) \to \mathbb{R}$,

$$\varphi_2(x) = \begin{cases} x, & x \neq 0 \\ 0, & x = 0^* \end{cases}$$

Then φ_1, φ_2 are compatible, because $\mathbb{R} \cap (\mathbb{R} - \{0\} \cap \{0^*\}) = \mathbb{R} - \{0\}$, which is open, and

$$\varphi_1 \circ \varphi_2^{-1}(x) = x, \quad \text{and} \quad \varphi_2 \circ \varphi_1^{-1}(x) = x \quad \forall x \in \mathbb{R} - \{0\}$$

and $\varphi_i(R \cap (\mathbb{R} - \{0\} \cup \{0^*\})) = \mathbb{R} - \{0\}$ for $i = 1, 2$. This implies that $(M, \{\varphi_1, \varphi_2\})$ is a manifold under this structure. As a note, notice that there do not exist disjoint open neighborhoods containing 0 and 0^* in this topology, and as such, the topology on M is not Hausdorff. **We will now restrict our discussion to manifolds that are Hausdorff.**

(b) Let $(M, \mathscr{A})$ be a manifold and let X be a set. Define a manifold structure on $M \times X$: for each chart $\varphi : U \to \mathbb{R}^d \in \mathscr{A}$ and $x \in X$, let $\varphi_x : U \times X \to \mathbb{R}^d$ by $\varphi_x(u, x) := \varphi(u)$. Then, $\tilde{\mathscr{A}} = \{\varphi_x | \varphi \in \mathscr{A}, x \in X\}$ is an atlas on $M \times X$, and if X is uncountable, then $M \times X$ is not second-countable [§]. **We will now restrict our discussion to manifolds that are second-countable.**

Definition. From here on in, a manifold is a tuple $(M, \mathscr{A})$ as in definition 3.1 such that the induced topology of M is Hausdorff and second-countable.

3.8 Compact sets and Manifolds

Proposition 12. *Let M be a manifold. Then,*

(a) All open sets $U \subset M$ are locally compact, i.e., for all $x \in U$ there exists a compact set $K \subset U$ such that $x \in K^0 \subset K \subset U$.

(b) M has an exhaustion by compact sets, i.e., there exist compact sets $K_1, K_2, ... \subset M$ such that $K_i \subset K_{i+1}$ and $M = \cup_i K_i$.

(c) M is para-compact, i.e., every open cover has a locally finite open refinement. I.e., for an open cover U of M, there exists an open refinement [¶] $\mathcal{V}$ of U such that $\mathcal{V}$ is locally finite (i.e., for all $x \in M$, there exists $W \subset_{open} M$ such that $x \in W$ and $V \cap W \neq 0$ for only finitely many $V \in \mathcal{V}$)

Proof.

(a) Let $U \subset M$ be open. Let $x \in U$, and let $\varphi : V \to \mathbb{R}^d$ be a chart with $x \in V \subset U$. Since $\varphi(V) \subset \mathbb{R}^d$ is open, there exists $B \subset_{open} \mathbb{R}^d$ such that $\varphi(x) \in B \subset \overline{B} \subset \varphi(V)$. Since φ is a homeomorphism, $\varphi^{-1}(\overline{B}) := K \subset U$, K compact, and $K^o = \varphi^{-1}(B)$ (because $K^0 = (\varphi^{-1}(\overline{B}))^0 = \varphi^{-1}(\overline{B}^o) = \varphi^{-1}(B)$). This implies that $x \in K^o \subset K \subset U$. Hence, we have local compactness.

(b) We want compact sets $K_1, K_2, ... | K_i \subset K_{i+1}, \cup_{i \in \mathbb{N}} K_i = M$. Let $\mathcal{B}$ be a countable base for M. Then, $\mathcal{B}' = \{B \in \mathcal{B} | \overline{B} \text{ is compact }\}$ is also a countable base for M, since for $x \in U \subset_{open} M$ by (q), there exists a compact set K such that $x \in K^o \subset K \subset U \Rightarrow \exists B \in \mathcal{B} : x \in B \subset K^o \subset K$, implying that $\overline{B} \subset K$ is compact (closed subsets of compact sets are compact), which implies that $B \in \mathcal{B}'$, with $x \in B \subset U$. We write $\mathcal{B}' = \{B_1, B_2, ...\}$. Then $K_1 = \overline{B}_1$, and by induction, we assume that we have $K_1, ..., K_n$ such that $B_i \subset K_i$ and $K_i \subset K_{i+1}$. Since K_n is compact, there exists some $s \in \mathbb{N}$ such that $K_n \subset B_1 \cup ... \cup B_s$. Let $t = max(s, n+1)$ and set $K_{n+1} := \overline{B}_1 \cup ... \cup \overline{B}_t$.

  (a) $n + 1 \leq t \Rightarrow B_{n+1} \subset K_{n+1}$
  (b) $K_n \subset B_1 \cup ... \cup B_s \subset B_1 \cup ... \cup B_t = K^o_{n+1}$

so we have $M = \cup_{i \in \mathbb{N}} B_i \subset \cup_{i \in \mathbb{N}} K_i \subset M$, as we wanted.

(c) Let $\mathcal{U}$ be an open cover of M. Let $K_1, K_2, ...$ be as in (b). Then, define $C_j := K_{j+1} - K^o_j$, which is compact, and let $W_j := K_{j+2} - K_{j-1}$, which is open, and note that $C_j \subset W_j$. Let B be any basis for the topology on M. We define a cover $\mathcal{C}_j$ for C_j by setting $\mathcal{C}_j := \{B \in \mathcal{B} | \exists x \in C_j, \exists U \in \mathcal{U} : x \in B \subset I \cap W_j\}$. Then $\mathcal{C}_j$ is a cover of C_j since $\mathcal{U}$ covers M and since $x \in W_j$ and $\mathcal{B}$ is a basis of M. Since C_j is compact, there exists a finite subcover $\mathcal{F}_j \subset \mathcal{C}_j$ which still covers C_j. Now set $\mathcal{V} = \cup_{i \in \mathbb{N}} F_i$. Note that $\mathcal{V}$ is a refinement of $\mathcal{U}$ (by the definition of $\mathcal{C}_j$). If $x \in M$, this implies there exists j such that $x \in K_{j+1} - K_j \subset K_{j+1} - K^o_j = C_j$. This implies there exists $W \in \mathcal{F}_j : x \in W \subset W_j$. Now, $W_j \cap W_\ell = \emptyset$ for all $\ell \neq j-2, ..., j+2$. Thus, we intersect non-trivially only finitely many of the open sets $V \subset \mathcal{F}_\ell \subset \mathcal{V}$. Since each $\mathcal{F}_k$ is finite, W_j intersects finitely many $V \in \mathcal{V}$.

□

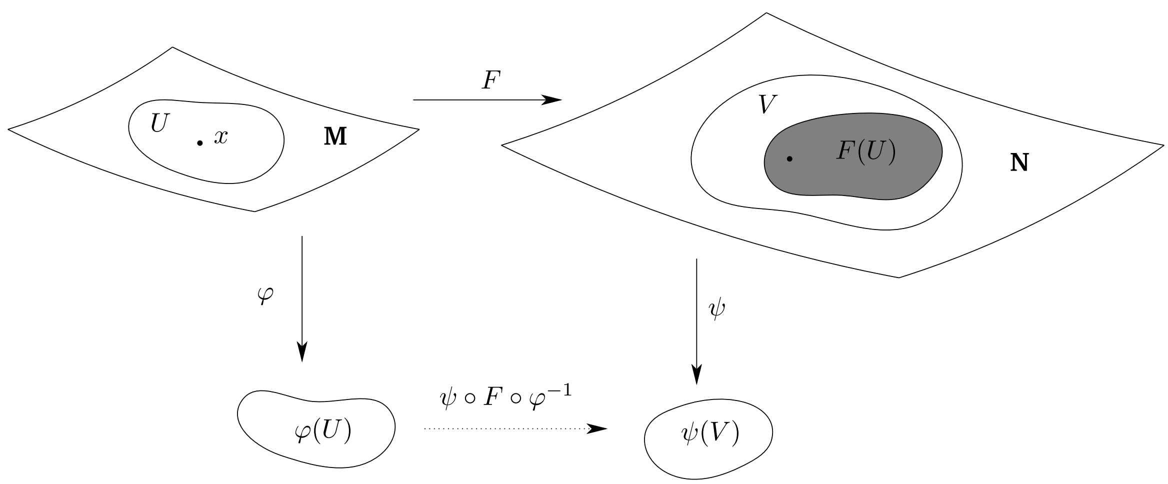
[§] Recall that Z is called **second countable** if and only if there exists a countable base of its topology, i.e., there exists a countable set $\mathcal{B} = \{B_i | B_i \subset_{open} Z, i \in \mathbb{N}\}$ such that for all open sets $U \subset Z$, there exists $N \in \mathbb{N}$ such that $U = \cup_{i \in N} B_i$. The main example of a second-countable space is $Z = \mathbb{R}^n$, with basis $\mathcal{B} = \{$ open balls of rational radius and center in $\mathbb{Q}^n\}$.

[¶] For all $V \in \mathcal{V}$, there exists $U \in U$ such that $V \subset U$

4 Maps between Manifolds

4.1 $\mathcal{C}^k$ maps

Definition. Let M and N be two $\mathcal{C}^k$-manifolds (not necessarily of the same dimension). A map $F : M \to N$ is called **of class** $\mathcal{C}^k$ if and only if for all $x \in M$ there exist charts $\varphi : U \to \varphi(U)$ of M, where $x \in U$, and a chart $\psi : V \to \psi(V)$ of N where $F(x) \in V$ and $F(U) \subset V$, and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a $\mathcal{C}^k$ map.



We write this as, $F \in \mathcal{C}^k(M,N)$. F is a $\mathcal{C}^k$ diffeomorphism, denoted $F \in Diff^k(M,N)$, if F is bijective and if $F \in \mathcal{C}^k(M,N)$ and $F^{-1} \in \mathcal{C}^k(M,M)$.

4.2 Properties of $\mathcal{C}^k$ maps

Lemma 13.

(a) If $F \in \mathcal{C}^k(M,N)$, then F is continuous.

(b) If $F \in \mathcal{C}^k(M,N)$, and $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ are any charts on M and N respectively, then $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is of class $\mathcal{C}^k$.

Proof. (a) If W be an open subset of N. We need to show that $F^{-1}(W)$ is an open subset of M. Let $x \in F^{-1}(W)$. Since $F \in \mathcal{C}^k(M,N)$, there exists $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ with $x \in U, F(x) \in V, F(U) \subset V$ where $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a $\mathcal{C}^k$ map. Since φ and ψ are homeomorphisms, this gives us that

$$F|_U = \underbrace{\psi^{-1}}_{continuous} \circ \underbrace{\left(\psi \circ F \circ \varphi^{-1}\right)}_{\in \mathcal{C}^k} \circ \underbrace{\varphi}_{continuous} \longrightarrow V$$

is continuous. Since W is an open subset of N, there exists $\tilde{\psi} : \tilde{V} \to \tilde{\psi}(\tilde{V})$ with $F(x) \in \tilde{V} \subset W$. Then,

$$(F|_U)^{-1}(V \cap \tilde{V}) = \underbrace{\varphi^{-1}}_{continuous} \circ \underbrace{\left(\psi \circ F \circ \varphi^{-1}\right)^{-1}}_{\in \mathcal{C}^k} \circ \underbrace{\psi(V \cap \tilde{V})}_{\substack{open,\ \psi\ \&\ \tilde{\psi} \\ compatible}}$$

which implies that $(F|_U)^{-1}(V \cap \tilde{V}) \subset_{open} F^{-1}(W)$ (recall that $F(x) \in V \cap \tilde{V}$), and as x is contained in an open set which itself is contained in $F^{-1}(W)$, it follows that $F^{-1}(W)$ is open.

(b) Now, let $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ be any charts of M and N, respectively. We need to show the following composition is a $\mathcal{C}^k$ map:

$$\varphi(U \cap F^{-1}(V)) \xrightarrow{\varphi^{-1}} U \cap F^{-1}(V) \xrightarrow{F} F(U) \cap V \xrightarrow{\psi} \psi(F(U) \cap V) \subset \psi(V)$$

Let $a \in \varphi(U \cap F^{-1}(V))$, and let $x = \varphi^{-1}(a)$. Also let $y = F(x) \in V$. Since $F \in \mathcal{C}^k(M,N)$, there exists $\tilde{\varphi} : \tilde{U} \to \tilde{\varphi}(\tilde{U})$, and $\tilde{\psi} : \tilde{V} \to \tilde{\psi}(\tilde{V})$, such that

$$(\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}) : \tilde{\varphi}(\tilde{U}) \to \tilde{\psi}(\tilde{V}) \in \mathcal{C}^k,$$

where $x \in \tilde{U}, F(x) \in \tilde{V}$, and $F(\tilde{U}) \subset \tilde{V}$. Then,

$$\psi \circ F \circ \varphi^{-1} = \underbrace{(\psi \circ \tilde{\psi}^{-1})}_{\in \mathcal{C}^k} \circ \underbrace{(\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1})}_{\in \mathcal{C}^k} \circ \underbrace{(\tilde{\varphi} \circ \varphi^{-1})}_{\in \mathcal{C}^k}$$

and so, $\psi \circ F \circ \varphi^{-1}|_{\varphi(U \cap \tilde{U} \cap F^{-1}(V))} \in \mathcal{C}^k$. As this works for an arbitrary $a \in \varphi(U \cap F^{-1}(V))$, we can change the restricted domain instead to $U \cap F^{-1}(V)$, implying that $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is of class $\mathcal{C}^k$.

□

4.3 Examples

(a) Let M be a manifold. Then, $id_M : M \to M \in \mathcal{C}^k(M,M)$.

(b) Let M, N be manifolds, and let $n \in N$. Let $F : M \to N$ denote the constant map $F(x) = n$ for all $x \in M$, then $F \in \mathcal{C}^k(M,N)$.

(c) Let M be a manifold, $\varphi : U \to \varphi(U)$ be a chart of M. Then, U has an induced manifold structure (*see* $3.6(b)$) and $\varphi(U)$, an open subset of $\mathbb{R}^d$, is a manifold $(\varphi(U), \{id_{\varphi(U)}\})$. Then, $\varphi : U \to \varphi(U)$ is a $\mathcal{C}^k$-diffeomorphism. In proving this note that

$$id \circ \varphi \circ \varphi^{-1} = id_{\varphi(U)}, \qquad \varphi \circ \varphi^{-1} \circ id = id_{\varphi(U)}$$

(d) Recall the manifolds $(\mathbb{R}, \{id_{\mathbb{R}})$ and $(\mathbb{R}, \{\varphi : x \to x^3\})$. The map $F : (\mathbb{R}, \{id\}) \to (\mathbb{R}, \varphi), F(x) = x^{1/3}$ is a smooth ($\mathcal{C}^\infty$) map. * This is actually almost trivially easy to see, we're in the situation where we have

*In fact, if $M = \mathbb{R}$, then any smooth structure on $\mathbb{R}$ is diffeomorphism to $(\mathbb{R}, \{id_{\mathbb{R}}\})$. More generally, any smooth structure on $\mathbb{R}^n$ with $n \neq 4$ is smoothly diffeomorphic to $(\mathbb{R}^n, \{id_{\mathbb{R}^n})$. In contrast, $\mathbb{R}^4$ has uncountably many smooth structures that are not diffeomorphic, called **fake** $\mathbb{R}^4$'s. Milno-Kerane showed that S^7 has exactly 28 non-diffeomorphic smooth structures.

$$\begin{tikzcd}[column sep=huge]
(\mathbb{R}, id_{\mathbb{R}}) \arrow[r, "x \mapsto x^{\frac{1}{3}}"] \arrow[d, "id_{\mathbb{R}}"] & (\mathbb{R}, \varphi) \arrow[d, "x \mapsto x^3"] \\
\mathbb{R} \arrow[r, dashed, "(id_{\mathbb{R}})^{-1} \circ (x^{\frac{1}{3}}) \circ (x^3) = id_{\mathbb{R}} \in \mathcal{C}^\infty"] & \mathbb{R}
\end{tikzcd}$$

(e) If $F \in \mathcal{C}^k(M, N)$ and $G \in \mathcal{C}^k(N, P)$, then $G \circ F \in \mathcal{C}^k(M, P)$. This follows from the fact that the composition of $\mathcal{C}^k$ maps is again, $\mathcal{C}^k$.

(f) If $F \in \mathcal{C}^k(M, N)$ and $\tilde{F} \in \mathcal{C}^k(\tilde{M}, \tilde{N})$, then $F \times \tilde{F} \in \mathcal{C}^K(M \times \tilde{M}, N \times \tilde{N})$. This follows from the fact that the product of $\mathcal{C}^k$ maps is again, $\mathcal{C}^k$.

(g) Let M, N be manifolds. Then, $proj_M : M \times N \to M$ and $proj_N : M \times N \to N$ are $\mathcal{C}^k$ maps.

Proof. Let $(x, y) \in M \times N$. Choose $\varphi : U \to \varphi(U), \psi : V \to \psi(V)$ such that $x \in U, y \in V$. Then, $proj_M(U \times V) \subset U$, and $\varphi \circ proj_M \circ (\varphi \times \psi)^{-1} = \underbrace{proj_{\varphi(U)}}_{\in \mathcal{C}^k} : \varphi(U) \times \psi(V) \to \varphi(U)$. □

(h) Let $F : M \to N \times P$, then $F \in \mathcal{C}^k(M, N \times P)$ if and only if $proj_N \circ F \in \mathcal{C}^k(M, N)$ and $proj_P \circ F \in \mathcal{C}^k(M, P)$.

(i) Let M be a submanifold of $\mathbb{R}^n$, and let $F : M \to \mathbb{R}$ where $(\mathbb{R}, \{id_{\mathbb{R}}\})$ has the canonical manifold structure. Then, the definitions for $\mathcal{C}^k(M, \mathbb{R})$ from definition 2.9 and 4.1 coincide [†].

Proof. $(2.9 \Rightarrow 4.1)$ Take $F : M \to \mathbb{R}$, $x \in M$ with the conditions conditions from definition 2.9, namely that there exists a map $\tilde{F} : U \to \mathbb{R} \in \mathcal{C}^k$ with $\tilde{F}|_{U \cap M} = F|_{U \cap M}$. Since M is a submanifold of $\mathbb{R}^n$, there exist $\tilde{U}, \tilde{V}$ which are open subsets of $\mathbb{R}^n$, and $\tilde{\varphi} : \tilde{U} \to \tilde{V}$ is a $\mathcal{C}^k$ diffeomorphism. We have that $x \in \tilde{U}$ and $\tilde{\varphi}(M \cap \tilde{U}) = \tilde{V} \cap (\mathbb{R}^d \times \{0\})$. By 3.4(a)[‡], $\varphi|_{\tilde{U} \cap M}$ is a chart of M. Then, with $\psi : \mathbb{R} \to \mathbb{R}$, $\psi = id_{\mathbb{R}}$, we have

$$id_{\mathbb{R}} \circ F \circ \varphi^{-1} = \underbrace{\tilde{F} \circ \tilde{\varphi}^{-1}}_{: \varphi(\tilde{U} \cap U) \to \mathbb{R} \in \mathcal{C}^k} \Big|_{\underbrace{\tilde{\varphi}(U \cap \tilde{U} \cap M)}_{\subset_{open} \mathbb{R}^d}} \in \mathcal{C}^k$$

$(4.1 \Rightarrow 2.9)$ Given definition 4.1, we have that $\varphi : U \to \varphi(U)$ (where of course, $U \subset M, \varphi(U) \subset \mathbb{R}^d$), is a chart such that

$$id_{\mathbb{R}} \circ F \circ \varphi^{-1} \in \mathcal{C}^k$$

Since M is a submanifold of $\mathbb{R}^n$, there exists $\tilde{\varphi} : \tilde{U} \to \tilde{V} \in \mathcal{C}^k$-diffeo (by 2.4(a)[§]) with $x \in \tilde{U}, \tilde{\varphi}(M \cap \tilde{U}) = \tilde{V} \cap \{\mathbb{R}^d \times \{0\}\}$. Since $\tilde{\varphi}|_{\tilde{U} \cap M}$ is also a chart of M as a manifold, we know that

$$\varphi \circ \tilde{\varphi}^{-1}|_{\tilde{\varphi}(\tilde{U} \cap U \cap M)}$$

is a $\mathcal{C}^k$ map. Let W be an open subset of $\mathbb{R}^{n-d}$, $\tilde{\tilde{V}}$ an open subset of $\mathbb{R}^d$ such that $\tilde{\tilde{V}} \times W \subset \tilde{\varphi}(\tilde{U} \cap U)$. Note that $\rho : \tilde{\tilde{V}} \times W \to \varphi(U)$ given by $\rho(x, w) = \varphi \circ \varphi^{-1}(x, 0) \in \mathcal{C}^k$. Take $\tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W) \subset_{open} \mathbb{R}^n$, then define

$$\tilde{F} : \tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W) \to \mathbb{R}$$

[†]Definition 2.9: Let M be a $\mathcal{C}^k$-submanifold of $\mathbb{R}^n$. Then, a function $f : M \to \mathbb{R}$ is **of class** $\mathcal{C}^k$ if, for all $x \in M$, there exists an open set $U \subset \mathbb{R}^n$ where $x \in U$ such that there also exists $\tilde{f} : U \to \mathbb{R} \in \mathcal{C}^k$ such that $\tilde{f}|_{U \cap M} = f|_{U \cap M}$.

[‡]Every d-dimensional submanifold of $\mathbb{R}^n$ is also a manifold in the sense of definition 3.1, because we can take the atlas $\mathscr{A} = \{\varphi_i' : U_i' \to V_i'\}$ as defined in Lemma 2.6.

[§]Definition [of a submanifold] 2.4(a): For all $x \in M$, there exist open subsets U, V in $\mathbb{R}^n$ with $x \in U$ and there exists a map $\varphi : U \to V$ such that φ is a $\mathcal{C}^k$-diffeomorphism, and $\varphi(M \cap U) = V \cap (\mathbb{R}^d \times \{\underbrace{(0, ..., 0)}_{n-d}\})$.

by

$$\tilde{F} = \underbrace{(F \circ \varphi^{-1})}_{\in \mathcal{C}^k} \circ \underbrace{\rho}_{\in \mathcal{C}^k} \circ \underbrace{\tilde{\varphi}}_{\in \mathcal{C}^k} |_{\tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W)} \in \mathcal{C}^k$$

and

$$\tilde{F}|_{M \cap \tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W)} = F \circ \varphi^{-1} \circ (\varphi \circ \tilde{\varphi}^{-1})|_{M \cap \tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W)} = F|_{M \cap \tilde{\varphi}^{-1}(\tilde{\tilde{V}} \times W)}$$

□

4.4 Partitions of Unity

Lemma 14. *Let $0 < r < s < \infty$ and let $x_0 \in \mathbb{R}^n$. Then, there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

1. $0 \leq f \leq 1$
2. $f|_{\overline{B}_r(x_0)} = 1$
3. $F|_{\mathbb{R} - B_s(x_0)} = 0$

*We call f a **bump function**.*

Proof. We proceed by completing 4 steps:

Step 1: Let $g : \mathbb{R} \to \mathbb{R}$,

$$g := \begin{cases} e^{-\frac{1}{x}}, x > 0 \\ 0, x \leq 0 \end{cases}$$

We claim that g is a smooth map. Clearly, g is smooth for all $x > 0$, and $x < 0$. It only remains to be shown that g is smooth at $x = 0$. First, we have to show (inductively) for $x > 0$ that the k^{th} derivative of g is

$$g^{(k)}(x) = p_k(x) \cdot \frac{e^{-\frac{1}{x}}}{x^{2k}}$$

where $p_k(x)$ is a polynomial of degree $\leq k$ (this is left as an exercise). We want to check that $\frac{d}{dx} g^{(k)}(0)$ exists and is equal to 0; we want

$$lim_{h \to 0} \frac{g^{(k)}(0 + h) - 0}{h} \stackrel{?}{=} 0$$

But we have that

$$\lim_{h \to 0^+} \frac{p^{(k)}(h) e^{-\frac{1}{h}} h^{2k} - 0}{h} = \lim_{h \to 0^+} p_k(h) \frac{e^{-\frac{1}{h}}}{h^{2k+1}} = {}^{\P} p_k(0) \cdot \lim_{h \to 0^+} \frac{e^{-\frac{1}{h}}}{h^{2k+1}} = 0$$

Which implies that $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.

Step 2: [This step covers the next case, of slightly higher complexity - where now $n = 1$, $x_0 = 0$, and $f : \mathbb{R} \to \mathbb{R}$ such that $0 \leq f \leq 1$, $f|_{\overline{B}_r(0)} = 1$, $F|_{\mathbb{R}^n - B_s(0)} = 0$.] Let

$$f(x) := \frac{g(s - x)}{g(s - x) + g(x - r)}$$

¶By L'Hopital's Rule

Since $g(x) \geq 0$ (since either $s-x > x$ or $x-r > 0$), this implies that $g(s-x)+g(x-r) \geq g(s-x)$, and so $1 \geq f(x)$ after dividing by the left hand side of the previous equation. As $g(x) \geq 0$, this tells us that $f(x) \geq 0$. For $x-r \leq 0$, this implies that$f(x) = \frac{g(s-x)}{g(s-x)+0} = 1$, and for $x \geq s$, we have that

$$s-x \leq 0 \Rightarrow f(x) = \frac{0}{0+g(x-r)} = 0$$

As such, $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. As a remark, note that our discussion above works for $x \geq 0$, to make this work for all x, simply let $f(x) = f(-x)$ for $x < 0$.

Step 3: [Now, we let $n \geq 1, x_0 = 0$] Let f be as in Step 2. Then, set $f_2 : \mathbb{R}^n \to \mathbb{R}$, $f_2(x) := f(|x|)$. It follows that f_2 has the required properties, f_2 is smooth since $|.|$ is smooth at $\mathbb{R}^n - \{0\}$, and $f_2(x) = 1$ in a neighborhood of 0.

Step 4: [Now we move up to the general case, for $n \geq 1$ and $x_0 \in \mathbb{R}^n$] Let $f_2 : \mathbb{R}^n \to \mathbb{R}$ be as in Step 3. Then, $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = f_2(x_0 - x)$ has the required properties.

□

4.5 Support of a Manifold, Partitions of Unity

Definition. Let M be a manifold.

(a) Let $f : M \to \mathbb{R}$. The set $supp(f) := \overline{\{x \in M | f(x) \neq 0\}}$ is called the **support** of f. By definition, the support of f is closed.

(b) A partition of unity on M is a collection of functions $\{\chi_i\}_{i \in I:}$, $\chi_i \in \mathcal{C}^\infty(M,\mathbb{R})$ such that

  (a) $0 \leq \chi_i(x) \leq 1, \forall i \in I, \forall x \in M$

  (b) $supp(\chi_i)$ is locally finite for all $i \in I$ (this means that for all $x \in M$, there exists an open subset $U \subset M$ with $x \in U$ such that $supp(\chi_i) \cap U \neq \emptyset$ for only finitely many χ_i).

  (c) $\sum_{i \in I} \chi_i(x) = 1$, for all $x \in M$ (this sum makes sense because it is actually a finite sum by the condition above).

(c) Let $\mathcal{U} = \{U_j\}_{j \in I}$ be an open cover of M and let $\{\chi_i\}_{i \in I}$ be a partition of unity. Then, we call the partition **subordinate** to U if for all $i \in I$ there exists $j \in J$ such that $supp(\chi_i) \subset U_j$.

4.6 Subordinate Partitions of Unity

Proposition 15. *Let $\mathcal{U} = \{U_j\}_{j \in I}$ be an open cover of M. Then, there exists a partition of unity $\{\chi_i\}_{i \in I}$ subordinate to $\mathcal{U}$.*

Proof. Let $K_1, K_2, ...$ be an exhaustion of M by compact sets as in proposition 3.7. This means that K_i is compact, $K_i \subset M$, $K_i \subset K_{i+1}^o$, and $M = \cup_i K_i$. Let $x \in M$, let $i_x = max\{i | x \in M - K_i\}$, and choose $j_x \in J$ such that $x \in U_{j_x}$. Then, $x \in U_{jX} \cap (K_{i_2}^o - K_{i_x}) \subset_{open} M$. Let $\varphi_x : V_x \to \mathbb{R}^d$ be a chart such that $V_x \subset U_{j_x} \cap (K_{i_2}^o - K_{i_x})$. Since $\varphi_x(V_x) \subset_{open} \mathbb{R}^d$, this implies the existence of $s > 0$ such that $\overline{B}_s(\varphi_x(x)) \subset \varphi(V_x)$. Let $f : \mathbb{R}^d \to \mathbb{R}$ such that $0 \leq f \leq 1, f|_{B_r(\varphi_x(x))} = 1$ for $r = \frac{s}{2}$, and $f|_{\mathbb{R}^d - B_s(\varphi_x(x))} = 0$. Then define $\psi_x : M \to \mathbb{R}$ as,

$$\psi_x(y) := \begin{cases} f \circ \varphi_x(y), & y \in V_x \\ 0, & else \end{cases}$$

Then $\psi_x \in \mathcal{C}^\infty(M,\mathbb{R})$, and ψ_x restricted to some neighborhood W_x is identically equal to 1, with

$$x \in W_x \subset supp(\psi_x) \subset \varphi_x^{-1}(\overline{B}_s(\varphi_x(x))) \subset V_x \subset U_{j_x} \cap (K_{i_2}^o - K_{i_x}) \tag{4.6.1}$$

Since $(K_{i+1} - K_i^o)$ is compact, we can cover it by finitely many $W_{x(i,1)}, ..., W_{x(i,r_i)}$. Define

$$W := \cup_{i=1}^{\infty}\{W_{x(i,1)}, ..., W_{x(i,r_i)}\}$$

Note that W is an open cover of M, and W is countable, so relabel these as $W = \{W_{x_1}, W_{x_2}, ...\}$. Let $\psi : M \to \mathbb{R}$ be given by

$$\psi(y) := \sum_{i=1}^{\infty} \psi_{x_i}(y)$$

Note that ψ is well-defined, since each $x \in M$ lies in some $x \in K_{i+2}^o - K_i$, and $supp(\chi_{x_i})$ intersects only finitely many of these $K_{i+2}^o - K_i$ by 4.6.1. Furthermore, $x \in W_{x_{i_0}}$ for some i_0, so that $\psi_{i_0}(x) = 1$, implying that $\psi(x) \geq 1$. Then, we define the partition of unity $\chi_i : M \to \mathbb{R}$ by $\chi_i(y) = \frac{\psi_{x_i}(y)}{\psi(y)} \in \mathcal{C}^\infty(M, \mathbb{R})$, and $0 \leq \chi_j \leq 1$, $supp(\chi_x) = supp(\psi_{x_i})$ is locally finite, and

$$\sum_{i=1}^{\infty} \chi_x(x) = \sum_{i=1}^{\infty} \psi_{x_i}(x)/\psi(x) = \frac{\sum_{i=1}^{\infty} \psi_{x_i}(x)}{\psi(x)} = \frac{\psi(x)}{\psi(x)} = 1 \forall x \in M$$

□

4.7 Smooth bump functions on manifolds

Corollary 16. *Let M be a manifold, let $x \in M$. Then let $A \subset M$ be a closed subset, $U \subset M$ be open such that $x \in A \subset U \subset M$. Then, there exists a smooth bump function $f \in \mathcal{C}^\infty(M, \mathbb{R})$ such that*

1. $0 \leq f \leq 1$
2. $supp(f) \subset U$
3. $f|_A = 1$

Proof. Let $\mathcal{U} = \{U, M - A\}$. Note that U is an open cover of M, which implies that there exists a partition of unity subordinate to $\mathcal{U}$: $\{\chi_i^I, \chi_j^{M-A}\}$ such that $supp(\chi_i^U) \subset U$, $supp(\chi_j^{M-A}) \subset M - A$. Let $f : M \to \mathbb{R}$ be the following smooth function,

$$f(x) := \sum_i \chi_i^U(x) \in \mathcal{C}^\infty(M, \mathbb{R})$$

This function satisfies,

1. $0 \leq f \leq 1$
2. $supp(\chi_i^U) \subset U \Rightarrow supp(f) \subset U$
3. $f|_A = 1 - \underbrace{\sum_j \chi_j^{M-J}}_{0 \ on \ A} = 1$, since $supp(\chi_j k^{M-A}) \subset M - A$.

□

5 The Tangent Space

This chapter is modeled after chapter 3 in Lee's book. We now restrict to $\mathcal{C}^\infty$ manifolds.

5.1 Definition

Definition. Let M be a $\mathcal{C}^\infty$ manifold, and denote (as usual), by $\mathcal{C}^\infty(M, \mathbb{R})$ the set of smooth functions from M to $\mathbb{R}$. Let $a \in M$. Then we define the **tangent space** of M at a to be the following:

$$T_a M := \{v : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R} | (1)\ v \text{ is linear, } (2)\ v \text{ is a derivation}\}$$

Explicitly, for $v \in T_a M$, our conditions are:

1. For all $r, s \in \mathbb{R}$, for all $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, $v(r \cdot f + s \cdot g) = r \cdot v(f) + s \cdot v(g)$.
2. For all $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, we have that

$$v_a(fg) = v_a(f) \cdot g(a) + f(a) \cdot v_a(g)$$

Remark. $T_a M$ is a $\mathbb{R}$-vector space, because we can define,

$$(rv)(f)_a := r \cdot v(f) \Rightarrow rv \in T_a M$$

which satisfies our two conditions above, and we have

$$(v + w)(f) := v(f) + w(f) \Rightarrow v + w \in T_a M$$

which also satisfies our two conditions, and finally satisfies the conditions for being an $\mathbb{R}$-vector space.

5.2 Properties

Lemma 17. *Let $v \in T_a M$.*

(a) If $c : M \to \mathbb{R}$ is a constant function, $c(x) = c$ for all $x \in M$, then $v(c) = 0$.

(b) If $f(a) = g(a) = 0$, then

$$v(fg) = 0$$

(c) Let $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ such that there exists an open neighborhood $U \subset M$ with $a \in U$ such that $f|_U = g|_U$, then $v(f) = v(g)$.

Proof. (a) First, if $c = 1 : M \to \mathbb{R}, x \mapsto 1$, then

$$v(1) = v(1 \cdot 1) = v(1) \cdot 1 + 1 \cdot v(1) = 2v(1) \Rightarrow v(1) = 0$$

(b) Similarly,

$$v(fg) = v(f)g(a) + f(a)v(g) = 0 + 0 = 0$$

(c) By Cor(4.7) with $A := \{a\} \subset_{closed} M$, $A \subset U \subset_{open} M$. Therefore, there exists a bump function $\chi \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\chi(a) = 1$, and $\chi|_{M-U} \equiv 0$. Note that $f(a) - g(a) = 0$, and $1 - \chi(a) = 0$. Therefore, by part (b),

$$v(0) = 0 = v((f-g)(1-\chi)) = v((f-g) - (f-g)\chi),$$

and since $f - g = 0$ on U, and $\chi = 0$ on $M - U$, the term $(f-g) \cdot \chi$ is identically 0 on all of M, so

$$0 = v(f-g) = v(f) - v(g)$$

and therefore, $v(f) = v(g)$.

$\square$

5.3 Differentials and Push forwards.

Definition. Let $(F : M \to N) \in \mathcal{C}^\infty(M, N)$.

(a) There is an induced map $F^* : \mathcal{C}^\infty(N, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})$ given by

$$F^*(f) := f \circ F$$

where $f \in \mathcal{C}^\infty(N, \mathbb{R})$. The picture of this is in Lee's book, chapter 3, within the first few pages.

(b) For $a \in M$, there is an induced map $dF_a : T_aM \to T_{F(a)}N$, which maps a linear derivation v to $dF_a(v)$, where for $f \in \mathcal{C}^\infty(N, \mathbb{R})$, $v \in T_aM$,

$$(\underbrace{dF_a(v)}_{\in T_{F(a)}N})(f) := v(F^*(f)) = v(f \circ F)$$

(check for yourself that this makes sense, v is a map from $\mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$, and since $f \in \mathcal{C}^\infty(N, \mathbb{R})$ and $F \in \mathcal{C}^\infty(M, N)$, $f \circ F$ is in $\mathcal{C}^\infty(M, \mathbb{R})$, and we now define a new map $dF_a(v) : \mathcal{C}^\infty(N, \mathbb{R})$ by taking something in $\mathcal{C}^\infty(N, \mathbb{R})$, a smooth map between M and N, and a map $v : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$ to get something that maps $\mathcal{C}^\infty(N, \mathbb{R}) \to \mathbb{R}$).

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, mapsto, "f\circ F"'] & N \arrow[dl, "f"] \\ \mathbb{R} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^\infty(M,\mathbb{R}) \arrow[d, mapsto, "v_a"'] & \mathcal{C}^\infty(N,\mathbb{R}) \arrow[l, "F^*"'] \arrow[dl, "dF_a(v)"] \\ \mathbb{R} & \end{tikzcd}$$

We also use the notation $dF_a = dF = F_*$, and it is called the **differential of** F **at a**. Note that $dF_a(v)$ is indeed in $T_{F(a)}N$, because for $r, s \in \mathbb{R}$ and $f, g \in \mathcal{C}^\infty(N, \mathbb{R})$,

(1)

$$\begin{aligned} dF_a(v)(rf + sg) &= v((rf + sg) \circ F) = v(r \cdot (f \circ F) + s \cdot (g \circ F)) \\ &= rv(f \circ F) + sv(g \circ F) = rdF_a(v)(f) + sdF_a(v)(g) \end{aligned}$$

(2)

$$\begin{aligned} dF_a(v)(f \cdot g) = v((f \cdot g) \circ F) = v((f \circ F)(g \circ F)) &= v(f \circ F)(g \circ F)(a) + (f \circ F)(a) \cdot v(g \circ F) \\ &= dF_a(v)(f) \cdot g(F(a)) + f(F(a)) \cdot dF_a(v)(g) \end{aligned}$$

5.4 Properties of F_*

Let $F \in \mathcal{C}^\infty(M, N), G \in \mathcal{C}^\infty(N, P)$.

(a) $dF_a : T_aM \to T_{F(a)}N$ is linear.

(b) $d(G \circ F)_a = dG_{F(a)} \circ dF_a : T_aM \to T_{F(a)}N \to T_{G(F(a))}P$.

(c) $d(id_M) = id_{T_aM}$ for all manifolds M and $a \in M$.

(d) If $F \in Diff^\infty(M, N)$ *, then dF_a is invertible, and

$$(dF_a)^{-1} = d(F^{-1})_{F(a)} : T_{F(a)}N \to T_aM.$$

(e) If there exists $a \in U$, which is an open subset of M, and $F(a) \in V \subset_{open} N$ such that $F|_U : U \to V$ is a $\mathcal{C}^\infty$ diffeomorphism, then $dF_a : T_aM \to T_{F(a)}N$ is an isomorphism. †

Proof. We proceed as follows:

(a) We have that,

$$dF_a(rv + sw)(f) = (rv + sw)(f \circ F) = rv(f \circ F) + sw(f \circ F) = rdF_a(v)(f) + s \cdot dF_a(w)(f),$$

and so for all $f \in \mathcal{C}^\infty(N, \mathbb{R})$, $dF_a(rv + sw) = rdF_a(v) + sdF_a(w)$.

(b) We apply, for $v \in T_aM$ and $f \in \mathcal{C}^\infty(P, \mathbb{R})$,

$$(dG_{F(a)} \circ dF_a)(v)(f) = dF_a(v)(f \circ G) = v(f \circ G \circ F) = d(G \circ F)_a(v)(f)$$

which gives us (b).

(c) We have,

$$d\ id_a(v)(f) = v(f \circ id) = v(f)$$

and so, $v \in T_aM \mapsto v$, which is (c).

(d) Because F is a diffeomorphism, $F \circ F^{-1} = id$, using (c) we have that

$$dF_a \circ dF^{-1}_{F(a)} = d(F \circ F^{-1})_{F(a)} = d\ id_{F(a)} = id_{T_{F(a)}N}.$$

and similarly,

$$dF^{-1}_{F(a)} \circ dF_a = id_{T_aM}$$

and so therefore, this shows that $(dF_a)^{-1} = dF^{-1}_{F(a)}$.

(e) First note that we only need to show that the inclusions $i_U : U \hookrightarrow M$ and $i_V : V \hookrightarrow N$ are maps whose differentials d_{i_U} and d_{i_V} are isomorphisms, because

$$F \circ i_U = i_V \circ F|_U : U \to N$$

This gives us that $dF \circ di_U = di_V \circ dF|_U$, which would give us that

$$dF = di_V \circ dF|_U \circ (di_U)^{-1}$$

is an isomorphism, since $F|_U$ is an isomorphism from (d). We now show that for $U \subset_{open} M$, $i_U : U \to M$ gives an isomorphism $di_U : T_aU \to T_aM$. We will show that di_U is (1) injective, and (2) surjective.

*Recall that this means that F is a bijection, $F \in \mathcal{C}^\infty(M, N)$, and $F^{-1} \in \mathcal{C}^\infty(N, M)$

†As a good exercise, consider $F : \mathbb{R} \to S^1$ in the normal covering space way, and apply this property. Which are the neighborhoods on which this works?

Lemma 18. *Let $f \in \mathcal{C}^\infty(U, \mathbb{R})$, and $a \in U$. Then, there exists a function $\tilde{f} : \mathcal{C}^\infty(M, \mathbb{R})$, and there exists $V \subset_{open} U$ such that $a \in V$ and $f|_V = \tilde{f}|_V$.*

We'll use this lemma to finish the proof of our property, then actually prove the lemma. We have to show (1) injectivity, and (2) surjectivity:

(1) **Injectivity:** let $di_U(v) = 0$ for some $v \in T_aU$, we would like to show that $v = 0$. By our assumption, $di_U(v)(f_0) = 0$ for all $f_0 \in \mathcal{C}^\infty(M, \mathbb{R})$ - but we still need to show that for any $f \in \mathcal{C}^\infty(U, \mathbb{R})$, $v(f) = 0$. Using our lemma, there exists $\tilde{f}$ and V such that $\tilde{f}|_V = f|_V$, and then $v(f) = v(\tilde{f}|_U)$ because $f, \tilde{f}$ coincide on V (see 5.2(c)[‡]). Finishing this line of thought,

$$v(f) = v(\tilde{f}|_U) = di_U(v)(\tilde{f}) = 0$$

and it follows that $v = 0$, as $v(f) = 0$ for all $f \in \mathcal{C}^\infty(U, \mathbb{R})$.

(2) **Surjectivity:** let $w \in T_aM$. We need to show that there exists $v \in T_aU$ such that $di_U(v) = w$. We define this $v \in T_aU$ as follows, by setting $v(f) := w(\tilde{f})$ where $\tilde{f} \in \mathcal{C}^\infty(M, \mathbb{R})$ comes from our lemma above. Note that v is well defined, since by 5.2(c), it is independent of of the chosen extension $\tilde{f}$. Also, v satisfies 5.1 (1) and (2), because w does. Finally:

$$di_U(v)(f) = v(f \circ i_U) = w(\tilde{f} \circ i_U) = w(f),$$

since f and $\tilde{f}|_U$ coincide on some neighborhood, V, and 5.2(c) assures us that if f and $\tilde{f}$ coincide on a neighborhood, then $w(f) = w(\tilde{f})$.

□

Now, we prove the lemma.

Proof. Let $f \in \mathcal{C}^\infty(U, \mathbb{R})$, and let $a \in M$. We need an $\tilde{f} \in \mathcal{C}^\infty(M, \mathbb{R})$ and a $V \subset_{open} U$ such that

$$\tilde{f}|_V = f|_V$$

Let $\varphi : W \to \mathbb{R}^d$ be a chart with $a \in W$. Let B_r be a ball centered at $\varphi(a)$ such that $B_r \subset \varphi(W)$ of radius r. Let $B_{r/2}$ be the ball of radius $\frac{r}{2}$ so that $\overline{B_{r/2}} \subset B_r$. By lemma 4.4, there exists a bump function $g : \mathbb{R}^d \to \mathbb{R}$ such that $g|_{\overline{B_{r/2}}} = 1$, and $g|_{\mathbb{R}^d - B_r} = 0$. Let $V := \varphi^{-1}(B_{r/2})$, and define $\tilde{f} : M \to \mathbb{R}$ by

$$\tilde{f} = \begin{cases} f(x) \cdot g(\varphi(x)) & x \in W \\ 0 & else \end{cases}$$

Then,

$$\tilde{f}|_V = \tilde{f}|_{\varphi^{-1}(B_{r/2})} = f|_V \cdot \underbrace{g \circ \varphi|_{\varphi^{-1}(B_{r/2})}}_{=1} = f|_V.$$

We now only need to show that $\tilde{f} \in \mathcal{C}^\infty(M, \mathbb{R})$.

Case 1: If $x \in W$, then using the chart $\varphi : W \to \varphi(W)$ shows that

$$id_{\mathbb{R}} \circ \tilde{f} \circ \varphi^{-1}|_{\varphi(W)} = \underbrace{f(\varphi^{-1}|_{\varphi(W)})}_{\in \mathcal{C}^\infty,\ 4.2(b)} \cdot \underbrace{g \circ \varphi \circ \varphi^{-1}}_{\in \mathcal{C}^\infty}|_{\varphi(W)}$$

(as a remark, I think you have to say something a little more about W [namely, that $U \subset W$], and φ has to be a chart of M. Really it's not a big deal, and I'm sure everything works out just fine, but there was some brushing of details under the rug here).

Case 2: If $x \notin W$, then $x \in M - \varphi^{-1}(\overline{B}_r) \subset_{open} M$, and $\tilde{f} = 0$ on $M - \varphi^{-1}(\overline{B}_r)$. Thus for any chart ψ, $id_{\mathbb{R}} \circ \tilde{f} \circ \psi^{-1} = 0 \in \mathcal{C}^\infty$.

□

[‡]Let $v \in T_aM$, $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ such that there exists an open neighborhood $U \subset M$ with $a \in U$ such that $f|_U = g|_U$, then $v(f) = v(g)$.

5.5 An isomorphism between $\mathbb{R}^n$ and $T_a\mathbb{R}^n$

Proposition 19. *Let $M = \mathbb{R}^n$, $a \in \mathbb{R}^n$. Define $D : \mathbb{R}^n \to T_a\mathbb{R}^n$, by*

$$D : v \mapsto D_v$$

where

$$D_v(f) := \frac{d}{dt} f(a + t \cdot v)\Big|_{t=0}$$

for all $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Then, D is a well-defined linear isomorphism between $\mathbb{R}^n$ and $T_a\mathbb{R}^n$. In particular, when v is the canonical unit basis vector e_i, we write

$$\frac{\partial}{\partial x^i}\Big|_a := \frac{\partial}{\partial x^i} := D_{e_i}\Big|_a := D_{e_i} \in T_a\mathbb{R}^n$$

It is,

$$\frac{\partial}{\partial x^i}\Big|_a (f) = D_{e_i}(f) = \frac{d}{dt} f(a + t \cdot e_i)\Big|_{t=0} = \frac{\partial f}{\partial x^i}(a)$$

where the rightmost expression is the usual partial derivative in $\mathbb{R}^n$

Proof. We prove this proposition in a number of steps, whose combination imply our proposition.

- D **is well-defined**: we want to show that things land where we think they land, i.e., that D_v is actually in $T_a\mathbb{R}^n$. We can show this by proving that D_v is (1) linear, and (2) a derivation.

  1. Simply,

     $$D_v(rf + sg) = \frac{d}{dt}(rf + sg)(a + tv)|_{t=0},$$

     the linearity of the directional derivative in $rf + sg$ implies (1).
  2. From the definition of D_v and properties of the directional derivative,

     $$\begin{aligned} D_v(fg) &= \frac{d}{dt}(fg)(a + tv)|_{t=0} \\ &= \frac{d}{dt} f(a + tv)\Big|_{t=0} \cdot g(a) + f(a) \cdot \frac{d}{dt} g(a + tv)|_{t=0} \\ &= D_v f \cdot g(a) + f(a) D_v(g), \end{aligned}$$

     and we have (2).

- D **is linear:** We have that, $D : \mathbb{R}^n \to T_a\mathbb{R}^n$, and for $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$,

  $$D_{rv+sw}(f) = \frac{d}{dt} f(a + t(rv + sw))|_{t=0} = \langle rv + sw, Df(a)\rangle^{\S}$$

  and $\langle , \rangle$ is linear in each component, so D must be linear.

- D **is injective:** let $D_v = 0$, i.e., $D_v(f) = 0$ for all $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. In the hope that we can show that $v = 0$, we write $v \in \mathbb{R}^n$ in terms of its components and the unit basis vectors; $v = \sum_{j=1}^n v^j e_j$, and let $f : \mathbb{R}^n \to \mathbb{R}$ which maps, $f(x^1, ..., x^n) = x^i \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Then, since $D_v(f) = \langle v, Df\rangle$,

  $$0 = D_v(f) = \sum_j v^j D_{e_j}(f) = \sum_j v^j \frac{\partial f}{\partial x^j}(a) = \sum_j v^j \frac{\partial x^i}{\partial x^j}(a) = \sum_j v^j \delta_{i,j} = v^i$$

  where $\delta_{i,j}$ is the usual Kronecker-delta function. Hence, $v = \sum_j v^j e_j = 0$, and we have that D is injective.

§ this follows from a very early definition from chapter 1; the definition of the directional derivative.

- D **is surjective**: let $w \in T_a\mathbb{R}^n$. We want $v \in \mathbb{R}^n$ such that $D_v = w$. Let $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, and expand f in terms of its Taylor series:

$$f(x) = f(a) + \sum_{j=1}^n \frac{\partial f}{\partial x^j}(a)(x^j - a^j) + \sum_{i,j} R_{i,j}(x^i - a^i)(x^j - a^j)$$

for some remainder function $R_{i,j}$. Note that $w(f(a)) = 0$, since $f(a)$ is constant, and we have 5.2(a). Note that $x \mapsto R_{i,j}(x^i - a^i)$, and $x \mapsto (x^j - a^j)$ are 0 at $x = 0$. Therefore, by 5.2(b), we have the product of two functions whose value at a is 0, so

$$w((R_{i,j}(x)(x^i - a^i))(x^j - a^j)) = 0$$

this implies that

$$w(f) = \sum_{j=1}^n \frac{\partial f}{\partial x^j}(a)w(x^j - a^j) = \sum_{j=1}^n \frac{\partial f}{\partial x^j}(a)w(x^j)$$

where the second equality comes from the linearity of w and the fact that $w(a^j) = 0$. Set $w(x^j) := v^j$. Then,

$$w(f) = \sum_{j=1}^n v^j D_{e_j}(f) = D_{\sum_{j=1}^n v^j e_j}(f)$$

and this implies that for all f,

$$w = D_{\sum_{j=1}^n v^j e_j}$$

and we've constructed an element $v \in \mathbb{R}^n$ such that $D_v = w$.

□

5.6 Some Important Formulas

Corollary 20. *Let M be a $\mathcal{C}^\infty$-manifold (smooth manifold) of dimension d, and let $a \in M$.*

(a) If $\varphi : U \to \mathbb{R}^d$ is a chart of M, $a \in U$, then $\varphi : U \to \varphi(U)$ is a diffeomorphism, and so

$$d\varphi_a : T_aM \to T_{\varphi(a)}\mathbb{R}^d ¶$$

is an isomorphism of $\mathbb{R}$-vector spaces by 5.4(e). In particular, T_aM is a d-dimensional $\mathbb{R}$-vector space. We define (for $a \in M$):

$$\left.\frac{\partial}{\partial x^i}\right|_a := d\varphi^{-1}_{\varphi(a)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(a)}\right)$$

so that $\{\left.\frac{\partial}{\partial x^i}\right|_a\}$ gives a basis of T_aM, i.e. every $v \in T_aM$ can be written as

$$v = \sum_j v^j \cdot \left.\frac{\partial}{\partial x^j}\right|_a$$

For $f \in \mathcal{C}^\infty(M, \mathbb{R})$:

$$\left.\frac{\partial}{\partial x^i}\right|_a(f) = d\varphi^{-1}_{\varphi(a)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(a)}\right)(f) = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(a)}(f \circ \varphi^{-1}) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(a)) \tag{5.6.1}$$

¶If you want to get technical, this is really

$$T_aM \cong T_aU \longrightarrow T_{\varphi(a)}\varphi(U) \cong T_{\varphi(a)}\mathbb{R}^d$$

(b) *Let $\varphi : U \to \mathbb{R}^d, \psi : V \to \mathbb{R}^d$ be two charts with $a \in U$, $a \in V$. We have the basis of T_aM given by $\{\frac{\partial}{\partial x^i}\Big|_a\}$ from φ, and $\{\frac{\partial}{\partial \tilde{x}^i}\Big|_a\}$ coming from ψ. Note that for $f : \psi(V) \to \mathbb{R}$,*

$$d(\psi \circ \varphi^{-1}) \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right)(f) = \frac{\partial}{\partial x^i}\Big|_{\varphi(a)} (f \circ \psi \circ \varphi^{-1}) = \frac{\partial (f \circ \psi \circ \varphi^{-1})}{\partial x^i}(\varphi(a))$$

by the chain rule,

$$= \sum_{j=1}^{d} \frac{\partial f}{\partial \tilde{x}^j}(\psi(a)) \cdot \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a))$$

$$\Rightarrow d(\psi \circ \varphi^{-1}) \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right) = \sum_{j=1}^{d} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_{\psi(a)}$$

this implies

$$\begin{aligned} \frac{\partial}{\partial x^i}\Big|_a &= d\varphi^{-1} \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right) = d\psi^{-1} \circ d(\psi \circ \varphi^{-1}) \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right) \\ &= d\psi^{-1} \left(\sum_{j=1}^{d} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_{\psi(a)} \right) = \sum_{j=1}^{d} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^j}(\varphi(a)) \cdot d\psi^{-1} \left(\frac{\partial}{\partial \tilde{x}^j}\Big|_{\psi(a)} \right) \\ &= \sum_{j=1}^{d} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_a \end{aligned}$$

in summary, we have the change of variables formula,

$$\frac{\partial}{\partial x^i}\Big|_a = \sum_{j=1}^{d} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_a \tag{5.6.2}$$

(c) *If $F \in \mathcal{C}^\infty(M, N)$ where M is a d-dimensional manifold and N is a k-dimensional manifold, then let $\varphi : M \to \mathbb{R}^d$ be a chart of M with $a \in U$, and let $\psi : V \to \mathbb{R}^k$ be a chart of N, where $F(a) \in V$. As in (b), we have that*

$$d(\psi \circ F \circ \varphi^{-1})_{\varphi(a)} \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right) = \sum_{j=1}^{k} \frac{\partial (\psi \circ F \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial x^j}\Big|_{\psi(F(a))}$$

which will be left as an exercise (it's really the same calculation). Therefore, we get equation (3),

$$dF \left(\frac{\partial}{\partial x^i}\Big|_a \right) = d\psi^{-1} \circ d\psi \circ dF \left(d\varphi^{-1} \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right) \right) = d\psi^{-1} \circ d(\psi \circ F \circ \varphi^{-1}) \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(a)} \right)$$

$$= \sum_{j=1}^{k} \frac{\partial (\psi \circ F \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot d\psi^{-1} \left(\frac{\partial}{\partial \tilde{x}^j}\Big|_{\psi(F(a))} \right) = \sum_{j=1}^{k} \frac{\partial (\psi \circ F \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_{F(a)}$$

in summary,

$$dF \left(\frac{\partial}{\partial x^i}\Big|_a \right) = \sum_{j=1}^{k} \frac{\partial (\psi \circ F \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_{F(a)} \tag{5.6.3}$$

5.7 Smooth Curves

(a) Let $I = (a, b) \subset \mathbb{R}$ be an open interval, and let M be a manifold. Then, a **smooth curve** α on M is an element $\alpha \in \mathcal{C}^\infty(I, M)$. For $t_0 \in I$, we can use

$$\frac{d}{dt}\Big|_{t_0} \in T_{t_0} I$$

(from 5.5 for $n = 1$) to define

$$\alpha'(t_0) = d\alpha\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\alpha(t_0)} M$$

and we call $\alpha'(t_0)$ the **velocity** of α at t_0.

(1) For $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we have

$$\alpha'(t_0)(f) = d\alpha\left(\frac{d}{dt}\Big|_{t_0}\right)(f) = \frac{d}{dt}\Big|_{t_0}(f \circ \alpha) = \frac{d(f \circ \alpha)}{dt}(t_0)$$

(2) For a smooth map $F : M \to N$, we have

$$dF(\alpha'(t_0)) = dF \circ d\alpha\left(\frac{d}{dt}\Big|_{t=0}\right) = d(F \circ \alpha)\left(\frac{d}{dt}\Big|_{t=0}\right) = (F \circ \alpha)'(t_0)$$

where $F \circ \alpha \in \mathcal{C}^\infty(I, N)$.

(b) In fact, we claim that for all $v \in T_a M$, there exists some $\epsilon > 0$ such that there exists some $\alpha \in \mathcal{C}^\infty((-\epsilon, \epsilon), M)$ such that $\alpha(0) = a$, and $\alpha'(0) = v$.

Proof. Let $\varphi : U \to \mathbb{R}^d$ be a chart of M, and $a \in U$. Let $v \in T_a M$, and write $v = \sum_{j=1}^d v^j \frac{\partial}{\partial x^j}\Big|_a$ coming from φ. Now define $\tilde{v} := \sum_{j=1}^d v^j e_j \in \mathbb{R}^d$, and let $\epsilon > 0$ such that for all $t \in (-\epsilon, \epsilon)$, we have that $\varphi(a) + t \cdot \tilde{v} \in \varphi(U)$. Define $\alpha : (-\epsilon, \epsilon) \to M$ by $\alpha(t) := \varphi^{-1}(\varphi(a) + t \cdot \tilde{v}) \in U$. Clearly, $\alpha(0) = \varphi^{-1}(\varphi(a)) = a$ and $\alpha \in \mathcal{C}^\infty((-\epsilon, \epsilon), M)$. Note, for $f \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$\alpha'(0)(f) = \frac{d(f \circ \alpha)}{dt}(0) = \frac{d(f \circ \varphi^{-1}(\varphi(a) + t\tilde{v}))}{dt}(0)$$

$$=^{\|} \sum_{j=1}^d \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(a)) \cdot \underbrace{\frac{d(\varphi(a) + t\tilde{v})^j}{dt}(0)}_{\tilde{v}^j = v^j}$$

$$=^{**} \sum_{j=1}^d \frac{\partial}{\partial x^j}\Big|_a (f) \cdot v^j = v(f) \Rightarrow \alpha'(0) = v$$

□

‖ By the chain rule
** By 5.6(1)

6
Vector Fields

(This section is modeled after Lee, Chapter 3 and 8).

6.1 The Tangent Bundle

Definition. Let M be a d-dimensional manifold. We define the **tangent bundle** of M to be the space

$$TM := \{(a, v) | a \in M, v \in T_a M\}$$

Note that there is a natural projection map $\pi : TM \to M$ which maps $(a, v) \mapsto a$.

Claim. We can give TM the structure of a $2d$-dimensional manifold. For $(\varphi : U \to \mathbb{R}^d) \in \mathcal{A}$, a chart of M, denote by

$$\varphi_{TM} : \pi^{-1}(U) \to \mathbb{R}^{2d}$$

the following map: let $(a, v) \in \pi^{-1}(U)$, i.e., $v \in T_a M$. Then we can write (by 5.6(a))

$$v = \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i}\Big|_a$$

for some $v^i \in \mathbb{R}$. We define

$$\varphi_{TM}(a, v) := \left(\varphi(a), v^1, v^2, ..., v^d\right) \in \mathbb{R}^{2d}$$

If we take $\mathcal{A} := \{\varphi_{TM} | \varphi \in \mathcal{A}\}$, this defines an atlas for TM, and $\pi \in \mathcal{C}^\infty(TM, M)$.

Proof. We show that $\mathcal{A}$ is an atlas in the same way that we've done in a number of times before.

- **φ_{TM} is a chart, i.e., (1) φ_{TM} has to be injective, and (2) $\varphi_{TM} U'$ is open, where $U' \subset \pi^{-1}(U)$.**
  1. Well, φ_{TM} has to be injective, because φ is injective, and $\frac{\partial}{\partial x^i}$ is a basis of $T_a M$- so the v^i are chosen uniquely with respect to v.
  2. Is $\varphi_{TM}(\pi^{-1}(U)) \subset_{open} \mathbb{R}^{2d}$? Well,

     $$\varphi_{TM}(\pi^{-1}(U)) = \varphi(U) \times \mathbb{R}^d \subset_{open} \mathbb{R}^{2d}$$

     and since $\varphi(U)$ is open in $\mathbb{R}^d$, we have our assertion.
- **We must show the compatibility of maps in the atlas:** Let $(\varphi : U \to \varphi(U)), (\psi : V \to \psi(V)) \in \mathcal{A}$. There are 3 compatibility conditions to show,

1. $\varphi_{TM}(\pi^{-1}(U) \cap \pi^{-1}(V))$ needs to be open in $\mathbb{R}^d$, but since

$$\varphi_{TM}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \varphi_{TM}(\pi^{-1}(U \cap V)) = \varphi(U \cap V) \times \mathbb{R}^d$$

and as $\varphi(U \cap V)$ is open in $\mathbb{R}^d$, the term on the right is open in $\mathbb{R}^{2d}$. We do the same thing for ψ_{TM}.

2. $\psi_{TM} \circ \varphi_{TM}^{-1} : \varphi(U \cap V) \times \mathbb{R}^d \to \psi(U \cap V) \times \mathbb{R}^d$, needs to be a smooth diffeomorphism. Really inspecting this map,

$$\left(\varphi\left(a\right), v^1, ..., v^d\right) \xrightarrow{\varphi_{TM}^{-1}} \left(a, \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i}\Big|_a\right)$$

using 5.6(2), the change of variable formula, and denoting the coordinate tangent vectors for ψ as $\frac{\partial}{\partial \tilde{x}^k}$,

$$\left(a, \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i}\Big|_a\right) = \left(a, \sum_{i=1}^{d} v^j \sum_{k=1}^{d} \frac{\partial\left(\psi \circ \varphi^{-1}\right)^k}{\partial x^i} \cdot \frac{\partial}{\partial \tilde{x}^k}\Big|_a\right) = \left(a, \sum_{i=1}^{d} \left(\sum_{k=1}^{d} v^i \frac{\partial\left(\psi \circ \varphi^{-1}\right)^k}{\partial x^i} \cdot \frac{\partial}{\partial \tilde{x}^k}\Big|_a\right)\right)$$

Taking ψ_{TM} of this, we get

$$= \left(\psi\left(a\right), \sum_{i=1}^{d} v^k \frac{\partial\left(\psi \circ \varphi^{-1}\right)^1}{\partial x^i}, ..., \sum_{i=1}^{d} v^i \frac{\partial\left(\psi \circ \varphi^{-1}\right)^d}{\partial x^i}\right)$$

Note that

$$(v^1, ..., v^d) \mapsto \left(\frac{\partial(\psi \circ \varphi^{-1})^k}{\partial x^i}\right)_{k,i} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$$

is a smooth map, which implies that $\psi_{TM} \circ \varphi_{TM}^{-1}$ is smooth, and

$$\left(\psi_{TM} \circ \varphi_{TM}^{-1}\right)^{-1} = \varphi_{TM}^{-1} \circ \psi_{TM}$$

is smooth by a similar argument.

- $\mathcal{A}$ **is an atlas:**

1. We have that

$$\bigcup_{\varphi_{TM} \in \mathcal{A}} \pi^{-1}(U) = \bigcup_{\varphi \in \mathcal{A}} U \times TM|_U = TM$$

2. φ_{TM}, ψ_{TM} are compatible, which was just shown.

- TM **is Hausdorff:** Let $(a, v) \neq (b, w) \in TM$. If $a \neq b$, then there exist charts $\psi : U \to \mathbb{R}^d$ and $\psi : V \to \mathbb{R}^d$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$, because M is Hausdorff. Then, take φ_{TM}, ψ_{TM}, and we have satisfied our condition in this case. Considering the other case, if $(a, v) \neq (a, w) \in T_aM$, then $v \neq w \in T_aM \cong \mathbb{R}^d$. Since $\mathbb{R}^d$ is Hausdorff, there exist neighborhoods U and V of v and w such that $U \cap V$ is empty.

- TM **is** 2^{nd}**-countable**: let $\mathcal{B}$ be a countable base for M. Define

$$\mathcal{B}_{\mathcal{A}} = \{B \in \mathcal{B} | \exists (\varphi : U \to \mathbb{R}^d) \in \mathcal{A} \; such \; that \; B \subset U\}$$

Then, the $\mathcal{B}_{\mathcal{A}}$ is a countable base for M. $\mathcal{B}_{\mathcal{A}}$ is a base because: for $a \in V \subset_{open} M$, there exists a chart $\varphi : U \to \mathbb{R}^d$, $a \in U \subset V$. Since $\mathcal{B}$ is a base, there exists $B \in \mathcal{B}$ such that $a \in B \subset U$. But then, $B \in \mathcal{B}_{\mathcal{A}}$, and $a \in B \subset U \subset V$. Write $B_{\mathcal{A}} = \{B_1, B_2, ...\}$, and choose $\varphi_i : U_i \to \mathbb{R}^d$ with $B_i \subset U_i$.

Since $(\varphi_i)_{TM}$ are charts of TM, we know that $\pi^{-1}(U_i)$ is homeomorphic to $\varphi(U_i) \times \mathbb{R}^d$ (by Lemma 3.5), and $\varphi(U_i) \times \mathbb{R}^d$ is 2^{nd} countable. Note that

$$TM = \bigcup_{i=1}^{\infty} \pi^{-1}(U_i),$$

and recall that a countable union of 2^{nd} countable space is again, 2^{nd} countable.

- $\pi : TM \to M$ **is smooth**: for φ and φ_{TM},

$$\varphi \circ \pi \circ \varphi_{TM}^{-1} = \varphi \circ \pi \left(\varphi^{-1}(x), ...\right) = \varphi(\varphi^{-1}(x)) = x \in \mathcal{C}^\infty$$

□

6.2 Induced (smooth) maps on tangent bundles

Corollary 21. *If $F \in \mathcal{C}^\infty(M, N)$, then there is an induced map $dF \in \mathcal{C}^\infty(TM, TN)$ given by*

$$dF(a, v) := (F(a), dF_a(v))$$

where $dF_a : T_aM \to T_{F(a)}M$.

Proof. For a chart of M $\varphi : U \to \mathbb{R}^d$, and $\psi : V \to \mathbb{R}^k$, a chart of N, we have:

$$\psi_{TM} \circ dF \circ \varphi_{TM}^{-1}(\varphi(a), v^1, ..., v^d) = \psi_{TM} \circ dF \left(a, \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i}\Big|_a\right) = \psi_{TM}\left(F(a), \sum_{i=1}^{d} v^i dF\left(\frac{\partial}{\partial x^i}\Big|_a\right)\right)$$

by (5.6(3)),

$$= \psi_{TM}\left(F(a), \sum_{i=1}^{d} v^i \sum_{\ell=1}^{k} \frac{\partial(\psi \circ F \circ \varphi^{-1})^\ell}{\partial x^i} \cdot \left(\frac{\partial}{\partial \tilde{x}^\ell}\Big|_{F(a)}\right)\right)$$

$$= \left(\psi(F(a)), \sum_{i=1}^{d} v^i \frac{\partial(\psi \circ F \circ \varphi^{-1})^1}{\partial x^i}, ..., \sum_{i=1}^{d} v^i \frac{\partial(\psi \circ F \circ \varphi^{-1})^k}{\partial x^i}\right)$$

where $\psi(F(a))$ is smooth, and

$$\left(\sum_{i=1}^{d} v^i \frac{\partial(\psi \circ F \circ \varphi^{-1})^1}{\partial x^i}, ..., \sum_{i=1}^{d} v^i \frac{\partial(\psi \circ F \circ \varphi^{-1})^k}{\partial x^i}\right) = \left(\frac{\partial(\psi \circ F \circ \varphi^{-1})^\ell}{\partial x^i}\right)_{k,i} \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$$

which is smooth in $(v^1, ..., v^d)$. □

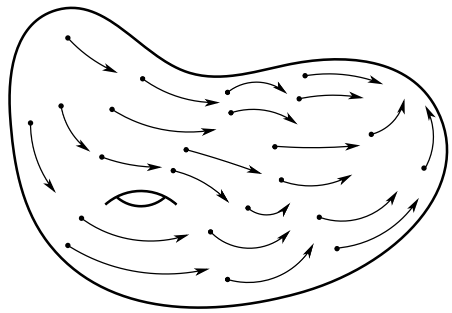
6.3 Recognizing tangent bundles as Cartesian products

Remark. Note that if M has a chart $\varphi : M \to \mathbb{R}^d$ where the domain of φ is all of M, then $\pi^{-1}(M) = TM$, so that

$$(\varphi^{-1} \times id_{\mathbb{R}^d}) \circ \varphi_{TM} : TM \to \varphi(M) \times \mathbb{R}^d \to M \times \mathbb{R}^d$$

is a diffeomorphism, by 4.3(c). In this case, TM is just the Cartesian product $M \times \mathbb{R}^d$. In general, this will not be the case, because we cannot identify different tangent spaces at different points of M with each other without any extra information.

6.4 Vector Fields



Definition. Let M be a manifold. A **vector field** X on M is a section

$$X : M \to TM$$

of π. In other words, $\pi \circ X(x) = x$. We write $X : a \mapsto (a, X_a)$ where $X_a \in T_a M$. For a subset $S \subset M$, we define a **vector field on** S to be a map $X : S \to TM$ such that $\pi \circ X(x) = x$.

Remark. If $f \in \mathcal{C}^\infty(U, \mathbb{R})$ where $U \subset_{open} M$, and X is a vector field defined on U, then X gives us a map $Xf : U \to \mathbb{R}$ given by $Xf(a) := X_a(f)$; which makes sense, as $X \in T_a M$ and $f \in \mathcal{C}^\infty(U, \mathbb{R})$. Note that the following properties are almost immediate, as X_a is a derivation:

1. $X(rf + sg) = rXf + s \cdot Xg$ for $r, s \in \mathbb{R}$, $f, g \in \mathcal{C}^\infty(U, \mathbb{R})$, and
2. $X(f \cdot g) = (Xf) \cdot g + f \cdot (Xg)$ as functions from $U \to \mathbb{R}$.

6.5 What is a smooth vector field?

Proposition 22. *Let X be a vector field on a manifold M. Then, the following conditions are equivalent:*

(a) $X \in \mathcal{C}^\infty(M, TM)$.

(b) For all $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $Xf \in \mathcal{C}^\infty(M, \mathbb{R})$.

(c) For every open subset $U \subset M$ and for all $f \in \mathcal{C}^\infty(U, \mathbb{R})$, $Xf \in \mathcal{C}^\infty(U, \mathbb{R})$.

(d) For $p \in U$, and all charts $\varphi : U \to \varphi(U)$, if we write X_p as

$$X_p = \sum_{j=1}^{d} X^j \frac{\partial}{\partial x^j}\Big|_p = X^1(p) \cdot \frac{\partial}{\partial x^1}\Big|_p + \dots + X^d(p) \cdot \frac{\partial}{\partial x^d}\Big|_p,$$

*then $X^j \in \mathcal{C}^\infty(U, \mathbb{R})$, $j = 1, \dots, d$, where we call the maps $X^j : U \to \mathbb{R}$ the **coordinate functions**.*

(e) For every point $p \in M$, there exists a chart $\varphi : U \to \mathbb{R}^d$ with $p \in U$, the coordinate functions $X^j : U \to \mathbb{R}^d$ are smooth; $X^j \in \mathcal{C}^\infty(U, \mathbb{R})$ for $j = 1, \dots, d$

*If any of these conditions is satisfied, then X is called a **smooth vector field.***

Proof. We proceed as follows:

$a \iff d \iff e$

First, we have for a chart $\varphi : U \to \mathbb{R}^d$ of M,

$$\varphi_{TM} \circ X \circ \varphi^{-1}(x) = \varphi_{TM}\left(\varphi^{-1}(x), \sum_{j=1}^{d} X^j\left(\varphi^{-1}(x)\right) \cdot \frac{\partial}{\partial x^j}\Big|_{\varphi^{-1}(x)}\right)$$

$$= (x, X^1(\varphi^{-1}(x)), ..., X^d(\varphi^{-1}(x)))$$

Now, $X \in \mathcal{C}^\infty(M, TM)$ if and only if it turns out that the map above is smooth. This map is smooth if and only $X^j \circ \varphi^{-1}$ is smooth for all $j = 1, ..., d$, which is the case if and only if $X^j \in \mathcal{C}^\infty(U, \mathbb{R})$. This is equivalent to (d) and (e) by definition 4.1 and lemma 4.2.

$e \Rightarrow c$

Let $f \in \mathcal{C}^\infty(U, \mathbb{R})$, and let $\varphi : V \to \varphi(V)$ be a chart (as in (e)) with $p \in V \subset U$. Then,

$$X_p = \sum_{j=1}^{d} X^j(p) \cdot \frac{\partial}{\partial x^j}\Big|_p, \quad \text{and} \quad Xf(p) = X_p(f) = \sum_{j=1}^{d} X^j(p) \cdot \frac{\partial}{\partial x^j}\Big|_p(f)$$

where by assuming (e), we have that $X^j \in \mathcal{C}^\infty(V, \mathbb{R})$.

Consider the map $f : U \to \mathbb{R}$ induced by X (only, we restrict to the j^{th} component), where

$$p \mapsto \frac{\partial}{\partial x^j}\Big|_p(f).$$

This map satisfies

$$id_{\mathbb{R}} \circ \frac{\partial}{\partial x^j}\Big|_p(f) \circ \varphi^{-1}(x) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(p),$$

which is the usual partial derivative in $\mathbb{R}^d$, by Cor. 5.6(1). The partial derivative is smooth, implying that $p \mapsto \frac{\partial}{\partial x^j}|_p(f) \in \mathcal{C}^\infty(M, \mathbb{R})$. This implies our claim, where we now consider instead of the j^{th} component, the sum of all such components from $j = 1...d$, and take products in order to get Xf.

$c \Rightarrow b$

Almost immediately, by setting $U := M$.

$b \Rightarrow e$

Assume (b); that for all $f \in \mathcal{C}^\infty(M, \mathbb{R})$, that $Xf \in \mathcal{C}^\infty(M, \mathbb{R})$. Let $p \in M$, and chose a chart of M, $\varphi : U \to \varphi(U)$, with $p \in U$. Write $X = \sum_{j=1}^{d} X^j \cdot \frac{\partial}{\partial x^j}|$. Let $f^j \in \mathcal{C}^\infty(U, \mathbb{R})$ given by $f^j(x) = (\varphi(x))^j$. By lemma 18 in section 5.4, there exists a smooth extension $\tilde{f}^j \in \mathcal{C}^\infty(M, \mathbb{R})$ such that for $p \in V \subset_{open} U$, f^j and $\tilde{f}^j$ agree; $\tilde{f}^j|_V = f^j|_V$. Then, we claim that on V, that

$$X\tilde{f}^j = \sum_{\ell=1}^{d} X^\ell \cdot \frac{\partial}{\partial x^j}\Big|_x(\tilde{f}^j) \overset{5.6(1)}{=} \sum_{\ell=1}^{d} X^\ell \cdot \frac{\partial(\tilde{f}^j \circ \varphi^{-1})}{\partial x^\ell} \circ \varphi$$

which implies that on $\varphi(V)$,

$$id_{\mathbb{R}} \circ X\tilde{f}^j \circ \varphi^{-1} = \left(\sum_{\ell=1}^{d} X^\ell \cdot \frac{\partial(\tilde{f}^j \circ \varphi^{-1})}{\partial x^\ell} \circ \varphi\right) \circ \varphi^{-1}$$

$$= \sum_{\ell=1}^{d} X^\ell \cdot \frac{\partial\left(\varphi(\varphi^{-1}(x))\right)^j}{\partial x^\ell} = \frac{\partial x^j}{\partial x^\ell} = \delta_{j\ell}$$

which is smooth. This calculation shows that $X\tilde{f}^j \in \mathcal{C}^\infty(V,\mathbb{R})$ so by taking sums and products we have shown that $Xf \in \mathcal{C}^\infty(U,\mathbb{R})$ for all $p \in M$. □

6.6 Smooth vector fields, and real-valued smooth maps.

Lemma 23. *Let $Z : \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R})$. Then, there exists X, a smooth vector field, with $Z(f) = Xf$ if and only if*

1. *Z is $\mathbb{R}$-linear, and*
2. *$Z(f \cdot g) = Z(f) \cdot g + f \cdot Z(g)$.*

Proof. We proceed as follows:

(⇒)

This part of the proof follows quickly by 6.4(1), or definition, because $Xf(p) = X_p(f)$ satisfies both (1) and (2).

(⇐)

Assume Z satisfies (1) and (2). Define $X : M \to TM$, $X_p \in T_pM$, $X_p(f) := Z(f)(p)$. Then X_p is a linear derivation, e.g., $X_p(f \cdot g) = Z(f \cdot g)(p) \overset{(2)}{=} Z(f)(p)g(p) + f(p) \cdot Z(g)(p) = X_p(f) \cdot g(p) + f(p) \cdot X_p(f)$. Lastly, X is smooth by Lemma 6.5($b \Rightarrow a$), and we are done. □

6.7 A module over the $\mathcal{C}^\infty$ structures; Lee brackets

Definition. Let M be a smooth manifold.

(a) Denote by $\mathbb{X}(M)$, the set of all smooth vector fields on M.

(b) For $X, Y \in \mathbb{X}(M)$, then $X + Y \in \mathbb{X}(M)$, by $(X+Y)(f) = Xf + Yf$.

(c) For $X \in \mathbb{X}(M)$, $f \in \mathcal{C}^\infty(M,\mathbb{R})$, we have that $f \cdot X \in \mathbb{X}(M)$, where $(f \cdot X)(g) = (f)(Xg)$ *.

(d) If $X, Y \in \mathbb{X}(M)$, $f \in \mathcal{C}^\infty(M,\mathbb{R})$. Define $Z : \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R})$ by taking $Z(f) := X(Y(f))$. Note that $Z \notin \mathbb{X}(M)$, because the derivation property (2) is not satisfied:

$$\begin{aligned} Z(f \cdot g)(p) &= X_p(Y(fg))(p) = X_p(Y(f) \cdot g + f \cdot Y(g))(p) = X_p(g \cdot Y(f))(p) + X_p(f \cdot Y(g))(p) \\ &= X_p(Y(f))(p) \cdot g(p) + Y(f)(p) \cdot X_p(g) + X_p(f) \cdot Y(g)(p) + f(p) \cdot X_p(Y(g))(p) \\ &= (Z(f) \cdot g)(p) + (f \cdot Z(g))(p) + \underbrace{(Y_pf)(X_pg) + (X_pf)(Y_pg)}_{\neq 0} \end{aligned}$$

*My notation was slightly misleading, as at first it looked like I was trying to take the image of Xg under f. I added the extra parenthesis to emphasize that this is incorrect - it might be more clear to consider this at a point on the manifold; for example,

$$(fX)_p(g) = \underbrace{f_p}_{\in\mathbb{R}}\underbrace{(Xg)_p}_{\in\mathbb{R}},$$

so, we still have something that is a smooth vector field.

But, defining $W(f) := X(Y(f)) - Y(X(f))$, $W : \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R})$ satisfies (1) and (2), so $W \in \mathbb{X}(M)$; by the three lines above, we would get

$$\begin{aligned}
W(fg)(p) &= X(Y(fg))(p) - (Y(X(fg))(p) \\
&= X(Yf)(p) \cdot g(p) + f(p) \cdot X(Y(g))(p) + (Y_p f)(X_p g) + (X_p F)(Y_p g) - (Y(Xf)(p) \cdot g(p) \\
&\quad + f(p) \cdot Y(X(g))(p) + (X_p f)(Y_p g) + (Y_p f)(X_p g)) \\
&= W(f)(p) \cdot g(p) + f(p) \cdot W(g)(p)
\end{aligned}$$

and linearity is clear. We call $[X,Y] := W \in \mathbb{X}(M)$ the **Lee bracket of X and Y.**

6.8 Properties of the Lee bracket

(a) $\mathbb{X}(M)$ is a module over $\mathcal{C}^\infty(M,\mathbb{R})$ with module structure $X+Y$ and $f \cdot X$ from section 6.7 (b) and (c).

(b) $[,]$ is bilinear over $\mathbb{R}$, i.e., for any $r,s \in \mathbb{R}$, we have that $[rX+sY,Z] = r[X,Z]+s[Y,Z]$, and $[X,rY+sZ] = r[X,Y]+s[X,Z]$.

(c) $[,]$ is anti-symmetric, i.e.,

$$[X,Y] = -[Y,X]$$

(d) $[,]$ satisfies the **Jacobi-identity:**

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

(e) For all $f,g \in \mathcal{C}^\infty(M,\mathbb{R})$,

$$[fX,gY] = fg[X,Y] + (fX(g)) \cdot Y - (gYf) \cdot X$$

Proof. (a) Clear

(b)

$$\begin{aligned}
[rX+sY,Z](f) &= (rX+sY)(Zf) - Z((rx+sY)(f)) \\
&= rX(Zf) + sY(Zf) - rZ(Xf) - sZ(Yf) \\
&= r[X,Z](f) + s[Y,Z](f)
\end{aligned}$$

(c) $[X,Y](f) = X(Yf) - Y(Xf) = -[Y,X](f)$

(d)

$$\begin{aligned}
&[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] \\
&= XYZ - XZY - YZX + ZYX + ZXY - ZYX - XYZ + YXZ + YZX - YXZ - ZXY + XZY = 0
\end{aligned}$$

(e)

$$\begin{aligned}
&[fX,gY](h) = fX(gY(h)) - gY(fX(h)) = \\
&fgX(Y)(h) + fX(g)(Y)(h) - gY(f)X(h) - gfY(X)(h) = fg[X,Y] + fX(g) \cdot Y - f \cdot Y(f)X(h)
\end{aligned}$$

$\square$

6.9 Diffeomorphisms and $\mathbb{X}(M)$.

Let $F \in Diff^{\infty}(M,N)$, then $F_* : \mathbb{X}(M) \to \mathbb{X}(N)$ is given by $X \in \mathbb{X}(M), F_*(X) \in \mathbb{X}(N), y \in N$ such that

$$F_*(N)(y) \in T_y N$$

if $F \in \mathcal{C}^{\infty}(M,N)$ is not injective and surjective, then F_* cannot be defined.

7

Immersions, Submersions, Embeddings, and Submanifolds

This chapter follows the discussion in Lee's book, mainly from chapters 4,5, and 6.

7.1 Introductory Definitions

Example. Let M^k and N^d be two manifolds of dimensions k and d respectively. Let $f \in \mathcal{C}^\infty(M, N)$ be a smooth function.

(a) Let $a \in M$. We call $rank(dF_a : T_aM \to T_{F(a)}M)$ the **rank of** F **at** a. We say that F has **constant rank** if there exists an r such that r is the rank at a for every $a \in M$.

(b) F is called a **smooth immersion** if for all $a \in M$, dF_a is inejctive; i.e., F has constant rank k.

(c) F is called a **smooth submersion** if for all $a \in M$, dF_a is surjective; i.e. F has constant rank d.

7.2 Local parameterizations of smooth immersions

Proposition 24. *Let $F : M^k \to N^d$ be a smooth immersion*. Then, for all $a \in M$, there exist charts $\varphi : U \to \mathbb{R}^k$ of M where $a \in U$, and $\psi : V \to \mathbb{R}^d$ of N with $F(a) \in V$ and $F(U) \subset V$ such that*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(F(U)); \qquad (x^1, ..., x^k) \in \mathbb{R}^k \mapsto (x^1, ..., x^k, \underbrace{0 \dots 0}_{d-k}) \in \mathbb{R}^d$$

Proof. Let $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^k$ be a chart of M at a, $a \in \tilde{U}$ and $\psi : \tilde{V} \to \mathbb{R}^d$ be a chart of N at $F(a), F(a) \in \tilde{V}$, $F(\tilde{U}) \subset \tilde{V}$. Without loss of generality, assume that $\tilde{\varphi}(a) = 0 \in \mathbb{R}^k$, and $\psi(F(a)) = 0 \in \mathbb{R}^d$. Since

$$d(\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1})_0 = \underbrace{d\tilde{\psi}_{F(a)}}_{isomorphism} \circ \underbrace{dF_a}_{injective} \circ \underbrace{d\tilde{\varphi}_0^{-1}}_{isomorphism}$$

is injective (rank k), we may assume after a rearrangement of the basis of $\mathbb{R}^d$ that

$$\left(\frac{\partial (\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1})^i}{\partial x^j}(0) \right)_{\substack{i=1\dots k \\ j=1\dots k}} \tag{7.2.1}$$

is invertible, i.e., it has rank k. Write

$$\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}(x) = (\underbrace{g(x)}_{\in \mathbb{R}^k}, \underbrace{h(x)}_{\in \mathbb{R}^{d-k}}) \in \mathbb{R}^d$$

*This notation only indicates that M is of dimension k, and N is of dimension d.

By what we just mentioned in equation 7.2.1 above, $Dg(0)$ is invertible. Then by the inverse function theorem, there exist open subsets of $\mathbb{R}^k$, U_0, U_1 such that that $0 \in U_0, 0 \in U_1$ and $g|_{U_0} \to U_1$ is a $\mathcal{C}^\infty$ diffeomorphism. Let $U := \tilde{\varphi}^{-1}(U_0)$, and let

$$\varphi := g|_{U_0} \circ \tilde{\varphi}|_U : U \xrightarrow{\tilde{\varphi}} U_0 \xrightarrow{g} U_1, \quad \in \mathcal{C}^\infty - diffeomorphism$$

Let $V_1 := \tilde{\psi}(\tilde{V}) \cap (U_1 \times \mathbb{R}^{d-k})$, and define a map,

$$\rho : V_1 \to \mathbb{R}^d; \qquad \rho(x,y) := (x, y - h(g^{-1}(x))).$$

Note that ρ is well-defined, since for $(x,y) \in V_1$, we must have that $x \in U_1$, and as such $g^{-1}(x) \in U_0$, and $h(g^{-1}(x))$ makes sense. Furthermore, ρ is smooth, since g and h are smooth and $\rho : V_1 \to \rho(V_1)$ is invertible with (smooth) inverse $\rho^{-1}(x,y) := (x, y + h(g^{-1}(x)))$ (it is easy to check that $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho = id$). With this, set $V = \tilde{\psi}^{-1}(V_1)$, and define $\psi : V \to \mathbb{R}^d$ by $\psi := \rho \circ \tilde{\psi}|_V$. Observe that $F(U) \subset V$, because

$$\tilde{\psi}(F(U)) = \tilde{\psi}(F(\tilde{\varphi}^{-1}(U_0))) \subset \tilde{\psi}(F(\tilde{U})) \subset \tilde{\psi}(\tilde{V}) \subset V_1.$$

Now, $\psi \circ F \circ \varphi^{-1} = (\rho \circ \tilde{\psi}|_V) \circ F \circ ((\tilde{\varphi}|_U)^{-1} \circ g^{-1}) : U \to \mathbb{R}^d$ is given by

$$U_1 \xrightarrow{g} U_0 \xrightarrow{(\tilde{\varphi}|_U)^{-1}} \tilde{\varphi}^{-1}(U_0) \subset \tilde{U} \xrightarrow{F} \tilde{V} \xrightarrow{\tilde{\psi}|_V} \psi(\tilde{V}) \subset V_1 \xrightarrow{\rho} \mathbb{R}^d$$

and for $x \in U_1 \subset \mathbb{R}^k$,

$$\psi \circ F \circ \varphi^{-1}(x) = \rho \circ \underbrace{\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}}_{(g,h)} \circ g^{-1}(x) = \rho(g(g^{-1}(x)), h(g^{-1}(x)))$$

$$= \rho(x, h(g^{-1}(x))) = (x, h(g^{-1}(x)) - h(g^{-1}(x))) = (x, 0)$$

□

7.3 Inverse Function Theorem for Manifolds

Theorem 25. *(a) (Local Version) Let $F \in \mathcal{C}^\infty(M,N)$, $a \in M$, and let $dF_a : T_aM \to T_{F(a)}N$ be invertible. Then, there exist sets U, V such that $a \in U \subset_{open} M$, $F(a) \in V \subset_{open} N$ where $F|_U : U \to V$ is a smooth diffeomorphism (this means that F is a local diffeomorphism if and only if dF_a is invertible for all $a \in M$).*

(b) (Global Version) Let $F \in \mathcal{C}^\infty(M,N)$, dF_a is invertible for every $a \in M$, and assume that f is injective. Then, $F : M \to F(M)$ is a smooth diffeomorphism.

Proof. (a) Well for starters,

$$dim(M) = dim(T_aM) = dim(T_{F(a)}N) = dim(N) = d$$

Let $\varphi : \tilde{U} \to \varphi(\tilde{U})$ be a chart at $a \in \tilde{U} \subset M$, and $\psi : \tilde{V} \to \psi(\tilde{V})$ be a chart at $F(a) \in \tilde{V} \subset N$ with $F(\tilde{U}) \subset \tilde{V}$. Since φ, ψ are smooth diffeomorphisms (by 4.3(h)), we have that

$$d(\psi \circ F \circ \varphi^{-1})|_{\varphi(a)} = d\psi_{F(a)} \circ dF_a \circ d\varphi^{-1}_{\varphi(a)}$$

is invertible. By the inverse function theorem in $\mathbb{R}^d$, there exist $\tilde{\tilde{U}}, \tilde{\tilde{V}} \subset_{open} \mathbb{R}^d$ such that

$$\psi \circ F \circ \varphi^{-1} : \tilde{\tilde{U}} \to \tilde{\tilde{V}}$$

is a smooth diffeomorphism. Set $U := \varphi^{-1}(\tilde{\tilde{U}}), V := \psi^{-1}(\tilde{\tilde{V}})$. Then,

$$F = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \xrightarrow{\varphi} \tilde{\tilde{U}} \xrightarrow{(\psi \circ F \circ \varphi^{-1})} \tilde{\tilde{V}} \xrightarrow{\psi^{-1}} \psi^{-1}(\tilde{\tilde{V}})$$

and as ψ^{-1}, φ, and $(\psi \circ F \circ \varphi^{-1})$ are all smooth diffeomorphisms, we have that $F|_U$ is also a smooth diffeomorphism.

(b) Note that $F(M)$ is open in N (because for all $F(a) \in N$, there exists $a \subset U \subset_{open} M$ such that there exists $F(a) \in V \subset_{open} N$ where $F|_U : U \to V$ is a diffeomorphism, and $F(a) \in V \subset F(M)$, implying that $F(M)$ is open). Then, $F : M \to F(M)$ is bijective. F is smooth, $F^{-1} : F(M) \to M$ is smooth (since for all $F(a) \in N$, there exist U, V as above, where $F^{-1} : V \to U$ is a diffeomorphism.)

□

7.4 Smooth Embeddings

Definition. Let $F \in \mathcal{C}^\infty(M, N)$. Then, F is called a **smooth embedding** if F is a smooth immersion and $F : M \to F(M)$ is a homomorphism, where $F(M) \subset N$ has the subspace topology.

Remark. Recall that $F : M \to N$ between topological spaces is a topological embedding if and only if $F : M \to F(M)$ is a homomorphism. Now, smooth embeddings occur if and only if we have a topological embedding *and* a smooth immersion.

7.5 Examples

(a) Let $U \subset_{open} M$ where M is a manifold Then, the inclusion $i : U \to M$ is a smooth embedding.

(b) Let $F : \left(\frac{\pi}{2} \to \frac{5\pi}{2}\right) \to \mathbb{R}^2$, $F(t) = (sin(2t), cos(t))$. Clearly, F is injective, dF_a is injective, and so F is a smooth immersion. However, $Im(F)$ is compact, where the domain of F is *not* compact, so $F : \left(\frac{\pi}{2} \to \frac{5\pi}{2}\right) \to F\left(\frac{\pi}{2} \to \frac{5\pi}{2}\right)$ is not a smooth embedding.

(c) Let $F \in \mathcal{C}^\infty(M, N)$ be injective and a smooth immersion. Assume that M is compact, then F^{-1} : $F(M) \to M$ is continuous (since for a closed subset A of M, we have that since M is closed that A is compact, and so $F(A)$ is compact in N, and by N being Hausdorff we have that $F(A)$ is closed in N; $(F^{-1})^{-1}$ maps closed sets to closed sets, so F^{-1} is continuous). We conclude that F is a smooth embedding.

(d) For manifolds M, N, $a \in M, b \in N$, the inclusion $i_b : M \hookrightarrow M \times N;\ x \mapsto (x, b)$ and $i_a : N \hookrightarrow M \times N;\ x \mapsto (a, x)$ are smooth embeddings (similarly, the projection maps $proj_M : M \times N \to M, proj_N : M \times N \to N$ are smooth submersions)

(e) The composition of smooth immersions (or smooth embeddings) are smooth immersions (or smooth embeddings).

Proof. We have that $d(F \circ G) = \underbrace{dF}_{injective} \circ \underbrace{dG}_{injective}$ and so we have injectivity for smooth immersions.

For embeddings, working in the situation where $M \xrightarrow{G} N \xrightarrow{F} P$, we have that if M is homeomorphic to $G(M) \subset N$ and N is homeomorphic to $F(N) \subset P$, that M is homeomorphic to $F(G(M))$. □

7.6 Immersed and embedded submanifolds

Definition. Let M be a manifold, and let $S \subset M$ be a subset. Then, $(S, \mathcal{A}_S)$ is called an **immersed submanifold** (**embedded submanifold**) if there is a manifold structure $(S, \mathcal{A}_S)$ such that the inclusion $i : S \hookrightarrow M$ is a smooth immersion (smooth embedding).

7.7 Topologies of embedded submanifolds

Remark. Note that if $S \subset M$ is an embedded submanifold, then the topology of S coming from $(S, \mathcal{A}_S)$ and from the subspace topology $S \subset M$ must coincide, because the inclusion $i : (S, \mathcal{A}_S) \hookrightarrow i(S) \subset M$ is a homeomorphism. For immersed submanifolds, this is **not** the case (see the example of the figure 8, as discussed last time).

7.8 Immersed & embedded submanifolds as images of injective smooth immersions and smooth embeddings

Proposition 26. *Let M and P be two manifolds, let $F \in \mathcal{C}^\infty(P, M)$ and let $S := F(P)$.*

(a) If F is an injective, smooth immersion, then S is an immersed submanifold.

(b) If F is an smooth embedding, then S is an embedded submanifold.

In other words: (immersed)/(embedded) submanifolds are exactly the images of (injective, smooth immersions)/ (smooth embeddings).

Proof. In both cases, we have that $F : P \to S$ is a bijection. For any chart $\varphi : U \to \varphi(U)$ of P, define a chart for S: $\varphi_S : F(U) \to \varphi(U)$, $\varphi_S := \varphi \circ (F|_{F(U)})^{-1}$. We claim that $\mathcal{A}_S := \{\varphi_S | \varphi \in \mathcal{A}_P\}$ defines an atlas for S.

- φ_S **is a chart:** (chart 1) since $\varphi_S = \varphi \circ F^{-1}$, φ_S is injective. (chart 2) We have that

  $$\varphi_S(F(U)) = \varphi(U) \subset_{open} \mathbb{R}^{dim(P)}$$

  and we have that φ_S is a chart.

- φ_S, ψ_S **are compatible:** (compatibility 1) We have $\varphi_S : F(U) \to \varphi(U)$, $\psi_S : F(V) \to \psi(V)$, and so

  $$\varphi_S(F(U) \cap F(V)) = \varphi_S(F(U \cap V)) = \varphi(U \cap V) \subset_{open} \mathbb{R}^{dim(P)}$$

  and (compatibility 2)

  $$\psi_S \circ \varphi_S^{-1} = \psi \circ F^{-1} \circ (\varphi \circ F^{-1})^{-1} = \psi \circ F^{-1} \circ F \circ \varphi^{-1} = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

  is a smooth diffeomorphism, because ψ and φ are charts in the atlas of P.

- $\mathcal{A}_S$ **is an atlas:** (atlas 1) We have that $\cup_{\varphi_S \in \mathcal{A}_S} F(U) = S$, (atlas 2) φ_2, ψ_S are compatible.

With this, we have that $F : (P, \mathcal{A}_P) \to (S, \mathcal{A}_S)$ is a smooth diffeomorphism (since $\varphi_S \circ F \circ \varphi^{-1} = id_{\varphi(U)}$ and $\varphi \circ F^{-1} \circ \varphi_S^{-1} = id_{\varphi(U)}$) and the inclusion $i : (S, \mathcal{A}_S) \hookrightarrow (M, \mathcal{A}_M)$ is the composition

$$i : (S, \mathcal{A}_S) \xrightarrow{F^{-1}} (P, \mathcal{A}_P) \xrightarrow{F} (M, \mathcal{A}_M)$$

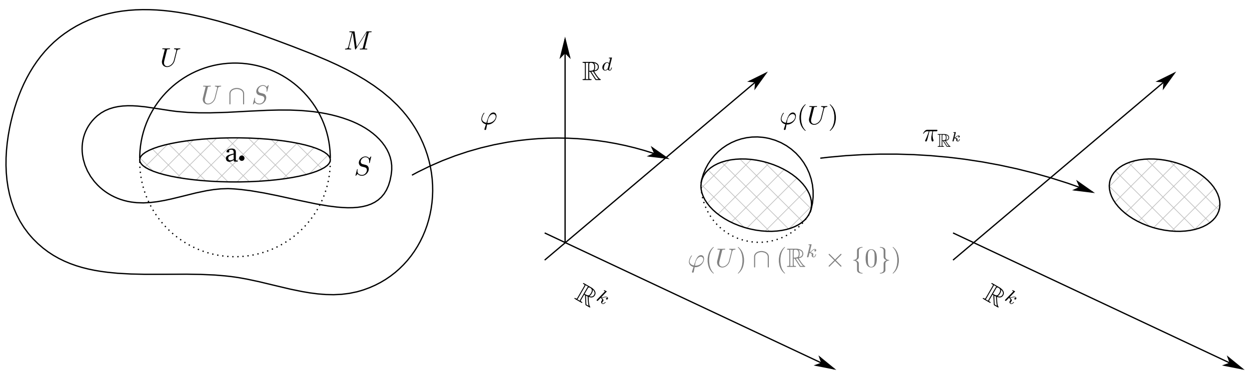
where F^{-1} is a smooth diffeomorphism, and F is a smooth immersion/embedding. As their composition is a smooth immersion (case (a))/smooth embedding (case (b)), we are done. $\square$

Note. For (a), the topology from $(S, \mathcal{A}_S)$ and from $S \subset M$ do not coincide (in general).

7.9 A property of embedded manifolds

Proposition 27. *Let M be a d-dimensional manifold and let $S \subset M$ be a subset of M. Then, S is an embedded submanifold if and only if there exists $k \leq d$ such that for all $a \in S$, there exists a chart $\varphi : U \to \mathbb{R}^d$ of M with $a \in U$ such that*

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \underbrace{\{0\}}_{\in \mathbb{R}^{d-k}}).$$



Remark. In the above situation, S has the atlas: $\mathcal{A}_S = \{\pi_{\mathbb{R}^k} \circ \varphi|_{U \cap S} : U \cap S \to \mathbb{R}^k | \varphi : U \to \mathbb{R}^d \in \mathcal{A}_M$, for φ as in the statement of our proposition, $\}$, where π_{R^k} is the projection from $\mathbb{R}^d$ to $\mathbb{R}^k$.

7.10 Equivalence between embedded submanifolds and submanifolds from definition 2.4(a)

Corollary 28. *A submanifold in the sense of definition $2.4(a)$is the same as an embedded submanifold of $(\mathbb{R}^n, \{id_{\mathbb{R}^n}\})$, in the sense of definition 7.6.*

Proof. ($\Leftarrow$) Let $\mathcal{A}_S$ be as in the statement. We need to show two things, (1) that $\mathcal{A}_S$ is an atlas, and (2) that the inclusion $i : (S, \mathcal{A}_S) \to (M, \mathcal{A}_S)$ is a smooth embedding.

1. For $\varphi : U \to \mathbb{R}^d$ with φ as in the second statement of the proposition, denote

$$\varphi_S = \pi_{\mathbb{R}^k} \circ \varphi : U \cap S \to \mathbb{R}^k.$$

- φ_S **is a chart:** (chart 1) φ_S is injective, since $\varphi(U \cap S) \subset \mathbb{R}^k \times \{0\}$; (chart 2)

$$\varphi_S(U \cap S) = \pi_{\mathbb{R}^k} \underbrace{(\underbrace{\varphi(U)}_{open\ in\ \mathbb{R}^d} \cap (\mathbb{R}^k \times \{0\})}_{open\ in \mathbb{R}^k \times \{0\}} \subset_{open} \mathbb{R}^k$$

- **compatibility:** take $\varphi_S : U \cap S \to \mathbb{R}^k, \psi_S : V \cap S \to \mathbb{R}^k$. (compatibility 1) we have that

$$\varphi_S((U \cap S) \cap (V \cap S)) = \pi_{\mathbb{R}^k}(\varphi(U \cap V) \cap \varphi(U \cap S)) = \pi_{\mathbb{R}^k}(\underbrace{\varphi(U \cap V) \cap \varphi(U)}_{\subset_{open} \mathbb{R}^d} \cap \underbrace{(\mathbb{R}^k \times \{0\})}_{\subset_{open} \mathbb{R}^d})$$

and (compatibility 2),

$$\psi_S \circ \varphi_S^{-1} : \varphi_S((U \cap S) \cap (V \cap S)) \hookrightarrow_{x \mapsto (x,0)} \varphi(U \cap V) \xrightarrow{\psi \circ \varphi^{-1}} \psi(U \cap V) \xrightarrow{\pi_{\mathbb{R}^k}} \psi_S((U \cap S) \cap (V \cap S))$$

the inclusion is smooth, the map $\psi \circ \varphi^{-1}$ is smooth, the projection map is smooth, and so $\psi_S \cap \varphi_S^{-1}$ is smooth and $(\psi_S \circ \varphi_S)^{-1} = \varphi_S \circ \psi_S^{-1}$ is also smooth, so $\psi_S \circ \varphi_S^{-1}$ is a smooth diffeomorphism.

- **Proving that these make up an atlas:** (atlas 1) We have that $\cup_\varphi (U \cap S) = S$ by assumption, and (atlas 2) is true; φ_S, ψ_S are compatible as was just shown.

2. Let $i : S \hookrightarrow M$, and consider $\varphi \in \mathcal{A}_M$ as in the second statement of our proposition with corresponding φ_S. Then,

$$\varphi \circ i \circ \varphi_S^{-1} = \varphi \circ i \circ (\pi_{\mathbb{R}^k} \circ \varphi)^{-1} : \pi_{\mathbb{R}^k}(\varphi(U)) \to \varphi(U); \quad x \mapsto (x, 0)$$

This implies that i is a smooth immersion. It remains to show that $i^{-1} : i(S) \to S$ is continuous. Let $V \subset_{open} S$ (in the $\mathcal{A}_S$ topology). We need to show that $V \subset_{open} i(S)$ where $i(S) \subset M$ has the subspace topology. Let $a \in V \subset_{open} (S, \mathcal{A}_S)$. This implies that there exists $\varphi_S : U \cap S \to \mathbb{R}^k$ such that $a \in U \cap S \subset V$. Therefore, $\varphi : U \to \mathbb{R}^d \in \mathcal{A}_M$, and $a \in U$, which is an open subset of M. Then, $a \in U \cap S \subset_{open} S$ (under the subspace topology of M). Since we can do this for all $a \in S$, we have that $V \subset S$ is open in the subspace topology of M.

($\Rightarrow$) Let $i : (S, \mathcal{A}_S) \hookrightarrow (M, \mathcal{A}_M)$ be a smooth embedding, and let $a \in S$. By proposition 7.2, there are charts $\varphi : U \to \mathbb{R}^d$ of M, $a \in U$, and $\psi : V \to \mathbb{R}^k$ of S where $a \in V$, with $V = i(V) \subset U$ such that for all $x \in \psi(V)$, the composition $\varphi \circ i \circ \psi^{-1}(x) = (x, 0)$. If necessary, restrict U and V such that $Im(\varphi \circ i \circ \psi^{-1}) = Im(\varphi) \cap (\mathbb{R}^k \times \{0\})$. Since S has the subspace topology of M, this implies that there exists a subspace $\tilde{V} \subset_{open} M$ such that $V = \tilde{V} \cap S$. Let $\tilde{U} := U \cap \tilde{V}$. Then $\varphi|_{\tilde{U}} : \tilde{U} \to \varphi(\tilde{U})$ is a chart of M such that

$$\varphi|_{\tilde{U}}(\tilde{U} \cap S) = \varphi(U \cap \tilde{V} \cap S) = \varphi(\tilde{U} \cap V) = \varphi(\tilde{U}) \cap Im(\varphi \circ i \circ \psi^{-1}) = \varphi(\tilde{U}) \cap (\mathbb{R}^k \times \{0\})$$

$\square$

7.11 Properly embedded submanifolds

Definition. Let $S \subset M$ be an embedded submanifold. Then, S is a **properly embedded submanifold**, if the inclusion $i : S \hookrightarrow M$ is a proper map, i.e., for every compact subset K of M, $i^{-1}(K) \subset S$ is a compact subset of S. Recall from topology that if you have any map $F : S \to M$ which is continuous, if S is compact and M is Hausdorff, then F is a proper map. Therefore, if $S \subset M$ is an embedded submanifold, and S is compact, then S is a properly embedded submanifold.

As an exercise, let $S \subset M$ be an embedded submanifold. Then, S is a properly embedded submanifold if and only if $S \subset M$ is closed.

7.12 Whitney Embedding Theorem

Theorem 29. *(Whitney Embedding Theorem) Let M be a smooth manifold, and assume that M is compact. Then, there exists an integer n such that there exists a proper smooth embedding $F : M \to \mathbb{R}^n$.*

Proof. For $a \in M$, let φ_a be a chart $\varphi_a : U_a \to \mathbb{R}^d$ of M (of dimension d) at a, i.e. $a \in U_a$. WLOG, assume that $\varphi_a(a) = 0$. Let $\epsilon(a) > 0$ be such that $B_{\epsilon(a)} \subset \varphi(U_a)$. Denote by $E_a := \varphi_a^{-1}(B_{\epsilon(a)})$ and $D_a := \varphi_a^{-1}(B_{\epsilon(a)/2})$. Note that the $\{D_a\}_{a \in M}$ is an open cover of M. Since M is compact, there exist $a_1, ..., a_r$ such that $D_{a_1} \cup ... \cup D_{a_r} = M$. Let $f_1, ..., f_r$ be functions, $f_j \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $0 \leq f_j \leq 1$, and $supp(f_j) \subset E_{a_j}$, $f_j|_{\overline{D_{a_j}}} = 1$. Note that $f_j \cdot \varphi_{a_j} : M \to \mathbb{R}^d$ is a smooth map (where φ_{a_j} is extended to M by 0 outside of U_{a_j}). Now let $n = r \cdot d + r$ and define $F : M \to \mathbb{R}^{rd+r}$,

$$F(x) := (f_1(x) \cdot \varphi_{a_1}(x), f_2(x) \cdot \varphi_{a_1}(x), \ldots, f_r(x) \cdot \varphi_{a_r}(x), f_1(x), ..., f_r(x)).$$

(if you get confused, just look at it until you remember that $f_1(x) \cdot \varphi_{a_1}(x) : M \to \mathbb{R}^d$.) We claim that F is a proper smooth embedding, which if we can prove, we are then done.

Is F injective? Let $F(x) = F(y)$. Since $x \in D_{a_j}$ for some a_j, it follows that $1 = f_j(x) - f_j(y)$ (since $F(x) = F(y)$), and so $y \in E_{a_j} \subset \varphi_{a_j}(U_{a_j})$. Also, $\varphi_{a_j}(x) = f_{a_j}(x) \cdot \varphi_{a_j}(x) = f_{a_j}(y) \cdot \varphi_{a_j}(y)$ (again, since $F(x) = F(y)) = \varphi_{a_j}(y)$. But since φ_{a_j} is injective, and both $x, y \in U_{a_j}$, it follows that $x = y$.

Is dF_a injective? For $x \in D_{a_j}$, we have that $f_j|\overline{D_{a_j}} = 1$, implying that $d(f_j \cdot \varphi_{a_j})_x = d(\varphi_{a_j}(x)$. To see this, apply $v \in T_xM$ to this, and use $5.2(c)$. Now, $d(\varphi_{a_j})_x$ is an isom. by 5.4(e), implying that $dF_x = (..., d(f_j \cdot \varphi_{a_j})_x, ...) = (..., \underbrace{d(\varphi_{a_j})}_{injective}, ...)$, so dF_x is injective. As such, F is an injective smooth immersion.

This implies by 7.5(c) that F is a smooth embedding, which implies by 7.11 that F is a proper smooth embedding (both of these use the fact that M is compact). □

7.13 Whitney's theorems

Theorem 30. *(a) (Whitney's Embedding Theorem; general version (see Lee)) Let M be a smooth manifold of dimension $d \geq 0$. Then there exists a proper smooth embedding $F : M \to \mathbb{R}^{2d+1}$.*

Remark. **Remarks on the Proof:** The proof of the above uses:

(a) The notion of a manifold with boundary.

(b) Sards' Theorem, which says: let $F \in \mathcal{C}^\infty(M, N)$, where M, N are smooth manifolds (with or without boundary). Let $C := \{x \in M | dF_x$ is not surjective $\}$ be the set of critical points of F. Then, $F(C)$ has measure 0 in N.

(b) (Whitney Embedding Theorem - Strong Version) Let M be a smooth manifold of dimension $d > 0$. Then, there exists a smooth embedding $F : M \to \mathbb{R}^{2d}$.

Remark. **Remark on the Proof:** In proving above, you first find a smooth immersion $F : M \to \mathbb{R}^{2d}$, and use the 'Whitney trick' to remove self-intersection. This is the starting point for surgery theory, which classifies manifolds of dimension ≥ 5. In fact, $M^2 \hookrightarrow \mathbb{R}^4$ and $M^4 \hookrightarrow \mathbb{R}^8$ are the best possible dimensions to embed 2-manifolds and 4-manifolds, respectively. But, every 3-manifold $M^3 \hookrightarrow M^5$.

(c) (Strong Whitney Embedding Immersion Theorem) Let M be a smooth manifold of dimension $d > 1$. Then there exists a smooth immersion $F : M \to \mathbb{R}^{2d-1}$.

(d) Generalization of (c) (Ralph Cohen, '85): Let M be a smooth manifold of dimension $d > 1$. Assume that M is compact, then there exists a smooth immersion $F : M \to \mathbb{R}^{2d-a(d)}$ where $a(d)$ is the number of '1's in the binary expansion of d.[†]

[†] See also, the Nash Embedding Theorem.

8 Cotangent Vector and Tensors

This chapter is modeled after Lee's book, chapters 11 and 12.

8.1 Dual Spaces

- Let V be a finite dimensional vector space (over $\mathbb{R}$). Then, the dual vector space V^* is the set of maps
$$V^* := \{\alpha : V \to \mathbb{R} | \alpha \text{ is a linear map } \}$$
- If $\{e_1, ..., e_d\}$ is a basis for V, then there is a dual basis $\{e_1^*, ..., e_d^*\}$ of V^* given by $e_j^*(e_\ell) = \delta_{j,\ell}$, $e_j^* : V \to \mathbb{R}$, induced linearly to all of V. Note that the isomorphism $V \to V^*$, $e^j \mapsto e_j^*$ is not canonical, i.e., it depends on the chosen basis $\{e_j\}$.
- However, the map $\theta : V \to V^{**}$,
$$\theta(v) : V^* \to \mathbb{R}, \qquad \theta(v)(\alpha) := \alpha(v)$$
is a canonical isomorphism, and is given by the evaluation map.
- If $A : V \to W$ is a linear map, then there is an induced linear map, $A^* : W^* \to V^*$, where $A^*(\beta)(v) = \beta(Av)$ for $v \in V$, as illustrated below.

$$\begin{tikzcd} W \arrow[d, "\beta\in W^*"'] & v \in V \arrow[l, "A"'] \arrow[l, bend right, "Av"'] \arrow[dl, dashed, "\beta(Av)\in V^*"] \\ \mathbb{R} & \end{tikzcd}$$

8.2 The Cotangent space of a manifold

Definition. Let M be a manifold of dimension d, and let $a \in M$. Then, the **cotangent space** of M at a is the dual space of T_aM, denoted by $T_a^*M := (T_aM)^*$.

Take M to be a manifold of dimension d, and $a \in M$: Let $\varphi : U \to \mathbb{R}^d$ be a chart, and denote (as usual) by $\{\frac{\partial}{\partial x^j}\Big|_j\}_{j=1...d}$ the basis of T_aM given by the chart φ. Denote by $dx^j|_a := \left(\frac{\partial}{\partial x^j}\Big|_a\right)^*$ a member of the **dual basis**, given by
$$(dx^j|_a)\left(\frac{\partial}{\partial x^k}\Big|_a\right) := \delta_{jk}$$

Remark. For another chart $\psi : V \to \mathbb{R}^d$ of M at a, with induced basis $\{\frac{\partial}{\partial \tilde{x}^j}\Big|_a\}$ of T_aM and $\{d\tilde{x}^j\}_j$ of T_a^*M, we have the transformation rules:

$$d\tilde{x}^k\left(\frac{\partial}{\partial x^i}\right) \overset{5.6(2)}{=} d\tilde{x}^k\left(\sum_{j=1}^{d}\frac{\partial(\psi\circ\varphi^{-1})^j}{\partial x^i}\cdot\frac{\partial}{\partial\tilde{x}^j}\right) = \sum_{j=1}^{d}\frac{\partial(\psi\circ\varphi^{-1})^j}{\partial x^i}\delta_{kj} = \frac{\partial(\psi\circ\varphi^{-1})^k}{\partial x^i}$$

summarizing,

$$d\tilde{x}^k\Big|_a = \sum_{j=1}^{d}\frac{\partial(\psi\circ\varphi^{-1})^k}{\partial x^j}(\varphi(a))\cdot dx^j\Big|_a \tag{8.2.1}$$

If $F \in \mathcal{C}^\infty(M,N)$, then there is an induced map $dF_a : T_aM \to T_{F(a)}N$, implying

$$dF_a^* : T_a^*N \to T_a^*M; \qquad dF_a^*(\beta_{F(a)})(X_a) = \beta_{F(a)}(dF_a(X_a))$$

in diagram form, this looks something like this:

$$\begin{tikzcd} X_a \in T_aM \arrow[r, "dF_a"] \arrow[dr, dashed, "\beta_{F(a)}(dF_a(X_a))"'] & T_{F(a)}N \arrow[d, "\beta_{F(a)}\in T^*_{F(a)}N"] \\ & \mathbb{R} \end{tikzcd}$$

If $\varphi : U \to \mathbb{R}^d$ is a chart of M at a, and $\psi : V \to \mathbb{R}^k$ is a chart of N at $F(a)$, and $F(U) \subset V$, then take basis $\{\frac{\partial}{\partial x^j}\}$ of T_aM, $\{dx^j\}$ basis of T_a^*M, basis $\{\frac{\partial}{\partial \tilde{x}^j}\}$ of $T_{F(a)}N$, $\{d\tilde{x}^j\}$ basis of $T^*_{F(a)}N$. Then, we have

$$dF_a^*\left(d\tilde{x}^j\Big|_{F(a)}\right)\left(\frac{\partial}{\partial x^i}\Big|_a\right) = d\tilde{x}^j\left(dF_a\left(\frac{\partial}{\partial x^i}\right)\right) \overset{5.6(3)}{=} d\tilde{x}^j\left(\sum_{\ell=1}^{k}\frac{\partial\left(\psi\circ F\circ\varphi^{-1}\right)^\ell}{\partial x^i}(\varphi(a))\cdot\frac{\partial}{\partial\tilde{x}^\ell}\right)$$

$$= \frac{\partial(\psi\circ F\circ\varphi^{-1})^j}{\partial x^i}(\varphi(a))$$

summarizing again,

$$dF_a^*\left(d\tilde{x}^j|_{F(a)}\right) = \sum_{i=1}^{d}\frac{\partial\left(\psi\circ F\circ\varphi^{-1}\right)^j}{\partial x^i}(\varphi(a))\cdot dx^i\Big|_a \tag{8.2.2}$$

8.3 Cotangent bundles, 1-forms

(a) Let $T^*M := \{(a,\beta) | a \in M, \beta \in T_a^*M\}$, called the **cotangent bundle**, and let

$$\pi : T^*M \to M, \qquad (a,\beta) \mapsto a.$$

For a chart $\varphi : U \to \mathbb{R}^d$ of M, define a chart $\varphi_{T^*M} : \pi^{-1}(U) \to \mathbb{R}^{2d}$,

$$\varphi\left(a, \sum_{j=1}^{d}\beta^i\cdot dx^j|_a\right) = \left(\varphi(a), \beta^1, ..., \beta^d\right)$$

where $\sum_{j=1}^{d}\beta^i\cdot dx^j|_a$ is in T_a^*M, and $\beta^i \in \mathbb{R}$.

Claim. Chart of this form define an atlas $\mathscr{A}_{T^*M}$, giving T^*M a $2d$ dimensional manifold structure, such that $\pi \in \mathcal{C}^\infty(T^*M, M)$.

Proof. Similar to the proof of 6.1, so it is left as an exercise. ☐

(b) A **covector field**, or **1-form** on M is a section $\beta : M \to T^*M$ of π, i.e., $\pi \circ \beta = id_M$ (this is what it means to be a section). The set of all smooth 1-forms is denoted by $\mathbb{X}^*(M)$. , and we call β **smooth** if $\beta \in \mathcal{C}^\infty(M, T^*M)$.

Claim. The following are equivalent:

(a) β is smooth.

(b) For every $X \in \mathbb{X}(M)$ (smooth vector fields on M), the map (which are now defining) $(\beta, X) : M \to \mathbb{R},\ (\beta, X)(a) := \beta_a(X_a)$ is smooth.

(c) For $\varphi : U \to \mathbb{R}^d$ a chart of M, $\beta|_U : U \to T^*M$, $\beta|_U(y) = \sum_{j=1}^d \beta^j(y)(dx^j|_y)$, $\beta^j \in \mathcal{C}^\infty(U, \mathbb{R})$.

Proof. Similar to the proof of 6.5, so left as an exercise. ☐

Remark. If we take $\beta \in \mathbb{X}^*(M)$, $f \in \mathcal{C}^\infty(M, \mathbb{R})$, and $X \in \mathbb{X}(M)$, then

$$\beta(f \cdot X)(a) = \beta_a((f \cdot X)|_a) = \beta_a(f(a) \cdot X_a) = f(a) \cdot \beta_a(X_a) = (f \cdot \beta(X))(a)$$

But, we had

$$X(fg)(a) = X_a(f)g(a) + f(a) \cdot X_a(g) = (X(f) \cdot g + f \cdot X(g))(a)$$

I am not entirely sure why this was pointed out, other than it may prompt the discussion of differentials below.

8.4 Differentials

Definition. Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$, then we define the **differential of** f **at** a, $df_a \in T_a^*M$, by setting

$$df_a(X_a) := X_a(f)$$

for $X_a \in T_aM$.

Claim. $df \in \mathbb{X}^*(M)$ is a smooth 1-form.

Proof. By 8.3(b), df is smooth if and only if $df_a(X_a) \in \mathcal{C}^\infty(M, \mathbb{R})$ for all smooth vector fields X. But, if $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and $X \in \mathbb{X}(M)$, this implies that $X(f) \in \mathcal{C}^\infty(M, \mathbb{R})$, and $X(f)(a) = X_a(f) = (df_a(X_a))$. Hence, df is smooth. ☐

Remark. (a) Let $\varphi : U \to \mathbb{R}^d$ be a chart. Then,

$$df_a\left(\frac{\partial}{\partial x^j}\Big|_a\right) = \frac{\partial}{\partial x^j}\Big|_a(f) \overset{5.6(1)}{=} \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(a)) \in \mathcal{C}^\infty(U, \mathbb{R})$$

(b) Recall that for $f \in \mathcal{C}^\infty(M, \mathbb{R})$, there is an induced map $df_a : T_aM \to T_{f(a)}\mathbb{R}$ whereas now, $df_a : T_aM \to \mathbb{R}$. In 5.3, we had for $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ that

$$df_a : T_aM \to T_{f(a)}\mathbb{R},$$

$$\Rightarrow df_a\left(\frac{\partial}{\partial x^j}\Big|_a\right)(g) = \frac{\partial}{\partial x^j}\Big|_a(g \circ f)$$

$$= \frac{\partial(g \circ f \circ \varphi^{-1})}{\partial x^j}(\varphi(a)) \overset{chain\ rule}{=} \frac{dg}{dx}(f \circ \varphi^{-1}(\varphi(a))) \cdot \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(a))$$

$$= \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(a)) \cdot \frac{d}{dx}\Big|_{f(a)}(g)$$

In summary,

$$\begin{tikzcd}
T_aM \arrow[r, "df_a\ as\ in\ 5.3"] \arrow[dr, "df_a\ as\ in\ 8.4"'] & T_{f(a)}\mathbb{R} \\
 & \mathbb{R}
\end{tikzcd}$$

(c) if $\varphi : U \to \mathbb{R}^d$ is a chart of M, denote by φ^j the j^{th} component of φ in $\mathbb{R}^d$, i.e., $\varphi = (\varphi^1, \varphi^2, ..., \varphi^d)$, where $\varphi^j : M \to \mathbb{R}$. Then,

$$\begin{aligned}
d\varphi^j_a \left(\frac{\partial}{\partial x^k}\Big|_a \right) &= \frac{\partial(\varphi^j \circ \varphi^{-1})}{\partial x^k}(\varphi(a)) \\
&= \frac{\partial x^j}{\partial x^k}(\varphi(a)) = \delta_{jk} \\
&\Rightarrow d\varphi^j_a = dx^j|_a
\end{aligned}$$

8.5 Maps between smooth 1-forms on manifolds

Definition. Let $F \in \mathcal{C}^\infty(M, N)$, (where F is not necessarily a diffeomorphism) then there is an induced map $dF_a : T_aM \to T_{F(a)}N$, so we get

$$dF_a^* : T^*_{F(a)}N \to T^*_a M$$

Claim. This implies that $F^* : \mathbb{X}^*(N) \to \mathbb{X}^*(M)$ where $F^*(\beta)_a := dF^*_a(\beta_{F(a)})$.

For $\beta \in \mathbb{X}^*(N)$, $X \in \mathbb{X}(M)$, it is

$$F^*(\beta)_a(X_a) = dF^*_a(\beta_{F(a)})(X_a) = \beta_{F(a)}(\underbrace{dF_a(X_a)}_{\in T_{F(a)}N})$$

Proof. For charts $\varphi : U \to \mathbb{R}^d$ of M, $a \in U$, and $\psi : V \to \mathbb{R}^k$ of N, $F(a) \in V, F(U) \subset V$, let

$$\beta = \sum_{j=1}^{k} \beta^j d\tilde{x}^j|_{F(a)} \in \mathbb{X}^*(V).$$

Then,

$$F^*(\beta) = dF^*_a \left(\sum_{j=1}^{k} \beta^j(F(a)) d\tilde{x}^j|_{F(a)} \right) \overset{dF^*_a\ llinear,\ 8.2(2)}{=} \sum_{j=1}^{k} \beta^j(F(a)) \cdot \sum_{i=1}^{d} \frac{\partial(\psi \circ F \circ \varphi^{-1})^j}{\partial x^i}(\varphi(a)) \cdot dx^i|_a$$

$$= \sum_{i=1}^{d} \underbrace{\left(\sum_{j=1}^{k} \beta^j(F(a)) \cdot \frac{\partial(\psi \circ F \circ \varphi^{-1})^j}{\partial x^i} \cdot (\varphi(a)) \right)}_{smooth,\ U \to \mathbb{R}} \cdot dx^i|_a$$

□

8.6 Lemma

Lemma 31. *Let $F \in \mathcal{C}^\infty(M, N)$, let $f \in \mathcal{C}^\infty(N, \mathbb{R})$ so that $df \in \mathbb{X}^*(N)$. Then,*

$$F^*(df) = d(f \circ F) \in \mathbb{X}^*(M)$$

Proof. For $a \in M, X_a \in T_aM$, it is,

$$F^*(df)_a(X_a) = dF^*_a(df_{F(a)})(X_a) = df_{F(a)}(dF_a(X_a)) = (dF_a(X_a))(f) = X_a(f \circ F) = d(f \circ F)_a(X_a)$$

□

8.7 Tensors

Definition. (a) let $V_1, ..., V_k, W$ be $\mathbb{R}$-vector spaces of finite dimension. A map $\alpha : V_1 \times V_2 \times ... \times V_k \to W$ is called **multilinear** if it is linear in each component:

$$\alpha(v_1, v_2, ..., rv_j + s\tilde{v}_j, ..., v_k) = r \cdot \alpha(v_1, ..., v_j, ..., v_k) + s \cdot \alpha(v_1, ..., v_k)$$

for all $r, s \in \mathbb{R}$ and $j = 1, ..., k$ (note that $rv_j + sv_j \in V_j$). The space of all multilinear maps $\alpha : V_1 \times ... \times V_k \to W$ is denoted by $L(V_1, ..., V_k; W)$. Note that $L(V_1, ..., V_k; W)$ is an $\mathbb{R}$-vector space (you define the sum of maps in the way you think you would, and scalar multiplication is similarly natural).

(b) Let V be a finite dimensional $\mathbb{R}$-vector space (later, we'll set $V := T_a M$). Then, a **tensor of type** (k, l) is a multilinear map

$$\underbrace{V^* \times \ldots \times V^*}_{k} \times \underbrace{V \times ... \times V}_{l} \to \mathbb{R}$$

The space of tensors of type (k, l) is denoted by

$$T^{(k,l)}(V) := T_k^\ell(V) := L(\underbrace{V^*, ..., V^*}_{k}, \underbrace{V, ..., V}_{\ell}; \mathbb{R})$$

In particular,

$$T^{(0,0)} = \mathbb{R}, \qquad T^{(0,1)}(V) = L(V; R) = V^*, \qquad T^{(1,0)} = L(V^*; \mathbb{R}) = V^{**} \cong V$$

where in the third equation, we send v from V into V^{**} by sending $v \mapsto eval(v)$.

8.8 $\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k \in L(V_1, ..., V_k; \mathbb{R})$

Lemma 32. *(a) If $\alpha_1 \in V_1^*$, $\alpha_2 \in V_2^*$, $\alpha_k \in V_k^*$, then*

$$\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k \in L(V_1, ..., V_k; \mathbb{R})$$

given by

$$(\alpha_1 \otimes ... \otimes \alpha_k)(v_1, ..., v_k) := \alpha_1(v_1) \cdot \alpha_2(v_2) \cdot ...\alpha_k(v_k)$$

(b) If $\{\alpha_1^j\}_j$ is a basis of V_1^*, $\{\alpha_2^j\}_j$ is a basis of V_2^*, ..., $\{\alpha_k^j\}_j$ is a basis of V_k^*, then

$$\{\alpha_1^{j_1} \otimes \alpha_2^{j_2} \otimes ... \otimes \alpha_k^{j_k}\}_{j_1, j_2, ..., j_k}$$

is a basis of $L(V_1, ..., V_k; \mathbb{R})$. In particular, if $dim(V_1) = d_1$, then $dim(L(V_1, ..., V_k; \mathbb{R})) = d_1 \cdot d_2 \cdot ... \cdot d_k$, and therefore,

$$dim(T^{k,\ell}(V)) = dim(L(V^*, ..., V^*, V, ..., V; \mathbb{R})) = dim(V)^{k+l}.$$

Proof. (a) We have,

$$(\alpha_1 \otimes ... \otimes \alpha_j \otimes ... \otimes \alpha_k)(v_1, ..., rv_j + s\tilde{v}_j, ..., v_k) = \alpha_1(v_1) \cdot ... \cdot (\alpha_j(rv_j + s\tilde{v}_j)) \cdot ... \cdot \alpha_k(v_k)$$

and since α_k is linear, and multiplication is linear,

$$= r(\alpha_1 \otimes ... \otimes \alpha_k)(v_1, ..., v_j, ..., v_k) + s(\alpha_1 \otimes ... \otimes \alpha_k)(v_1, ..., \tilde{v}_j, ..., v_k).$$

(b) Let $\{v_p^j\}_j$ be a basis of V_p such that $\alpha_p^j(v_p^i) = \delta_{ij}$. In showing the span, let $\beta \in L(V_1, ..., V_k; \mathbb{R})$, and define $\beta_{j_1,...,j_k} := \beta(v_1^{j_1}, ..., v_k^{j_k}) \in \mathbb{R}$. Claim:

$$\beta = \sum_{j_1,...,j_k} \beta_{j_1,...,j_k} \; \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k}$$

To show this, we apply the right hand map to $(v_1^{j_1}, ..., v_k^{j_k})$, and get

$$\sum_{j_1,...,j_k} \beta_{j_1,...,j_k} \; \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k}(v_1^{j_1}, ..., v_k^{j_k}) = \sum_{j_1,...,j_k} \beta_{j_1,...,j_k} \alpha_1^{j_1}(v_1^{j_1}) \cdot ... \cdot \alpha_k^{j_k}(v_k^{j_k}) = \beta_{j_1,...,j_k}$$

which is equal to $\beta(v_1^{j_1}, ..., v_k^{j_k})$, by definition. Hence, as multilinear maps are determined by their image of a basis, and from their equality, we conclude that our claim must hold. We perform a similar check to show linear dependence, if

$$\sum_{j_1,...,j_k} c_{j_1,..,j_k} \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k} = 0$$

to show that $c_{j_1,...,j_k} = 0$, we have

$$0 = \sum_{j_1,...,j_k} c_{j_1,...,j_k} \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k}(v_1^{i_1}, ..., v_k^{i_k}) = \sum_{j_1,...,j_k} c_{j_1,...,j_k} \underbrace{\alpha_1^{j_1}(v_1^{i_1})}_{\delta_{j_1,i_1}} ... \alpha_k^{j_k}(v_k^{i_k}) = c_{i_1,...i_k}$$

□

8.9 Defining the tensor product $V_1 \otimes V_2$

Definition. Let $V_1, ..., V_k$ be finite dimensional $\mathbb{R}$-vector spaces. Denote by $\mathcal{F}$ the free $\mathbb{R}$-vector space generated by $V_1 \times V_2 \times ... \times V_k$ (i.e., elements of $\mathcal{F}$ are finite sums $\sum_j c_j \beta_j$, where each $\beta_j \in V_1 \times ... \times V_k$).

Let $\mathcal{R} \subset \mathcal{F}$ be the linear subspace generated by elements of the form

$$\begin{cases} (v_1, ..., rv_\ell, ..., v_k) - r \cdot (v_1, ..., v_\ell, ..., v_k) \in \mathcal{F} \\ (v_1, ..., v_\ell + \tilde{v}_\ell, ..., v_k) - (v_1, .., v_\ell, ..., v_k) - (v_1, ..., \tilde{v}_\ell, ..., v_k) \in \mathcal{F} \end{cases}$$

then define the tensor product of $V_1, ..., V_k$ as

$$V_1 \otimes V_2 \otimes ... \otimes V_k := \mathcal{F}/\mathcal{R}$$

we claim that there is a canonical (independent of choices) isomorphism

$$V_1 \otimes ... \otimes V_k \cong L(V_1^*, V_2^*, ..., V_k^*; \mathbb{R})$$

In particular,

$$T^{(k,\ell)}(V) = L(V^*, .., V^*, V, ..., V, \mathbb{R}) \cong L(V^*, ..., V^*, V^{**}, ..., V^{**}; \mathbb{R}) = V \otimes ... \otimes V \otimes V^* \otimes ... \otimes V^*$$

Proof. Since $V_\ell \cong V_\ell^{**}$ via a canonical isomorphism, setting $W_\ell := V_\ell^*$, the claim is equivalent to proving $W_1^* \otimes ... \otimes W_k^* \cong L(W_1, .., W_k; \mathbb{R})$. Define $\Phi : W_1^* \times ... \times W_k^* \to L(W_1, ..., W_k; \mathbb{R})$,

$$\Phi(\alpha_1, .., \alpha_k)(w_1, ..., w_k) := \alpha_1(w_1) \cdot ... \cdot \alpha_k(w_k)$$

in other words, $\Phi(\alpha_1, ..., \alpha_k) = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k$. Since $\mathcal{F}$ is generated by the set $W_1^* \times ... \times W_k^*$, we can extend Φ by linearity to $\Phi^{\mathcal{F}} : \mathcal{F} \to L(W_1, ..W_k; \mathbb{R})$. We claim that $\Phi^{\mathcal{F}}|_{\mathcal{R}} = 0$. This implies that $\Phi^{\mathcal{F}} : \mathcal{F} \to L(W_1, ..., W_k; \mathbb{R})$ induces a map (which happens to be an isomorphism,)

$$\Phi^{\otimes} : \mathcal{F}/\mathcal{R} \to L(W_1, ..., W_k; \mathbb{R})$$

given by

$$[\alpha_1, ..., \alpha_k] \mapsto \alpha_1 \otimes ... \otimes \alpha_k.$$

we need to show that $\Phi^\otimes$ is an isomorphism. Let $\{\alpha_p^j\}$ be a basis of W_p^*. Then (by definition) a basis of the free vector space $\mathcal{F}$ is given by $\{(\sum_{j_1} c_{j_1}^1 \alpha_1^{j_1}, ..., \sum_{j_k} c_{j_k}^k \alpha_k^{j_k})\}_{c_{j_1}^1,...,c_{j_k}^k}$. On $\mathcal{F}/\mathcal{R}$, we have

$$\left[\sum c_{j_1}^1 \alpha_1^{\alpha_1}, ..., \sum c_{j_k}^k \alpha_k^{j_k}\right] = \sum_{j_1,...,j_k} c_{j_1}^1 ... c_{j_k}^k \left[\alpha_1^{j_1}, ..., \alpha_k^{j_k}\right]$$

so that $[\alpha_1^{j_1}, ..., \alpha_k^{j_k}]$ span $\mathcal{F}/\mathcal{R}$. First, we show that $\Phi^\otimes$ is surjective: we have

$$\Phi^\otimes[\alpha_1^{j_1}, ..., \alpha_k^{j_k}] = \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k}$$

which is a basis of $L(W_1, .., W_k; \mathbb{R})$ by 8.8(b). Now, we show that $\Phi^\otimes$ is injective: assume that

$$\Phi^\otimes\left(\sum_{j_1,...,j_k} c_{j_1,...,j_k}[\alpha_1^{j_1}, ..., \alpha_k^{j_k}]\right) = 0.$$

We need to show $c_{j_1}, ..., c_{j_k} = 0$. Let $\{w_p^{i_p}\}$ be a basis of W_p such that $\alpha_p^{j_p}(w_p^{i_p}) = \delta_{j_p i_p}$. Then,

$$\begin{aligned} 0 = \Phi^\otimes\left(\sum_{j_1,...,j_k} c_{j_1,...,j_k}[\alpha_1^{j_1}, ..., \alpha_k^{j_k}]\right)(w_1^{i_1}, ..., w_k^{i_k}) \\ = \sum c_{j_1,...,j_k} \alpha_1^{j_1} \otimes ... \otimes \alpha_k^{j_k}(w_1^{i_1}, ..., w_k^{i_k}) \\ = \sum_{j_1,...,j_k} c_{j_1,...,j_k} \alpha_1^{j_1}(w_1^{i_1}) \cdot ... \cdot \alpha_k^{j_k}(w_k^{i_k}) \end{aligned}$$

since the α's are just δ_{j_k,i_i}, we have

$$= c_{i_1,...,i_k} = 0$$

for all $i_1, ..., i_k$. □

The summary of the rest of the proof is that $V_1 \otimes ... \otimes V_k \cong L(V_1^*, ..., V_k^*, \mathbb{R})$, given by $[v_1, .., v_k] \mapsto \theta(v_1) \otimes ... \otimes \theta(v_k)$ where $\theta(v_1) \in V^{**}$, and we have $[v_1, ..., rv_j + s\tilde{v}_j, ..., v_k] = r[v_1, ..., v_k, .., v_k] + s[v_1, ..., \tilde{v}_j, ..., v_k]$. Under this identification, we simply write: $v_1 \otimes ... \otimes (rv_j + s\tilde{v}_j) \otimes ... \otimes v_j = rv_1 \otimes ... \otimes v_j \otimes ... \otimes v_k + sv_1 \otimes ... \otimes \tilde{v}_j \otimes ... \otimes v_j$, and we have our claim.

8.10 Linear maps and $T^{(0,\ell)}$

Definition. Let $A : V \to W$ be a linear map between finite dimensional $\mathbb{R}$-vector spaces. Then, there is an induced map $A^* : T^{(0,\ell)}(W) \to T^{(0,\ell)}(V)$;

$$A^* : L(\underbrace{W, ..., W}_{\ell}; \mathbb{R}) \to L(\underbrace{V, ..., V}_{\ell}; \mathbb{R})$$

$$A^*(\alpha)(v_1, ..., v_\ell) := \alpha(A(v_1), ..., A(v_\ell))$$

8.11 Tensors in the context of manifolds.

Definition. Let M be a smooth manifold of dimension d, and let $a \in M$. A **tensor of type** (k, ℓ) **on** M **at** a is an element of $T^{(k,\ell)}(T_aM)$ which by definition, is a multilinear map

$$\underbrace{T_a^*M \times, ..., \times T_a^*M}_{k} \times \underbrace{T_aM \times, ..., \times T_aM}_{\ell} \to \mathbb{R}.$$

For a chart $\varphi : U \to \mathbb{R}^d$ of M with $a \in U$, we can write $\alpha_a \in T^{(k,\ell)}(T_aM)$ as (by 8.8(b), 5.6(a), 8.2(b)):

$$\alpha_a = \sum_{i_1 \ldots i_k, j_1, \ldots, j_k}^{d} \alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}\Big|_a \left(\frac{\partial}{\partial x^{i_1}}\Big|_a\right) \otimes ... \otimes \left(\frac{\partial}{\partial x^{i_k}}\Big|_a\right) \otimes (dx^{j_1}|_a) \otimes ... \otimes (dx^{j_k}|_a)$$

we call $\alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} \in \mathbb{R}$ the components (or component functions) of $\alpha_a \in T^{(k,\ell)}(T_aM)$.

8.12 Tensor Bundles

Let M be a smooth manifold of dimension d.

(a) Let

$$T^{(k,\ell)}(TM) := \{(a, \alpha) | a \in M, \alpha_a \in T^{(k,\ell)}(T_aM)\},$$

let $\pi : T^{(k,\ell)}(TM) \to M$ where $\pi(a, \alpha_a) = a$, and call it the **tensor bundle of** M **of (mixed) type** (k, ℓ). Then, $T^{(k,\ell)}(TM)$ is a manifold of dimension $d + d^{k+\ell}$, given by charts as follows: for each chart $\varphi : U \to \mathbb{R}^d$ of M, define $\varphi_{T^{(k,\ell)}(TM)} : \pi^{-1}(U) \to \mathbb{R}^{d+d^{k+\ell}}$, by mapping

$$(a, \sum_{i_1 \ldots i_k, j_1, \ldots, j_k}^{d} \alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}\Big|_a \left(\frac{\partial}{\partial x^{i_1}}\Big|_a\right) \otimes ... \otimes \left(\frac{\partial}{\partial x^{i_k}}\Big|_a\right) \otimes (dx^{j_1}|_a) \otimes ... \otimes (dx^{j_k}|_a)) \mapsto (\varphi(a), \underbrace{\alpha^{1,\ldots,1}_{1,\ldots,1}, ..., \alpha^{d,\ldots,d}_{d,\ldots,d}}_{d^{k+\ell} \ components})$$

Then, $\mathcal{A}_{T^{(k,\ell)}} = \{\varphi_{T^{(k,\ell)}} | \varphi \in \mathcal{A}_M\}$ is an atlas of $T^{(k,\ell)}(TM)$ such that π is smooth.

Proof. Similar to 6.1 and 8.3(a) □

(b) A **tensor field of mixed type (**k, ℓ**)** is a section $\alpha : M \to T^{(k,\ell)}(TM)$ of π, i.e.,

$$\pi \circ \alpha = id_M$$

The following are equivalent:

(1) α is smooth

(2) $\alpha \in \mathcal{C}^\infty(M, T^{(k,\ell)}(TM))$

(3) For all $\rho^1, ..., \rho^k \in \mathbb{X}^*(M)$, and for all $X^1, ..., X^\ell \in \mathbb{X}(M)$, the following is also smooth:

$$\alpha_a(\rho^1_a, ..., \rho^k_a, X^1_a, ..., X^\ell_a) \in \mathcal{C}^\infty(M, \mathbb{R}),$$

(4) For all charts $\varphi : U \to \mathbb{R}^d$, the component functions $a \mapsto \alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_\ell}|_a$ are smooth; $\in \mathcal{C}^\infty(M, \mathbb{R})$

Proof. Similar to the proof of 6.5. □

The set of all smooth tensor fields of type (k, ℓ) is denoted by

$$\mathcal{T}^{(k,\ell)}(M) = \{\alpha \in \mathcal{C}^\infty(M, T^{(k,\ell)}(TM)) | \pi \circ \alpha = id_M\}.$$

8.13 Tensor fields and multilinear maps

Lemma 33. *(a) Tensor fields $\alpha \in \mathcal{T}^{(k,\ell)}(M)$ are precisely the $\mathcal{C}^\infty(M,\mathbb{R})$-multilinear maps*

$$\alpha : \underbrace{\mathbb{X}^*(M) \times ... \times \mathbb{X}^*(M)}_{k} \times \underbrace{\mathbb{X}(M) \times ... \times \mathbb{X}(M)}_{\ell} \to \mathcal{C}^\infty(M,\mathbb{R})$$

i.e. for all $\rho^1, ..., \rho^k \in \mathbb{X}^*(M), x^1, ..., x^\ell \in \mathbb{X}(M)$, for all $f \in \mathcal{C}^\infty(M,\mathbb{R})$,

$$\alpha(\rho^1, ..., f \cdot \rho^j, ..., \rho^k, x^1, .., x^\ell) = f \cdot \alpha(\rho^1 ...., \rho^j, ..., \rho^k, x^1, ..., x^\ell)$$

and

$$\alpha(\rho^1, ..., \rho^j + (\rho^j)', ..., x^\ell) = \alpha(\rho^1, ..., \rho^j, ..., x^\ell) + \alpha(\rho^1, ..., (\rho^j)', ..., x^\ell)$$

(and similarly for x^i's).

(b) $\mathcal{T}^{(k,\ell)}(M)$ is a $\mathcal{C}^\infty(M,\mathbb{R})$ module given by taking $\alpha, \tilde{\alpha} \in \mathcal{T}^{(k,\ell)}(M)$, then $(\alpha + \tilde{\alpha})_a := \alpha_a + \tilde{\alpha}_a$, and for $f \in \mathcal{C}^\infty(M,\mathbb{R})$,

$$(f \cdot \alpha)_a = f(a) \cdot \alpha_a$$

(c) The map $\otimes : \mathcal{T}^{(k,\ell)}(M) \times \mathcal{T}^{(k',\ell')} \to \mathcal{T}^{(k+k',\ell+\ell')}$,

$$(\alpha \otimes \beta)(\rho^1, ..., \rho^{k+k'}, X^1, ..., X^{\ell+\ell'})$$

$$= \alpha(\rho^1, ..., \rho^k, X^1, ..., X^\ell) \cdot \beta(\rho^{k+1,...,k+k'}, .X^{\ell+1}, ..., X^{\ell+\ell'})$$

is $\mathcal{C}^\infty(M,\mathbb{R})$-linear and is associate.

Proof. (a) If $\alpha \in \mathcal{T}^{(k,\ell)}(M)$, then α maps into $\mathcal{C}^\infty(M,\mathbb{R})$ by 8.12(b). It is $\mathcal{C}^\infty(M,\mathbb{R})$ multilinear, because α is defined point-wise:

$$\alpha(\rho^1, ..., X^\ell)(a) = \alpha_a(\rho^1_a, ..., X^\ell_a)$$

and this implies the claim. Conversely, if $\alpha : \mathbb{X}^*(M)^k \times \mathbb{X}(M)^\ell \to \mathcal{C}^\infty(M,\mathbb{R})$ is $\mathcal{C}^\infty(M,\mathbb{R})$- multilinear, then we claim that $\alpha \in \mathcal{T}^{(k,\ell)}(M)$. To see this, we need to show that

$$\alpha(\rho^1, ..., \rho^k, X^1, .., X^\ell)(a)$$

only depends on $\rho^1_a, ..., \rho^k_a, X^1_a, ..., X^\ell_a$. First, we show that it only depends on a local neighborhood of a: assume that $\rho^j|_U = \tilde{\rho}^j|_U$ for $a \in U$, where U is an open neighborhood of M. Choose a bump function $f \in \mathcal{C}^\infty(M,\mathbb{R})$ such that $f(a) = 1$, and $f|_{M-U} = 0$. Then, the function $f \cdot (\rho^j - \tilde{\rho}^j) = 0$. Therefore, $0 = \alpha(.., f(\rho^j - \tilde{\rho}^j), ...)(a)$, and by $\mathcal{C}^\infty(M,\mathbb{R})$,

$$= \underbrace{f(a)}_{=1} \cdot \alpha(..., \rho^j - \tilde{\rho}^j, ...)(a) = \alpha(..., \rho^j)(a) - \alpha(..., \tilde{\rho}^j, ...)(a)$$

Second, α only depends on $\rho^1_a, ..., \rho^k_a, X^1_a, .., X^\ell_a$. Let $\varphi : U \to \mathbb{R}^d$ be a chart of M at $a \in U$. Fix j. On U, $\rho^j = \sum_{i=1}^d g_i dx^i$, $g_i \in \mathcal{C}^\infty(U,\mathbb{R})$.

Then, there exists $f_i \in \mathcal{C}^\infty(M,\mathbb{R}), i = 1, ..., d$ such that there exists a neighborhood V of a where

$$w^j|_V = \sum_{i=1}^d f_i dx^i$$

and $\sum_i f_i dx^i$ can be extended to a global covector field as in Lemma 5.4($\star$). Then,

$$\alpha(..., \rho^j, ...)(a) = \alpha(..., \sum_i f_i dx^i, ...)(a) = \sum f_i(a) \cdot \underbrace{\alpha(..., dx^i, ...)(a)}_{G(a)}$$

and $f_i, G \in \mathcal{C}^\infty(M, \mathbb{R})$. Therefore, α_a given by

$$\alpha_a(\rho_a^1, ., .., \rho_a^k, X_a^1, ..., X_a^\ell)$$

is well defined. Note that $\alpha_a : M \to T^{(k,\ell)}(TM)$ is smooth by criterium 8.12(b). This implies the claim (realize that this is less of a hard proof, and more of a sketch).

(b) clear

(c) clear

□

8.14 Covariant Tensors

We now restrict to **covariant tensors,** i.e., tensors of type $(0, \ell)$.

(a) For $F \in \mathcal{C}^\infty(M, N)$, and $a \in M$, there is an induced pullback map

$$dF_a^* : T^{(0,\ell)}(T_{F(a)}N) \to T^{(0,\ell)}(T_a M)$$

given by

$$(dF_a^*(\alpha_{F(a)}))(X_a^1, ..., X_a^\ell) := \alpha_{F(a)}(dF_a(X_a^1), ..., dF_a(X_a^\ell))$$

(b) There is also an induced map $F^* : \mathcal{T}^{(0,\ell)}(N) \to \mathcal{T}^{(0,\ell)}(M)$ by

$$F^*(\alpha)_a := dF_a^*(\alpha_{F(a)})$$

(c) We have: $F^*(f \cdot \alpha) = (f \circ F) \cdot F^*(\alpha)$ for $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $\alpha \in \mathcal{T}^{(0,\ell)}(N)$.

Proof.

$$\begin{aligned} F^*(f \cdot \alpha)_a(X^1, ..., X^\ell) &= dF_a^*((f \cdot \alpha)_{F(a)})(X^1, ..., X^\ell) = (f|_{F(a)} \cdot \alpha_{F(a)})(dF_a(X^1), ..., dF_a(X^\ell)) \\ &= f(F(a)) \cdot dF_a^*(\alpha_{F(a)})(X^1, ..., X^\ell) = ((f \circ F) \cdot F^*(\alpha))_a(X^1, ..., X^\ell) \end{aligned}$$

□

(d) For $\alpha \in \mathcal{T}^{(0,\ell)}(N), \beta \in \mathcal{T}^{(0,\ell')}(N)$, we have $\alpha \otimes \beta \in \mathcal{T}^{(0,\ell+\ell')}(N)$, and it is

$$F^*(\alpha \otimes \beta) = F^*(\alpha) \otimes F^*(\beta)$$

Proof.

$$\begin{aligned} F^*(\alpha \otimes \beta)_a(X^1, ..., X^{\ell+\ell'}) = (\alpha \otimes \beta)_{F(a)}(dF_a(X^1), ..., dF_a(X^{\ell+\ell'})) &= \alpha_{F(a)}(dF_a(X^1), ..., dF_a(X^\ell)) \cdot \beta_{F(a)}(dF_a(X^{\ell+1}), ..., d \\ &= F^*(\alpha)_a(X^1, ..., X^\ell) \cdot F^*(\beta)(X^{\ell+1}, ..., X^{\ell+\ell'}) \\ &= (F^*(\alpha) \otimes F^*(\beta))_a(X^1, ..., X^{\ell+\ell'}) \end{aligned}$$

□

(e) In local coordinates for some chart $\psi : V \to \psi(V)$ of N,

$$\alpha = \sum_{j_1, \dots, j_\ell} \alpha_{j_1, \dots, j_\ell} d\tilde{x}^{j_1} \otimes ... \otimes d\tilde{x}^{j_\ell} \in \mathcal{T}^{(0,\ell)}(V)$$

then

$$(1) \quad F^*(\alpha) \overset{(c),(d)}{=} \sum_{j_1, \dots, j_\ell} (\alpha_{j_1, \dots, j_\ell} \circ F) \cdot F^*(d\tilde{x}^{j_1}) \otimes ... \otimes F^*(d\tilde{x}^{j_\ell})$$

by 8.6,

$$= \sum_{j_1, \dots, j_\ell} (\alpha_{j_1, \dots, j_\ell} \circ F) \cdot d(\tilde{x}^{j_1} \circ F) \otimes ... \otimes d(\tilde{x}^{j_\ell} \circ F)$$

8.15 8.15 Important Tensors in Differential Geometry

Remark.

- Riemannian metric,

$$g = \sum g_{ij} dx^i \otimes dx^j \in \mathcal{T}^{(0,2)}(M)$$

- Riemannian curvature, $R \in \mathcal{T}^{(1,3)}(M)$
- Ricci curvature $Ric \in \mathcal{T}^{(0,2)}(M)$
- Scalar curvature $S \in \mathcal{T}^{(0,0)}(M) = \mathcal{C}^\infty(M, \mathbb{R})$
- Einstein Field equations,

$$Ric - \frac{1}{2} S \cdot g = T$$

where T is the **energy-stress tensor**. These are the field equations for general relativity, and this latest equation lives in $\mathcal{T}^{(0,2)}(M)$.

9 Differential Forms

This chapter is based on Lee's book, chapter 12 and 14 (mainly 14, and a little bit of 12).

9.1 Symmetric tensors

Definition. Let V be a finite dimensional vector space of dimension d.

(a) Denote by $T^\ell(V^*) := T^{(0,\ell)}(V) = L(\underbrace{V, ..., V}_{\ell}; \mathbb{R}) \cong \underbrace{V^* \otimes ... \otimes V^*}_{\ell}$

(b) $\alpha \in T^\ell(V^*)$ is called **symmetric** if and only if

$$\alpha(v_1, ..., v_i, .., v_j, ..., v_\ell) = \alpha(v_1, .., v_j, ..., v_i, ..., v_\ell) \quad \forall i, j$$

The space of symmetric tensors is denoted by $S^\ell(V^*) \subset T^\ell(V^*)$.

(c) $\alpha \in T^\ell(V^*)$ is called **anti-symmetric** or **alternating** if

$$\alpha(v_1, ..., v_i, .., v_j, ..., v_\ell) = -\alpha(v_1, .., v_j, ..., v_i, ..., v_\ell) \quad \forall i, j$$

The space of alternating tensors is denoted by $\Lambda^\ell(V^*) \subset T^\ell(V^*)$. $\Lambda^\ell := \oplus_{\ell \geq 0} \Lambda^\ell(V^*)$ is called the exterior algebra of V^*, and we will see later that it is indeed, an algebra.

Claim. The following are equivalent:

1. α is alternating
2. For all v, $\alpha(, ..v, ..., v, ...) = 0$
3. If $v_1, ..., v_\ell$ are linearly dependent, then $\alpha(v_1, ..., v_\ell) = 0$.

Proof. (1) $\iff$ (2)

We think that $1 \Rightarrow 2$ is clear, because 'it is its own negative'. Also, the $\Leftarrow$ direction of the proof is similarly easy, in that

$$0 = \alpha(, ..., v_i + v_j, ..., v_i + v_j, ...) = \alpha(, ..., v_i, ..., v_i) + \alpha(..., v_i, ..., v_j) + \alpha(, ..., v_j, ..., v_j) + \alpha(, ..., v_j, ..., v_i)$$

as two of the above are 0 by assumption, we have our claim.

(2) $\iff$ (3)

Again, we say that the $\Leftarrow$ direction is clear. In the other direction, assume $v_1 = \sum_{j \geq 2} c_j \cdot v_j$ (which we can do without loss of generality). Then,

$$\alpha(v_1, v_2, ..., v_\ell) = \alpha(\sum_{j=2}^{\ell} c_j v_j, v_2, ..., v_\ell)$$

by multilinearity,

$$= \sum_{j=2}^{\ell} c_j \cdot \alpha(v_j, v_2, ..., v_\ell) = 0$$

since $(v_j, v_2, ..., v_\ell) = 0$. $\square$

9.2 A basis for $\Lambda^\ell(V^*)$

Proposition 34. *Let $\{e^j\}_{j=1}^d$ be a basis of V and let $\{\epsilon^j\}_{j=1}^d$ be the dual basis of V^*, i.e., $\epsilon^j(e_i) = \delta_{ij}$. For a multindex $I = (i_1, ..., i_\ell)$ (where $1 \leq i_j \leq d$), we define $\epsilon^I \in T^\ell(V^*)$ by*

$$\epsilon^I(v_1, ..., v_\ell) := det \begin{pmatrix} \epsilon^{i_1}(v_1) & ... & \epsilon^{i_1}(v_\ell) \\ \vdots & & \vdots \\ \epsilon^{i_\ell}(v_1) & \dots & \epsilon^{i_\ell}(v_\ell) \end{pmatrix} = det(\epsilon^{i_j}(v_k))_{j,k}$$

Claim.
1. $\epsilon^I \in \Lambda^\ell(V^*)$
2. $\{\epsilon^I | I = (i_1, ..., i_\ell)$ with $i_1 < i_2 < ... < i_\ell\}$ forms a basis of $\Lambda^\ell(V^*)$.
3. $dim(\Lambda^\ell(V^*)) = \binom{d}{\ell} = \frac{d!}{\ell!(d-\ell)!}$ for $1 \leq \ell \leq d$, $dim(\Lambda(V^*)) = 0$ whenever $\ell > d$.

Proof.
1. $$\epsilon^I(..., v, .., v, ..) = det \begin{pmatrix} ... & \epsilon^{i_1}(v) & ... & \epsilon^{i_1}(v) & ... \\ & \vdots & & \vdots & \\ ... & \epsilon^{i_\ell}(v) & ... & \epsilon^{i_\ell}(v) & ... \end{pmatrix}$$

   as this matrix has the same column twice, and hence the determinant is equal to 0. This implies the claim, by 9.1(c)(3).

2. We need to show that $\{\epsilon^I | I = (i_1, ..., i_\ell)$ with $i_1 < i_2 < ... < i_\ell\}$ spans the exterior algebra, and is linearly independent. For the span, let $\alpha \in \Lambda^\ell(V^*)$. Then for any $I = (i_1, ..., i_\ell)$ (not necessarily increasing) define $\alpha_I := \alpha(e^{i_1}, e^{i_2}, ..., e^{i_\ell}) \in \mathbb{R}$. Note by 9.1 that $\alpha_{(...,i,...,i,...)} = 0$, and for a permutation $\sigma \in \Sigma_\ell$ (the symmetric group), we have that

   $$\alpha_{(i_{\sigma(1)}, ..., i_{\sigma(\ell)})} = \alpha(e^{i_{\sigma(1)}}, ..., e^{i_{\sigma(\ell)}}) = sign(\sigma)\alpha(e^{i_1}, ..., e^{i_\ell})$$

   where $sign(\sigma) \in \{\pm 1\}$ is the sign of the permutation. Then we claim that the following is true, where the sum on the right is the sum over all such multindexes J,

   $$\alpha = \sum_{J=(j_1<...<j_\ell)} \alpha_J \cdot \epsilon^J. \tag{9.2.1}$$

   To see this, apply $(e^{i_1}, ..., e^{i_\ell})$ for some multindex $I = (i_1 < i_2 < ..., i_\ell)$. Then $\alpha(e^{i_1}, ..., e^{i_\ell}) = \alpha_I$, and

   $$\sum_{J=(j_1<...<j_\ell)} \alpha_j \epsilon^J(e^{i_1}, ..., e^{i_\ell}) = \sum_{J=(j_1<...<j_\ell)} \alpha_j \cdot det \begin{pmatrix} \epsilon^{j_1}(e^{i_1})....\epsilon^{j_1}(e^{i_\ell}) \\ \vdots \\ \epsilon^{j_\ell}(e^{i_1})...\epsilon^{j_\ell}(e^{i_\ell}) \end{pmatrix}$$

The claim is that the determinant on the right is equal to $\delta_{j_1 i_1} \cdot \delta_{j_2 i_2} \cdot \ldots \cdot \delta_{j_\ell i_\ell}$. For a general $I = (i_1, \ldots, i_\ell)$, note that both sides of 9.2.1 are alternating, (and both sides applied to $(\ldots, v, \ldots, v, \ldots)$ give us 0), so that 9.2.1 holds when applies to any $(e^{i_1}, \ldots, e^{i_\ell})$. As such, we have the span condition. Now, we show linear independence: assume that

$$\sum_{J=(j_1<\ldots<j_\ell)} c_J \cdot \epsilon^J = 0,$$

we need to show that every $c_J = 0$. Apply

$$\sum_{J\ incr.} c_j \cdot \epsilon^J$$

to $(e^{i_1}, \ldots, e^{i_\ell})$ for $i_1 < \ldots < i_\ell$, then

$$\sum_{J\ incr.} c_j \cdot \epsilon^J(e^{i_1}, \ldots, e^{i_\ell}) = \sum_{J\ incr} \cdot det\left(\left(\epsilon^j(e_{i_s})\right)_{j,s}\right)$$

and the determinant is equal to $\delta_{j_1 i_1} \cdot \ldots \cdot \delta_{j_\ell i_\ell}$, so the sum is equal to c_I.

3. Note that increasing indicies $(i_1 < i_2 < \ldots < i_\ell)$ correspond to ℓ-element subsets $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, d\}$. There are $\binom{d}{\ell}$ many ℓ-element subsets of $\{1, \ldots, d\}$. If $\ell > d$, then $\Lambda^\ell(V^*) = \{0\}$ by 9.1(c)(3).

$\square$

9.3 The wedge product

Definition. (a) Define the **alternation map**

$$Alt : T^\ell(V^*) \to \Lambda^\ell(V^*)$$

given by

$$(Alt(\alpha))(v_1, \ldots, v_\ell) := \frac{1}{\ell!} \sum_{\sigma \in \Sigma_\ell} sign(\sigma) \cdot \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(\ell)})$$

As an exercise, check that $Alt(\alpha)(\ldots, v_i, \ldots, v_j, ..) = -Alt(\alpha)(, \ldots, v_j, \ldots, v_i, \ldots)$, and therefore, $Alt(\alpha) \in \Lambda^\ell(V^*)$ is well defined.

(b) The **wedge product** or **exterior product** is a map

$$\wedge : \Lambda^\ell(V^*) \times \Lambda^{\ell'}(V^*) \to \Lambda^{\ell+\ell'}(V^*)$$

where $\alpha \in \Lambda^\ell(V^*), \beta \in \Lambda^{\ell'} V^*$, given by

$$\alpha \wedge \beta := \frac{(\ell + \ell')!}{\ell! \ell'!} \cdot Alt(\alpha \otimes \beta) \in \Lambda^{\ell+\ell'}(V^*)$$

where $\alpha \otimes \beta \in T^{\ell+\ell'}(V^*)$.

(c) We have

$$\epsilon^{(i_1,\ldots,i_\ell)} \wedge \epsilon^{(j_1,\ldots,j_{\ell'})} = \epsilon^{(i_1,\ldots,i_\ell,j_1,\ldots,j_{\ell'})}$$

(d) The wedge product is bilinear; for all $r, s \in \mathbb{R}$,

$$(r\alpha + s\beta) \wedge \gamma = r(\alpha \wedge \gamma) + s(\beta \wedge \gamma)$$

and

$$\alpha \wedge (r\beta + s\gamma) = r(\alpha \wedge \beta) + s(\alpha \wedge \gamma)$$

(e) The wedge product is associative,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

(f) The wedge product is anti-commutative, if $\alpha \in \Lambda^\ell(V^*)$ and $\beta \in \Lambda^{\ell'}(V^*)$, then

$$\alpha \wedge \beta = (-1)^{\ell \cdot \ell'} \cdot \beta \wedge \alpha$$

Proof. We have

$$Alt(\alpha \otimes \beta)(v_1, ..., v_{\ell+\ell'}) = \sum_{\sigma \in \Sigma_{\ell+\ell'}} sign(\sigma) \cdot \alpha(v_{\sigma(1)}, ..., v_{\sigma(\ell)}) \cdot \beta(v_{\sigma(\ell+1)}, ..., v_{\sigma(\ell+\ell')}) \in \mathbb{R}$$

$$= \sum_{\sigma \in \Sigma_{\ell+\ell'}} sign(\sigma) \cdot \beta(v_{\sigma(\ell+1)}, ..., v_{\sigma(\ell+\ell')}) \cdot \alpha(v_{\sigma(1)}, ..., v_{\sigma(\ell)})$$

$$= \sum_{\sigma \in \Sigma_{\ell+\ell'}} sign(\sigma) \cdot \beta(v_{\sigma(\tilde{1})}, ..., v_{\sigma(\tilde{\ell'})}) \cdot \alpha(v_{\sigma(\tilde{\ell'+1})}, ..., v_{\sigma(\tilde{\ell+\ell'})})$$

where σ and $\tilde{\sigma}$ are related by $\tilde{\sigma} = \sigma \circ \tau$, where

$$\tau(j) = \begin{cases} j + \ell, j \leq \ell' \\ j - \ell', j > \ell' \end{cases}$$

Note that $sign(\tau) = \ell \cdot \ell'$. Therefore, this gives us that the sum above is equal to

$$= (-1)^{\ell \cdot \ell'} \cdot Alt(\beta \otimes \alpha)$$

□

(g) For all $\rho_1, ..., \rho^\ell \in V^* = T^1(V^*) = \Lambda^1(V^*)$,

$$\underbrace{(\rho^1 \wedge ... \wedge \rho^\ell)}_{\in \Lambda^\ell(V^*)}(v_1, ..., v_\ell) = det\left(\left(\rho^j(v_i)\right)_{i,j}\right)$$

Proof. The formula is true for $\rho^j = e^{i_j} \in V^*$ by definition 9.2. The claim follows, since both sides are multi-linear in ρ^j's (by (d) above, and the determinant is also multi-linear). □

Proof. Though select properties were proven, this will be on the next homework sheet. □

9.4 The interior product

Definition. Let $v \in V$. Then, we can define the **interior product**,

$$i_v : \Lambda^\ell(V^*) \to \Lambda^{\ell-1}(V^*)$$

given by

$$(i_v(\alpha))(v_1, ..., v_{\ell-1}) = \alpha(v, v_1, ..., v_{\ell-1})$$

Notationally, we also write $i_v(\alpha) = v \neg \alpha$ (this should be upside down) It has the following properties:

(a) $i_v \circ i_v = 0$

(b) For $\alpha \in \Lambda^\ell(V^*)$, $\beta \in \Lambda^{\ell'}(V^*)$:

$$i_v(\alpha \wedge \beta) = i_v(\alpha) \wedge \beta + (-1)^\ell \cdot \alpha \wedge i_v(\beta)$$

Proof. (a) clear

(b) exercise

□

9.5 Linear maps, exterior algebras, and the wedge product

Lemma 35. *(a) If $A : V \to W$ is a linear map, then there is an induced map*

$$A^* : \Lambda^\ell(W^*) \to \Lambda^\ell(V^*)$$

where $\Lambda^\ell(W^*) \subset T^{(0,\ell)}(W)$, and $\Lambda^\ell(V^*) \subset T^{(0,\ell)}(V)$ given by

$$(A^*(\beta))(v_1, ..., v_\ell) = \beta(Av_1, ..., Av_\ell)$$

It is true that $A^*(\beta \wedge \gamma) = A^*(\beta) \wedge A^*(\gamma)$.

Proof. A^* is defined on $T^{(0,\ell)}$ by 8.10 (and is anti-symmetric), $A^*(\beta \otimes \gamma) = A^*\beta \otimes A^*\gamma$, and $Alt \circ A^* = A^* \circ Alt$, which you can check on β and some $(v_1, ..., v_\ell)$. □

(b) If $A : V \to W$ and $B : W \to Z$ are linear maps, then $(B \circ A)^* = A^* \circ B^* : \Lambda^\ell(Z^*) \to \Lambda^\ell(V^*)$.

Proof.

$$(A^* \circ B^*)(\gamma)(v_1, .., v_\ell) = A^*(B^*\gamma)(v_1, ..., v_\ell) = (B^*\gamma)(Av_1, ..., v_\ell) = \gamma(BAv_1, .., BAv_\ell) = (B \circ A)^*(\gamma)(v_1, ..., v_\ell)$$

□

9.6 A manifold structure on the exterior bundle, $\Lambda^\ell(T^*M)$

Definition. Let M be a manifold of dimension d. Let

$$\Lambda^\ell(T^*M) := \{(a, \alpha) | a \in M, \alpha \in \Lambda^\ell(T^*_a M)\}.$$

be the **exterior bundle** of M, and let $\pi : \Lambda^\ell(T^*M) \to M$ where $\pi(a, \alpha) = a$.

Claim. $\Lambda^\ell(T^*M)$ is a manifold of dimension $d + \binom{d}{\ell}$.

Proof. Let $\alpha : M \to \Lambda^\ell(T^*M)$ be a section of π, i.e.,

$$\pi \circ \alpha = id_M$$

Let $\varphi : U \to \mathbb{R}$ be a chart of M. Then, $\{dx^i|_a\}_{i=1,\dots,d}$ is a basis of $T^*_a M$ so that

$$\left\{(dx^{i_1})|_a \wedge ... \wedge (dx^{i_\ell})|_a\right\}_{I=(i_1<\dots<i_\ell)}$$

forms a basis of $\Lambda^\ell(T^*_a M)$ (recall that $dx^i \wedge dx^j = -dx^j \wedge dx^i$, and so $dx^i \wedge dx^i = 0$). Then α can be written (locally) as

$$\alpha = \sum_{I=(i_1<\dots<i_\ell)} \alpha_{i_1 \dots i_\ell} dx^{i_1} \wedge ... \wedge dx^{i_\ell}$$

For any $\varphi : U \to \mathbb{R}^d$ in the atlas of M, $\mathcal{A}_M$, define

$$\varphi_{\Lambda^\ell} : \pi^{-1}(U) \to \mathbb{R}^{d+\binom{d}{\ell}}$$

by

$$\varphi_{\lambda^\ell}(a, \alpha) = (\varphi(a), \{\alpha_{i_1,\dots,i_\ell}\}_{i_1<\dots<i_\ell})$$

then $\mathcal{A}_{\Lambda^\ell} = \{\varphi_{\Lambda^\ell} | \varphi \in \mathcal{A}_M\}$ is an atlas and $\pi \in \mathcal{C}^\infty(\Lambda^\ell(T^*M), M)$. □

9.7 Differential ℓ-forms

Definition. (a) Let $\alpha : M \to \Lambda^\ell(T^*M)$ be a section of π. Then the following are equivalent:

(a) α is smooth.

(b) $\alpha \in \mathcal{C}^\infty(M, \Lambda^\ell(T^*M))$.

(c) $\alpha_{i_1,\dots,i_\ell} \in \mathcal{C}^\infty(M, \mathbb{R})$ for all multi-indices $I = (i_1, .., i_\ell)$.

Proof. We have the corresponding property for tensors, see 8.12(b). □

A smooth section $\alpha \in \mathcal{C}^\infty(M, \Lambda^\ell(T^*M))$ is called a **(differential) ℓ-form** of M, the set of all ℓ-forms is denoted by,

$$\Omega^\ell(M) = \Omega^\ell_{DR}(M) := \{\alpha \in \mathcal{C}^\infty(M, \Lambda^\ell(T^*M)) | \pi \circ \alpha = id_M\} =: \Gamma(M, \Lambda^\ell(T^*M)^*$$

The wedge product gives us a map,

$$\wedge : \Omega^\ell(M) \times \Omega^{\ell'}(M) \to \Omega^{\ell+\ell'}(M)$$

Remark. $\Omega^\ell(M)$ is a $\mathbb{R}$-vector space.

Define,

$$\Omega^*(M) = \Omega^*_{DR}(M) := \oplus_{\ell \geq 0} \Omega^\ell(M)$$

which is called the **deRhamn algebra of** M, where the algebra structure given by

$$\wedge : \Omega^*(M) \times \Omega^*(M) \to \Omega^*(M)$$

Remark. It was asked that we better flesh-out the map that the wedge product induces, so for all $a \in M$,

$$\wedge : \Lambda^\ell(T^*_a M) \times \Lambda^{\ell'}(T^*_a M) \to \Lambda^{\ell+\ell'}(T^*_a M),$$

$$\left(\sum_I \alpha_{i_1,\dots,i_\ell} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \right) \wedge \left(\sum_J \beta_{j_1,\dots,j_\ell} dx^{j_1}, \dots, , dx^{j_\ell} \right)$$
$$= \sum_{I,J} (\alpha_{i_1,\dots,i_\ell} \cdot \beta_{j_1,\dots,j_\ell}) \cdot dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j'_\ell}$$

(b) For a smooth map $F \in \mathcal{C}^\infty(M, N)$, we have an induced map

$$F^* : \Omega^\ell(N) \to \Omega^\ell(M)$$

such that

$$F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta) \tag{9.7.1}$$

Proof. Pullbacks exist for tensors $\mathcal{T}^{(0,\ell)}(M)$ by 8.14. The alternating property is preserved (pointwise) by 9.5(a), and 9.7.1 follows pointwise from 9.5(a). □

*I cannot find this notation elsewhere, so I fear that I made a typo. I have allowed it to remain, for the time being.

9.8 The exterior derivative d_{DR}

(a) Let $U \subset \mathbb{R}^n$ be an open subset. Define the **exterior derivative** $d = d_{DR}$,

$$d_{DR} : \Omega^\ell(U) \to \Omega^{\ell+1}(U)$$

as follows: for $\alpha \in \Omega^\ell(U)$, write

$$\alpha = \sum_{I=(i_1\dots i_\ell)} \alpha_{i_1\dots i_\ell} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} = \sum_I \alpha_I dx^I$$

where $\alpha_I = \alpha_{i_1,\dots i_\ell}$ and $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_\ell}$ (here, I is just an index, and need not be increasing). Let

$$d\alpha = d\left(\sum_I \alpha_I dx^I\right) := \sum_I (d\alpha_I) \wedge dx^I$$

Note that this is well defined, because

$$\alpha_I \in \mathcal{C}^\infty(U,\mathbb{R}) \Rightarrow d\alpha_I \in \mathbb{X}^*(U) = \mathcal{T}^{(0,1)}(U) = \Omega^1(U),$$

and the definition is independent of your choice of multi-indices, since for $dx^I = (-1)^\epsilon dx^J$, we have

$$d(\alpha \cdot (-1)^\epsilon \cdot dx^J) = d((-1)^\epsilon \cdot \alpha) \wedge dx^J = d\alpha \wedge dx^I = d(\alpha \cdot dx^I)$$

(b) Example: U is still an open subset of $\mathbb{R}^n$. If $f \in \Omega^0(U) = \mathcal{C}^\infty(U,\mathbb{R})$:

$$df \overset{8.4}{=} \sum_j \frac{\partial f}{\partial x^j} dx^j$$

$\alpha \in \Omega^1(U)$ implies $\alpha = \sum_{i=1}^n f_i dx^i$. Then,

$$d\alpha = d\left(\sum_i f_i dx^i\right) = \sum_i df_i \wedge dx^i = \sum_i \sum_j \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i<j} \frac{\partial f}{\partial x^j} dx^j \wedge dx^i + \sum_{i>j} \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \sum_{i>j} \left(\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i}\right) dx^j \wedge dx^i$$

Lastly, take $\alpha \in \Omega^2(U)$, so $\alpha = \sum_{i<j} f_{ij} dx^i \wedge dx^j$, giving us

$$d\alpha = d\left(\sum_{i<j} f_{ij} dx^i \wedge dx^j\right) = \sum_{i<j} df_{ij} \wedge dx^i \wedge dx^j = \sum_{i<j} \sum_k \frac{\partial f_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j$$

$$= \sum_{k<i<j} (\dots) + \sum_{i<k<j} (\dots) + \sum_{i<j<k} (\dots) = \sum_{k<i<j} \left(\frac{\partial f_{ij}}{\partial x^k} - \frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} dx^k \wedge dx^i \wedge dx^j\right)$$

9.9 Properties of the exterior derivative.

Proposition 36. *With notation from 9.8, we have the following:*

(a) d is $\mathbb{R}$-linear

(b) For all $\alpha \in \Omega^\ell(U)$, $\beta \in \Omega^{\ell'}(U)$, we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^\ell \cdot \alpha \wedge d\beta$$

(c) $d \circ d = 0$

(d) *For all $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open, and for all $F : U \to V$ smooth with all $\alpha \in \Omega^\ell(V)$:*

$$F^*(d\alpha) = d(F^*(\alpha))$$

Proof. (a) Note that $d(r \cdot f) = r \cdot d(f)$ for all $r \in \mathbb{R}$, and $f \in \mathcal{C}^\infty(U, \mathbb{R})$.

(b) First, for $f, g \in \Omega^0(U) = \mathcal{C}^\infty(U, \mathbb{R})$, we have

$$d(f \cdot g) = \sum_j \frac{\partial(f \cdot g)}{\partial x^i} dx^i = \sum_i \left(\frac{\partial f}{\partial x^i} \cdot g + f \cdot \frac{\partial g}{\partial x^i} dx^i \right) = \left(\sum_i \frac{\partial f}{\partial x^i} dx^i \right) \cdot g + f \cdot \left(\sum_i \frac{\partial g}{\partial x^i} dx^i \right)$$

$$= (df)g + f(dg) = (df) \wedge g + f \wedge (dg)$$

In general, $\alpha \in \Omega^\ell(U)$, $\alpha = \sum_i \alpha_i dx^i$, $\beta \in \Omega^{\ell'}$, $\beta = \sum_J \beta_J dx^J$, we have

$$d(\alpha \wedge \beta) = d(\sum_{I,J} \alpha_I \cdot \beta_J dx^I \wedge dx^J) = \sum_{I,J} d(\alpha_I \cdot \beta_J) \wedge dx^I \wedge dx^j$$

so from the previous case,

$$= \sum_{I,J} (d\alpha_I \cdot \beta_J + \alpha_I \cdot d\beta_J) \wedge dx^i \wedge dx^J = \sum_{I,J} (d\alpha_I \wedge dx^I \wedge \beta_J \wedge dx^J + (-1)^\ell \alpha_I \cdot dx^I \wedge d\beta_J \wedge dx^I)$$

$$= \left(\sum_I d\alpha_I \wedge dx^I \right) \wedge \left(\sum_J \beta_J dx^J \right) + (-1)^\ell \cdot \left(\sum_I \alpha_I dx^I \right) \wedge \left(\sum_J d\beta_J \wedge dx^J \right)$$

$$= d\alpha \wedge \beta + (-1)^\ell \alpha \wedge d\beta$$

(c) For $f \in \mathcal{C}^\infty(U, \mathbb{R}) = \Omega^0(U)$, we have that

$$d^2(f) = d(df) = d\left(\sum_i \frac{\partial f}{\partial x^i} dx^i \right) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \overset{9.8(b)}{=} \sum_{j<i} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^j \wedge dx^i = 0$$

For

$$\alpha = \sum_I \alpha_I dx^I \in \Omega^\ell(U),$$

we have that

$$d^2(\alpha) = d\left(\sum_I d\alpha_I \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_\ell} \right)$$

$$\overset{(b)}{=} \sum_I \underbrace{d^2\alpha_I}_{0} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_\ell} - \sum_I d\alpha_I \wedge \underbrace{d^2 x^{i_1}}_{0} \wedge \ldots \wedge dx^{i_\ell} + \ldots + \sum_I d\alpha_I \wedge dx^{i_1} \wedge \ldots \wedge \underbrace{d^2 x^{i_\ell}}_{0} = 0$$

(d)

$$F^* \circ d(f dx^i \wedge \ldots \wedge dx^{i_\ell}) = F^*(df \wedge dx^i \wedge \ldots \wedge dx^{i_\ell}) \overset{9.5(a)}{=} (F^* df) \wedge (F^* dx^{i_1}) \wedge \ldots \wedge (F^* F^* dx^{i_\ell})$$

$$\overset{8.6}{=} d(f \circ F) \wedge d(x^i \circ F) \wedge \ldots \wedge d(x^{i_\ell} \circ F) = d((f \circ F) \cdot d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_\ell} \circ F)) \overset{9.5(a)}{=} d(F^*(f \cdot dx^{i_1} \wedge \ldots \wedge dx^{i_\ell}))$$

The claim then follows by the $\mathbb{R}$-linearity of F^* and d.

$\square$

9.10 The local definition of the exterior derivative

Definition. Let M be a smooth manifold. Define $d : \Omega^\ell(M) \to \Omega^{\ell+1}(M)$ as follows: for $\alpha \in \Omega^\ell(M)$, $d\alpha \in \Omega^{\ell+1}$ is defined locally on a chart $\varphi : U \to \varphi(U)$ by setting $d\alpha|_U \in \Omega^{\ell+1}(U)$ to be

$$d\alpha|_U := \varphi^* \circ d \circ (\varphi^{-1})^*(\alpha|_U)$$

This is well defined, since for another chart $\psi : V \to \psi(V)$ we have

$$\psi^* \circ d \circ (\psi^{-1})^*(\alpha) = \varphi^* \circ (\varphi^{-1})^* \circ \psi^* \circ d \circ (\psi^{-1})^*(\alpha) = \varphi^* \circ d \circ (\psi \circ \varphi^{-1})^* \circ (\psi^{-1})^*(\alpha) = \varphi^* \circ \alpha \circ (\varphi^{-1})^*(\alpha).$$

where some of these equalities are taken from 9.5(b) and 9.9(d).

9.11 More properties of the exterior derivative

Proposition 37. *The map d satisfies the following:*

(a) d *is* $\mathbb{R}$*-linear*

(b) $d(\alpha \wedge b) = d\alpha \wedge \beta + (-1)^\ell \cdot \alpha \wedge d\beta$ *for* $\alpha \in \Omega^\ell(M)$

(c) $d^2 = 0$

(d) $\forall f \in \Omega^0(M)$*,* $x \in \mathbb{X}(M)$*,* $df(x) = X(f)$*,*

(e) $\forall F \in \mathcal{C}^\infty(M, N)$*,* $\alpha \in \Omega^\ell(N)$*:* $F^*(d\alpha) = d(F^*\alpha)$

Proof. (a) Clear from 9.9(a)

(b)

$$\begin{aligned} d(\alpha \wedge \beta) &\overset{def}{=} \varphi^* d(\varphi^{-1})^*(\alpha \wedge \beta) \overset{9.7(b)}{=} \varphi^* d((\varphi^{-1})^*\alpha \wedge (\varphi^{-1})^*\beta) \\ &\overset{9.9(b)}{=} \varphi^*((d(\varphi^{-1})^*\alpha) \wedge (\varphi^{-1})^*\beta) + (-1)^\ell((\varphi^{-1})^*\alpha) \wedge d((\varphi^{-1})^*\beta)) \\ &\overset{9.7(b)}{=} d\alpha \wedge \beta + (-1)^\ell \alpha \wedge d\beta \end{aligned}$$

(c)

$$d^2 = \varphi^* d(\varphi^{-1})^* \varphi^* d(\varphi^{-1})^* = \varphi^* d^2 (\varphi^{-1})^* = 0$$

(d) Locally:

$$df\left(\frac{\partial}{\partial x^i}\right) = \varphi^* d((\varphi^{-1})^* f)(\frac{\partial}{\partial x^i}) = \varphi^* \left(\sum_j \frac{\partial(f \circ \varphi^{-1})}{\partial x^j} dx^j\right)\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \circ \varphi$$

(e) Locally: for charts φ of M, ψ of N, $F(U) \subset V$ (again, I'm just using the normal notation here):

$$F^*(d\alpha) = F^* \circ \psi^* \circ d(\psi^{-1})^*(\alpha) = \varphi^* \circ (\varphi^{-1})^* \circ F^* \circ \psi^* \circ d \circ (\psi^{-1})^*(\alpha) \overset{9.9(d)}{=} \varphi^* \circ d \circ (\psi \circ F \circ \varphi^{-1})^* \circ (\psi^{-1})^*(\alpha) = \varphi^* d(\varphi^{-1})^* F^* \alpha$$

Remark. d is the unique map satisfying $(a) - (d)$.

□

9.12 Person addendum

The notes above have now covered a significant number of different manifold structures, all of which are deeply inter-related. I thought it might be worth it (for my personal perspective) to draw a 'chart' of some kind, which simply summarizes some of these ideas. The graph below isn't supposed to be taken as anything that follows conventional notation, rather, I was hoping it might provide some kind of mnemonic device. If you're not me, you probably want to skip this.

$$
\begin{tikzcd}
a \in M \arrow[r, dashed] \arrow[dr, dashed] & T_a M \arrow[r, dashed] \arrow[d, dashed, "*"] \arrow[dd, dashed, bend right, blue] & TM \arrow[d, dashed, "*"] & \mathbb{X}(M) \arrow[l, dashed, red] \arrow[d, dashed, "*"] \\
& T_a^* M \arrow[r, dashed] \arrow[d, dashed] \arrow[dd, dashed, bend right, blue] & T^* M & \mathbb{X}^*(M) \arrow[l, dashed, red] \\
& T^{(k,\ell)}(T_a M) \arrow[r, dashed] \arrow[d, dashed] & T^{(k,\ell)}(M) \arrow[uur, dashed, gray] \arrow[ur, dashed, gray] & \mathcal{T}^{(k,\ell)} \arrow[l, dashed, red] \\
& T^{\ell}(T_a^* M) \supset \Lambda^{\ell}(T_a^* M) \arrow[r, dashed] & \Lambda^{\ell}(T_a^* M) & \Omega^{\ell}(M) = \Omega^{\ell}_{DR} M \arrow[l, dashed, red]
\end{tikzcd}
$$

10 Orientation and Integration

10.1 Orientations

Definition. Let V be a vector space of dimension $d \geq 1$. An ordered basis is a d-tuple $(e_1, ..., e_d)$ such that $\{e_i\}_{i=1,...,d}$ is a basis of V. Two ordered bases $(e_1, ..., e_d)$ and $(f_1, ..., f_d)$ are called **equivalent** if the linear map $A : V \to V$, $A(e_i) = f_i$ for all $i = 1, .., d$ satisfies $det(A) > 0$ (it can be shown that this is an equivalence relation). An **orientation** on V is an equivalence class of ordered bases $[(e_1, ..., e_d)]$. A vector space V has exactly two orientations: $[(e_1, ..., e_d)]$, and $[(-e_1, ..., e_d)] = [(e_2, e_1, e_3, ..., e_d)]$.

10.2 Linear maps and induced maps on exterior algebras

Lemma 38. *Let V be a vector space of dimension d, so that $\Lambda^d(V^*)$ has dimension $\binom{d}{d} = 1$. If $A : V \to V$ is a linear map, then the induced map $A^* : \Lambda^d(V^*) \to \Lambda^d(V^*)$ from 9.5.;*

$$A^*(\beta)(v_1, .., v_d) = \beta(Av_1, ..., Av_d)$$

is given by

$$A^*(\beta) = det(A) \cdot \beta \in \Lambda^d(V^*)$$

Proof. Let $\{e_i\}_{i=1}^d$ be a basis of V with dual basis $\{\alpha_i\}_{i=1}^d$ of V^*. Then, $\alpha_1 \wedge ... \wedge \alpha_d$ spans $\Lambda^d(V^*)$, so that

$$\beta = c \cdot \alpha_1 \wedge ... \wedge \alpha_d.$$

Then,

$$\beta(e_1, ..., e_d) = c \cdot \alpha_1 \wedge ... \wedge \alpha_d(e_1, ..., e_d) = c \cdot det((\alpha^i(e_j))_{ij}) \overset{9.3(g)}{=} c,$$

and if $A(e_j) = \sum_{\ell=1}^n A_j^\ell e_\ell$,

$$A^*(\beta)(e_1, ..., e_d) = \beta(Ae_1, ..., Ae_d) = c \cdot det\left(\left(\alpha^i(Ae_j)\right)_{ij}\right) = c \cdot det\left(\left(\sum_\ell A_j^\ell \alpha^i(e_\ell)\right)_{i,j}\right) = c \cdot det(A)$$

We claim that this is true when applied to $(e_1, .., e_d)$. Since both sides are alternating, this implies true for when applied to $(e_{\sigma(1)}, ..., e_{\sigma(d)})$. Since both sides are multilinear, this implies the claim when applied to any $(v_1, ..., v_d)$. $\square$

10.3 Determining an orientation with $\Lambda^d(V^*) - \{0\}$

Corollary 39. *Let $d \geq 1$. An orientation consists of a choice of component of $\Lambda^d(V^*) - \{0\}$. In fact, $\beta \in \Lambda^d(V^*) - \{0\}$ determines the orientation $[(e_1, ..., e_d)]$ so that $\beta(e_1, ..., e_d) > 0$.*

Proof. If $A(e_i) = f_i$, then

$$\beta(f_1, ..., f_d) = \beta(Ae_1, ..., Ae_d) = (A^*\beta)(e_1, ..., e_d) = det(A) \cdot \beta(e_1, ..., e_d) \lesseqgtr 0$$

where the inequality is determined by the sign of A's determinant. Thus, β determines a well-defined orientation. If β' is in the same component as β, then $\beta' = c \cdot \beta, c > 0$. thus $\beta'(e_1, ..., e_d) = c \cdot \beta(e_1, ..., e_d) > 0$. This implies that β' determines the same basis, while $-\beta$ determines $[(-e_1, ..., e_d)]$.

□

10.4 Equivalent conditions for being orientable

Proposition 40. *Let M be a manifold of dimension d. The following three conditions are equivalent:*

1. *There exists a choice of orientations $[(e_1^a, ..., e_d^a)]$ of T_aM for each $a \in M$ such that for all $a \in M$ there exist $U_a \subset_{open} M$, $a \in U_a$, such that there exist $X_1^{(a)}, ..., X_d^{(a)} \in \mathbb{X}(U_a)$ such that for all $x \in U_a$, $X_1^{(a)}|_x, ..., X_d^{(a)}|_x$ are linear independent and*

$$(X_1^{(a)}|_x, ..., X_d^{(a)}|_x) \sim (e_1^x, ..., e_d^x)$$

2. *There exists a nowhere vanishing d-form $\omega \in \Omega^d(M)$*
3. *There exists a non-maximal atlas $\mathscr{A}_a = \{\varphi : U \to \mathbb{R}^d\}$ of M such that for all $(\varphi, U), (\psi, V) \in \mathscr{A}_a$,*

$$det\left(\frac{\partial(\psi \circ \varphi^{-1})^j}{\partial x^i}\Big|_{\varphi(U\cap V)} \right)_{i,j} > 0$$

*If M satisfies one of these conditions, then M is called **orientable**. A choice of $[(e_1^a, ..., e_d^a)]$ (or of ω, or of $\mathscr{A}_a$) is an **orientation** of M. Furthermore, we call a chart $\varphi : U \to \mathbb{R}^d$ **positively oriented** if*

$$\left(\frac{\partial}{\partial x^1}\Big|_p, ..., \frac{\partial}{\partial x^d}\Big|_p \right)$$

*induces the chosen orientation at each $p \in U$. Also, $\mathscr{A}_a$ is called an **orienting atlas.***

Proof. We approach this proof by showing that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$(1 \Rightarrow 2)$

Let $U_a, X_1^a, ..., X_d^a$ as in (1). Define $\rho_a \in \Omega^d(U_a)$,

$$\rho(X_1^a, ..., X_d^a) = 1$$

This is smooth, because in local coordinates, write

$$X_i^a = \sum_{i=1}^{d} A_i^j \frac{\partial}{\partial x^j} \in \mathbb{X}(U_a)$$

with $A_i^j \in \mathcal{C}^\infty(U_a, \mathbb{R})$, if $A : \frac{\partial}{\partial x^i} \mapsto X_i^a$, then

$$\rho_a(X_1^a, ..., X_d^a) = 1 = \rho_a\left(A\frac{\partial}{\partial x^1}, ..., A\frac{\partial}{\partial x^d}\right) = A^*(\rho_a)\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^d}\right) = det(A) \cdot \rho_a\left(\frac{\partial}{\partial x^1} ..., \frac{\partial}{\partial x^d}\right)$$

$$\Rightarrow \rho_a\left(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^d}\right) = \frac{1}{det(A)} \in \mathcal{C}^\infty(U_0,\mathbb{R})$$

implying ρ_a is smooth by 8.13(a). Let χ_a be a partition of unity subordinate to $\{U_a\}_{a\in M}$. Then define,

$$\omega := \sum_{a\in M}\chi_a\cdot\rho_a$$

where $\chi_a\cdot\rho_a$ is a d form on M (is in $\Omega^d(M)$), because $supp(\chi_a)\subset U_a$. We have that $\omega\in\Omega^d(M)$ is nowhere vanishing, from the following reasoning: consider

$$\omega(e_1^y,...,e_d^y) = \sum_{a\in M}\chi_a\cdot\rho_a(e_1^y,...,e_d^y),$$

and let $A_a: T_yM\to T_yM$, $A_a: X_i^a|_y\to e_i^y$, where $X_i^a\in\mathbb{X}(U_a)$. Then $det(A)>0$, which implies that

$$\begin{aligned}\omega(e_1^y,...,e_d^y) &= \sum_{a\in M}\chi_a\cdot\rho_a(e_1^y,...,e_d^y)\\ &= \sum_{a\in M}\chi_a\cdot\rho_a(A_a\cdot A_a^{-1}e_i^y,...,A_aA_a^{-1}e_i^y)\\ &= \sum_a\chi_aA_a^*(\rho_a)(X_1^a|_y,...,X_d^a|_y)\\ &= \sum_a\chi_a\cdot det(A)\cdot\rho_a(X_1^a,...,X_d^a) = \sum_a\underbrace{\chi_a}_{\geq 0}\cdot\underbrace{det(A_{(a)})}_{>0}\end{aligned}$$

and there exists a $\chi_a(y)>0$, so this sum is not equal to 0. $\square$

($2\Rightarrow 3$**)**

Let the orienting atlas be $\mathscr{A}_{\overline{a}} = \{\varphi\in\overline{a}|\omega\left(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^d}\right)>0\}$ where $\frac{\partial}{\partial x^1}$ are induced by the chart φ (this is the oriented atlas, and I may have meant to write this as, $\mathcal{O}_{\overline{\mathcal{A}}}$). This is an atlas; in fact it covers all of M (because if φ is a chart, write ω in the basis given by the chart implying that $\omega = f\cdot dx^1\wedge...\wedge dx^d$, $f\in\mathcal{C}^\infty(U,\mathbb{R})$, $f\neq 0$). Assuming that U is connected, then $f>0$ or $f<0$. This implies either

$$\omega\left(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^d}\right) = f\cdot\underbrace{dx^1\wedge...\wedge dx^d\left(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^d}\right)}_{=1} = f>0$$

implying that φ is in the orienting atlas, or take a chart $\varphi^\#: U\to\mathbb{R}^d$, $\varphi^\#(x) = (-\varphi^1(x),\varphi^2(x),...,\varphi^d(x)$, implying

$$\omega\left(-\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^d}\right) = -f<0$$

which would imply that $\varphi^\#$ is in the orienting atlas. We now need to check the determinant property of (3): for $\varphi,\psi\in\mathscr{A}_{\overline{\mathscr{A}}}$,

$$\omega_a\left(\frac{\partial}{\partial x^1}\Big|_a,...,\frac{\partial}{\partial x^d}\Big|_a\right) = \omega_a\left(A\frac{\partial}{\partial\tilde{x}^1}\Big|_a,...,A\frac{\partial}{\partial\tilde{x}^d}\Big|_a\right)$$

where $A: T_aM\to T_aM$, $A\left(\frac{\partial}{\partial\tilde{x}^i}\right) = \frac{\partial}{\partial x^i}|_a$. By 5.6(2), A has matrix representation

$$A_{ij} = \left(\frac{\partial(\psi\circ\varphi^{-1})^j}{\partial x^i}\varphi(a)\right)_{ij}$$

this implies that

$$\omega_a\left(\frac{\partial}{\partial x^1}|_a,...,\cdot\frac{\partial}{\partial x^d}|_a\right) \overset{10.2}{=} det(A)\cdot\underbrace{\omega_a\left(A\frac{\partial}{\partial\tilde{x}^1}|_a,...,A\frac{\partial}{\partial\tilde{x}^d}|_a\right)}_{>0}$$

implying $det(A)>0$, which is the claim.

($3 \Rightarrow 1$**)**

Let $a \in M$, let $\varphi : U \to \mathbb{R}^d$ be a chart of M, $a \in U$. Then define

$$[(e_1^a, ..., e_d^a)] := \left[\left(\frac{\partial}{\partial x^1}\Big|_a, ..., \frac{\partial}{\partial x^d}\Big|_a \right) \right]$$

This is well-defined, because for another chart $\psi \in \mathcal{O}_{\mathscr{A}}$, we have

$$\frac{\partial}{\partial x^i}\Big|_a \overset{5.6(2)}{=} \left(\left(\frac{\partial \left(\psi \circ \varphi^{-1}\right)^j}{\partial x^i} \right) \varphi(a) \right)_{i,j} \cdot \left(\frac{\partial}{\partial \tilde{x}^j} \right)_j = A \cdot \left(\frac{\partial}{\partial \tilde{x}^j} \right)_j$$

and by the assumption in (3),

$$det\,(A) = det \left(\left(\frac{\partial \left(\psi \circ \varphi^{-1}\right)^j}{\partial x^i} \right)_{i,j} \right) > 0$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial x^1}\Big|_a, ..., \frac{\partial}{\partial x^d}\Big|_a \right) \right] = \left[\left(\frac{\partial}{\partial \tilde{x}^1}\Big|_a, ..., \frac{\partial}{\partial \tilde{x}^d}\Big|_a \right) \right]$$

Clearly $X_i^a := \frac{\partial}{\partial x^i} \in \mathbb{X}(U)$ is a smooth vector field giving the correct orientation (by construction). $\square$

10.5 Recall (Integration in $\mathbb{R}^n$)

10.5.1 Domains of integration

If $D \subset \mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ such that the boundary $\partial D := \overline{D} - D^o$ has measure 0 in $\mathbb{R}^n$ (i.e., for every $\epsilon > 0$ there exist countable open rectangles $R_1, ...,: \partial D \subset \cup_{j=1}^{\infty} R_j$ and $\sum_{j=1}^{\infty} vol(R_j) < \epsilon$), and if $f : D \to \mathbb{R}$ is a continuous function, then one can show that

$$\int_D f(x) dx^1 ... dx^n$$

is defined. In this case, D is called a **domain of integration**,

10.5.2 Examples

(1) If $B \subset \mathbb{R}^n$ is an n-ball, then the boundary of B is S^{n-1}, which has measure 0 in $\mathbb{R}^n$. So, B is a domain of integration.

(2) If $R \subset \mathbb{R}^n$ is an n-dimensional rectangle, then R is also a domain of integration.

(3) If D is a finite union and or intersection of domains of n-balls or rectangles, then D is a domain of integration.

(4) Let $D = \sqcup_{j=1}^{\infty} U_j$, where $U_j \subset (0, 1)$ is open. We set up these U_j's such that D is some kind of crazy Cantor set. This set is not a domain of integration.

10.5.3 Change of variables formula

Let $D, E \subset \mathbb{R}^n$ be domains of integration. Let $\phi : D^o \to E^o$ be a smooth diffeomorphism such that ϕ extends to a neighborhood U of $\overline{D}$ and a neighborhood V of $\overline{E}$ as a smooth diffeomorphism. Let $f : D \to \mathbb{R}$ be continuous. Then,

$$\int_E f(x) dx^1 ... dx^n = \int_D (f \circ \psi) \cdot |det(D\phi)| dx^1 ... dx^n$$

where

$$D\phi = \left(\frac{\partial \phi^i}{\partial x^j}\right)_{ij}$$

is the Jacobian. Note that this is equal to

$$\pm \int_D (f \circ \phi) \cdot det(D\phi) dx^1 \dots dx^n$$

where $+$ when $det(A)$ is positive, and $-$ when $det(A)$ is negative.

10.6 Integrating differential forms

Let $E \subset \mathbb{R}^n$ be a domain of integration. Let $\overline{E} \subset V$ open, and let $\omega \in \Omega^n(V)$. Then, we can write $\omega = f \cdot \underbrace{dx^1 \wedge ... \wedge dx^n}_{in\ this\ order}$. We define,

$$\int_E \omega := \int f \cdot dx^1 \wedge ... \wedge dx^n := \int_E f \ dx^1 ... dx^n$$

which sort of feels like cheating. Let $\phi : D \to E$ be as in 10.5(c). Then,

$$\phi^*(\omega) \subset \Omega^n(V)$$

Claim.

$$\int_E \omega = \pm \int_D \phi^*(\omega)$$

where we get $+$ if $det(D\phi) > 0$ and 0 if $det(D\phi) < 0$.

Proof.

$$(\phi^*\omega)\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right) = \phi^*(f \cdot dx^1 \wedge ... \wedge dx^n)\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right) = f \circ \phi \cdot \phi^*(dx^1 \wedge ... \wedge dx^n)\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right)$$

$$= f \circ \phi \cdot det(D\phi) dx^1 \wedge ... \wedge dx^n \underbrace{\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right)}_{=1}$$

$$= f \circ \phi \cdot det(D\phi)$$

therefore, $\phi^*\omega = f \circ \phi \cdot det(D\phi) \ dx^1 \wedge ... \wedge dx^n$, and the claim follows from 10.5(c). ☐

10.7 Integrating differential forms with compact support

Let M be an oriented manifold of dimension $d \geq 1$.

(a) Let $\omega \in \Omega^d(M)$ such that $supp(\omega) = \overline{\{x \in M | \omega_x \neq 0\}} \subset M$, is compact. Assume that there exists a chart $\varphi : U \to \mathbb{R}^d$ such that $\varphi(U) = B$, a d-dimensional ball in $\mathbb{R}^d$, and such that $supp(\omega) \subset U$. Then define,

$$\int_M \omega := \pm \int_B (\varphi^{-1})^*(\omega)$$

where we have $+$ if φ is positively oriented, and $-$ if φ is negatively oriented. Note: the integral over B is defined, and this definition is independent of the chosen chart, expanding on this, take another chart $\psi : V \to \mathbb{R}^d$, $supp(\omega) \subset U \cap V$. Then there exist domains of integration D and E such that

$$\varphi(supp(\omega)) \subset D \subset \varphi(U \cap V)$$

$$\psi(supp(\omega)) \subset E \subset \psi(U \cap V)$$

because $\varphi(supp(\omega))$ is compact, implying $\varphi(supp(\omega)) \subset \cup_{i=1}^k B_i$, which is a domain of integration (similarly with ψ). Then for $\phi := \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$, we have

$$\int_{\psi(V)} (\psi^{-1})^*\omega = \int_E (\psi^{-1})^*(\omega) \overset{10.6}{=} \pm \int_D \phi^*(\psi^{-1})^*\omega = \pm \int_{\varphi(U)} (\varphi^{-1})^*\psi^*(\psi^{-1})^*\omega = \pm \int_{\varphi(U)} (\varphi^{-1})^*\omega$$

where we get $+$ when φ and ψ have the same orientation, and get $-$ when ψ and φ have the opposite orientation.

(b) Let $\omega \in \Omega^d(M)$ such that $supp(\omega)$ is compact. Let $\varphi_i : U_i \to B_i \subset \mathbb{R}^d$ be charts of M covering $supp(\omega)$. Since $supp(\omega)$ is compact, there exist finitely many of these, $\varphi_1, ..., \varphi_k$ such that the support sits inside of their union. Let χ_i be a partition of unity subordinate to this cover, $U = \{U_1, ..., U_k, U_0\}$ where $U_0 = M - supp(\omega)$. Then, $\chi_i \cdot \omega \in \Omega^d(M)$, $supp(\chi_i\omega) \subset U_i$ which is a single chart. Then define,

$$\int_M \omega := \sum_{j=1}^k \int_M \chi_i \cdot \omega$$

This is well-defined, meaning each

$$\int_M \chi_i\omega \in \mathbb{R}$$

was defined in (a). This is a finite sum of elements in $\mathbb{R}$, so there isn't a convergence issue. Also, $\int_M \omega$ is independent of the chosen cover and partition of unity. For another cover $V_1, ..., V_\ell$, and partition of unity $\rho_1, ..., \rho_\ell$ subordinate to $V_1, ..., V_\ell$, it is $\int_M \rho_j \cdot \chi_i\omega$ is well-defined (since $supp(\rho_i\chi_i\omega) \subset U_i$ and of V_j). Therefore,

$$\sum_{i=1}^k \int_M \chi_i\omega = \sum_{i=1}^k \int_M \sum_{j=1}^\ell \rho_j \cdot \chi_i\omega = \sum_{i=1}^k \sum_{j=1}^\ell \int_M \rho_j\chi_i\omega = \sum_{j=1}^\ell \int_M \sum_{i=1}^k \chi_i \cdot \rho_j\omega = \sum_{j=1}^\ell \int_M \rho_j \cdot \omega$$

10.8 Regular domains

Let M be a manifold of dimension $d \geq 1$. A **regular domain** $D \subset M$ is a subset such that for all $a \in \partial D = \overline{D} - D^o$, there exists a chart $\varphi : U \to \mathbb{R}^d$ such that for some $2r \in \mathbb{R}$,

1. $\varphi(U) = (-r, r)^d$ (the open d-dimensional cube with side length $2r$)
2. $\varphi(U \cap \overline{D}) = (-r, r)^{d-1} \times [0, r)$.
3. $\varphi(U \cap \partial D) = (-r, -r)^{d-1} \times \{0\}$.

In particular, $\varphi(U \cap (M - D^o)) = (-r, r)^{d-1} \times (-r, 0]$. Note:

1. $\overline{D} = D^o \cup \partial D$, $D^o \subset openM$, and if $D \neq \emptyset$ then $D^o \neq \emptyset$, and ∂D is an embedded submanifold of M of dimension $(d-1)$, by proposition 7.9.
2. If $\omega \in \Omega^d(M)$ with compact support, then we can still define $\int_D \omega$.
   (a) If $supp(\omega) \subset U \subset D^o$, where $\varphi : U \to \varphi(U) = (-r, r)^d$ is a chart of M, then $\int_D \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^*(\omega)$, which is well-defined as in 10.7(a).
   (b) If $supp(\omega) \subset U$ for some chart $\varphi : U \to (-r, r)^d$, as in the three conditions above, then define $\int_D \omega = \pm \int_{\varphi(U \cap \overline{D})} (\varphi^{-1})^*\omega$. By (2), this (I think I mean the domain of integration) is the same as $(-r, r)^{d-1} \times [0, r)$, and the $\pm$ is given by φ having positive or negative orientation. This is well-defined, because if we take another chart $\psi : V \to (-r, r)^d$ also as in the 3 conditions

above, with $supp(\omega) \subset U \cap V$, then there exist $\tilde{D}, \tilde{E}$ which are domains of integration such that $\varphi(supp(\omega) \cap \overline{D}) \subset \tilde{D} \subset \varphi(U \cap \overline{D})$ and $\psi(supp(\omega) \cap \overline{D}) \subset \tilde{E} \subset \psi(V \cap \overline{D})$. *
This implies

$$\int_{\psi(V \cap \tilde{D})} (\psi^{-1})^*(\omega) = \int_{\overline{E}} (\psi^{-1})^*(\omega) = \pm \int_{\tilde{D}} (\psi \circ \varphi^{-1})^*(\psi^{-1})^* \omega$$

$$= \pm \int_{\tilde{D}} (\varphi^{-1})^*(\omega) = \pm \int_{\varphi(U \cap \tilde{D})} (\varphi^{-1})^*(\omega)$$

where the $\pm$ comes from whether $\psi \circ \varphi^{-1}$ is positively or negatively oriented.

(c) For $supp(\omega) \subset_{cpt} M$, let $\varphi_i : U \to \mathbb{R}^d$ be charts as in (a) or (b) covering $supp(\omega) \cap \overline{D}$. Since $supp(\omega) \cap \overline{D}$ is compact, there are finitely many $\varphi_1, ..., \varphi_k$. Let $\{\chi_i\}_j$ be a partition of unity subordinate to $\{U_1, ..., U_k, U_0\}$ where $U_0 = M - (supp(\omega) \cap \overline{D})$. Then define, as before,

$$\int_D \omega := \sum_{j=1}^{k} \int_D \chi_i \cdot \omega$$

This is well-defined, as in 10.7 (i.e., it is independent of the chosen cover and partition of unity).

3. If M has an orientation, then ∂D has an induced orientation as follows: let $v = -\frac{\partial}{\partial x^d}|_a \in T_a M$ for some chart $\varphi : U \to \mathbb{R}^d$ as in the first three conditions of this section. We call v an **outward pointing** tangent vector. Let $e_i^a \in T_a(\partial D)$ for $i = 1, ..., d-1$ be a basis. Then, $(e_1^a, ..., e_{d-1}^a)$ represents the orientation of $T_a(\partial D)$ if and only if $(v, e_1^a, ..., e_{d-1}^a)$ represents the orientation of $T_a M$. This is well-defined: i.e., it is independent of the chosen chart. Let $\psi : V \to \mathbb{R}^d$ be another chart as s in this discussion, satisfying the 3 conditions at the beginning of this section. Recall from 5.6(2) that the change of variable

$$\frac{\partial}{\partial x^d}|_a = \sum_{j=1}^{d} \frac{\partial(\psi \circ \varphi^{-1})^j}{\partial x^d}(\varphi(a)) \cdot \frac{\partial}{\partial \tilde{x}^j}\Big|_a$$

Since $\psi : \partial D \to (-r, r)^{d-1} \times \{0\}$, we have for all $j < d$ that

$$\frac{\partial}{\partial \tilde{x}^j}\Big|_a \in T_a(\partial D) = span(e_1^a, ..., e_{d-1}^a)$$

Now for $j = d$, note that $\psi \circ \varphi^{-1} : (-r, r)^{d-1} \times [0, r) \to (-r, r)^{d-1} \times [0, r)$, therefore

$$\psi \circ \varphi^{-1}(\varphi(a) + (0, ..., t)) \in \begin{cases} (-r, r)^{d-1} \times (0, r) \ for \ t > 0 \\ (-r, r)^{d-1} \times \{0\} \ for \ t = 0 \\ (-r, r)^{-d1} \times (-r, 0) \ for \ t < 0 \end{cases}$$

this implies

$$\frac{\partial(\psi \circ \varphi^{-1})^d}{\partial x^d}(\varphi(a)) = \frac{d}{dt}(\psi \circ \varphi^{-1}(\varphi(a) + (0, ..., 0, t)) > 0 \Rightarrow$$

$$\frac{\partial}{\partial x^d}|_a = \underbrace{\sum_{j=1}^{d-1} r_i \cdot e_i^a}_{\in T_a(\partial D} + c \cdot \frac{\partial}{\partial \tilde{x}^d}|_a$$

*these $\tilde{D}, \tilde{E}$ can be constructed as in 10.7, e.g. for $\tilde{D}$ there exist finitely many $B_1, ..., B_k$ that cover $\varphi(supp(\omega))$ and $B_j \subset (-r, r)^d$. Then take $\tilde{D} := (B_1 \cup ... \cup B_k) \cap ((-r, r)^{d-1} \times [0, r))$, which is indeed, a domain of integration.

where $c > 0$. This implies that if $A : T_aM \to T_aM$, $A(e^0_j) = e^a_j$, $A(-\frac{\partial}{\partial \tilde{x}^d}|_a) = -\frac{\partial}{\partial x^d}|_a$, then A has a matrix representation given by

$$A = \begin{bmatrix} c & r_1 & \dots & r_{d-1} \\ 0 & 1 & 0 & \\ \vdots & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

implying that $det(A) = c > 0$, and so

$$[(-\frac{\partial}{\partial \tilde{x}^d}|_a, e^a_1, ..., e^a_{d-1})] = \left[\left(-\frac{\partial}{\partial x^d}\bigg|_a, e^a_1, ..., e^a_{d-1}\right)\right]$$

10.9 10.9 Example

$M = \mathbb{R}^n$, $D = \{x \in \mathbb{R}^n | x^n \geq 0\}$, implying that $\partial D = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Let $a \in \partial D$. Denote by $[(e_1, ..., e_n)]$ the standard orientation of $T_a\mathbb{R}^n$ given by $e_j = \frac{\partial}{\partial x^j}|_a$. We claim that the induced orientation of $T_a\mathbb{R}^{n-1}$ is $[(-1)^n \cdot e_1, ..., e_{n-1})]$.

Proof. Let $v = -\frac{\partial}{\partial x^n} = -e_n$ be the outward pointing tangent vector. Then: $[(v, (-1)^n e_1, e_2, ..., e_{n-1})] = [(-e_n, (-1)^n e_1, ..., e_{n-1})] = [((-1)^{n-1} \cdot (-1)^n e_1, e_2, ..., e_{n-1}, -e_n)] = [(e_1, ..., e_{n-1}, e_n)]$. □

10.10 Stoke's Theorem

Let M be an oriented manifold of dimension d. Let D be a regular domain of M. Let $\omega \in \Omega^{d-1}(M)$ with compact support. Then:

$$\int_D d\omega = \int_{\partial D} \omega$$

where ω on the right is interpreted as $i^*(\omega) \in \Omega^{d-1}(\partial D)$, with i being the inclusion $i : \partial D \hookrightarrow M$.

Proof. Transcribing this proof became difficult, so I omit it due to a large number of anticipated errors. This proof can be found in Lee's book. □

10.11 Proposition

Let M be an oriented manifold of dimension d, $\omega \in \Omega^d(M)$ with compact support, let $D \subset M$ be a regular domain. We want to calculate $\int_D \omega$. Let (φ_i, U_i) be positively oriented charts of M for $i = 1, ..., k$ such that

1. For all $i = 1, ..., k$, $\varphi_i(U_i)$ is a domain of integration,
2. There exists a continuous extension of φ^{-1}:
$$\tilde{\varphi_i}^{-1} : \overline{\varphi_i(U_i)} \to \overline{U_i}$$
3. $supp(\omega) \subset \overline{U_1} \cup ... \cup \overline{U_k}$
4. For all $i \neq j$, $U_i \cap U_j = \emptyset$

Then:

$$\int_D \omega = \sum_{i=1}^{k} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\omega)$$

Proof. This is a sketch of the proof: first, assume there exists $\varphi : U \to \mathbb{R}^d$ positively oriented such that $supp(\omega) \subset U$ and that $\varphi(U)$ is a domain of integration and that $\overline{U}$ is compact, and $\overline{\varphi(U)}$ is compact, and that φ extends to a diffeomorphism $\tilde{\varphi} : V \to \mathbb{R}^d$ with $\overline{U} \subset V$. Then: $\partial(U \cap U_i)$ has measure 0 (because $\partial(\varphi(U \cap U_i)) = \varphi(\partial(U \cap U_i))$ by assumption on the extension of φ, and $\partial(U \cap U_i) \subset \partial U \cup \partial U_i$ (U, U_i open), and ∂U and ∂U_i have measure 0 since φ, φ_i are domains of integration.) This implies that

$$\int_D \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega = \sum_{j=1}^{k} \int_{\varphi(U \cap U_j)} (\varphi^{-1})^*(\omega)$$

because $\varphi(supp(\omega)) \subset \overline{\varphi(U \cap U_1)} \cup ... \cup \overline{\varphi(U \cap U_k)}$, while for all $i \neq j$, $\varphi(U \cap U_i) \cap \varphi(U \cap U_j) = \emptyset$ (by (4)). These images are open, and so

$$\overline{\varphi(U \cap U_i)} \cap \overline{\varphi(U \cap U_j)} \subset \partial(\varphi(U \cap U_i)) \cup \partial(\varphi(U \cap U_j))$$

but either set on the right has measure 0, and so it follows that the left hand side has measure 0 as well. Then,

$$\int_{\varphi(U \cap U_j)} (\varphi^{-1})^* \omega = \int_{\varphi_j(U \cap U_j)} (\varphi \circ \varphi_j^{-1})^* (\varphi^{-1})^* \omega = \int_{\varphi_j(U_j)} (\varphi_j^{-1})^*(\omega)$$

where the last equality follows from $supp(\omega) \subset U$ implying that $\varphi_j(supp(\omega)) \subset \varphi_j(U \cap U_j)$. □

10.12 Example

Let $\varphi^{-1} : (0, \pi) \times (0, 2\pi) \to S^2$, given by

$$\varphi^{-1}(\alpha, \beta) = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \beta)$$

Note: $\overline{\varphi^{-1}((0, \pi), (0, 2\pi))} = S^2$, and this implies

$$\int_{S^2} \omega = \int_0^{\pi} \int_0^{2\pi} (\varphi^{-1})^*(\omega) d\alpha d\beta$$

11

DeRham Theory

YES OH SWEET LORD WE'RE BACK TO TOPOLOGY (Lee, chapter 17 and 18).

11.1 Cochains

(a) A **cochain complex complex over** $\mathbb{R}$ consists of $\mathbb{R}$-vector spaces $C^j,\ j \in \mathbb{Z}$ and linear maps $d_j : C^j \to C^{j+1}$ such that $d_j \circ d_{j-1} = 0$ for all $j \in \mathbb{Z}$.

(b) $\alpha \in C^k$ is called **closed** if $d_k(\alpha) = 0$, in this case, α is called a **cocycle,** and the space of all cocycles is denoted by Z^k.

(c) $\alpha \in C^k$ is called **exact** if $\alpha = d_{k-1}(\beta)$ for some $\beta \in C^{k-1}$. In this case, α is called a **coboundary,** the space of all coboundaries is denoted by B^k.

(d) We define the cohomology in degree k as $H^k := Z^k/B^k$.

(e) A map of chain complexes $\{C_j, d_j\}$ and $\{\tilde{C}_j, \tilde{d}_j\}$ consists of linear maps $F_j : C^j \to \tilde{C}^j$ such that $\tilde{d}_j \circ F_j = F_{j+1} \circ d_j$.

11.2 Definition

Let M be a smooth manifold. Let $C^k := \Omega^k(M)$ for $k \geq 0$, $C^k = 0$ for $k < 0$, and let $d_k : C^k \to C^{k+1}$ be the DeRham exterior derivative from 9.10: $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Define,

$$H^k_{DR} := Z^k/B^k = \{\alpha \in \Omega^k(M) | d\alpha = 0\}/\{\alpha \in \Omega^k(M) | \exists \beta \in \Omega^{k-1}(M) : \alpha = d\beta\}$$

called the **DeRham cohomology of** M **in degree** k**.**

Note:

(a) $H^k_{DR}(M) = \{0\}$ for $k < 0$, and $H^k_{DR}(M) = \{0\}$ for $k > dim(M)$, because $\Omega^k(M) = \{0\}$ for $k > m$.

(b) If $F \in \mathcal{C}^\infty(M, N)$, then this gives us that

$$F^* : \Omega^k(N) \to \Omega^k(M)$$

by 9.7(b). We have $d(F^*\alpha) = F^* d\alpha$ by 9.11(c), and this implies that F^* gives a morphism of chain complexes, which therefore maps $F^* : Z^k(N) \to Z^k(M)$ (since $d\alpha = 0 \Rightarrow dF^*\alpha = F^* d\alpha = 0$) and similarly $F^* : B^k(N) \to B^k(M)$ (since $\alpha = d\beta \Rightarrow F^*\alpha = F^* d\beta = dF^*\beta$). Hence, we get a map

$$F^* : H^k_{DR}(N) \to H^k_{DR}(M); \quad \text{given by:} \quad F^*([\alpha]) := [F^*(\alpha)]$$

which is well defined, i.e., independent of representative α, because for $[\alpha] = [\alpha']$, then $\alpha - \alpha' \in B^k(N)$, if and only if $\alpha - \alpha' = d\beta$ for some $\beta \in B^{k-1}(N)$, but this implies that $F^*\alpha - F^*\alpha' = F^*d\beta = dF^*\beta$, and so $F^*\alpha - F^*\alpha' \in B^k(M)$, which is equivalent to saying that $[F^*\alpha] = [F^*\alpha']$. Note: $id : M \to M$ is a trivial map on the DeRham cohomology, and the composition of these maps on cohomology works out well; $(F \circ G)^* = G^* \circ F^*$.

11.3 11.3 Definition

Definition. 1. For $v_0, ..., v_k \in \mathbb{R}^n$ let

$$[v_0, ..., v_k] := \text{the affine span of } v_0, ..., v_k = \{\sum_{i=0}^{k} t_j \cdot v_j \ \Big| t_j \in [0,1] \text{ and } \sum_{j=0}^{k} t_j = 1\}$$

2. Let $e_0 := 0 \in \mathbb{R}^k$, and for $j = 1, ..., k$ let $e_j := (0, ..., 1, .., 0) \in \mathbb{R}^k$, where the 1 is of course, in the j^{th} coordinate. Define the **standard k-simplex** to be

$$\sigma^k := [e_0, ..., e_k]$$

3. There are **face maps** $f_j^k : \sigma^k \to \sigma^{k+1}$, for $j = 0, ..., k+1$ given by

$$f_k^k[e_0, ..., e_k] \to [e_0, ..., \hat{e_j}, ..., e_{k+1}];$$

$$e_0 \mapsto e_0, ..., e_{j-1} \mapsto e_{j-1}, e_j \mapsto e_{j+1}, ..., e_k \mapsto e_{k+1}$$

extending f_j^k linearly over $e_0, ..., e_k$,

$$f_j^k \left(\sum_{i=0}^{k} t_i e_i \right) := \sum_{i=0}^{k} t_i f_j^k(e_i)$$

4. Let M be a manifold. A k-simplex in M is a continuous map $\sigma : \sigma^k \to M$. A k**-chain** in M is a finite linear combination of k-simplicies:

$$c = \sum_{i=1}^{\ell} c_k \sigma_i, \quad c_i \in \mathbb{R}, \ \ \sigma_i \text{ a } k\text{- simplex.}$$

Denote by $C_k(M, \mathbb{R}) := \{c | c \text{ is a } k\text{-chain in } M\}$. This is the free $\mathbb{R}$-vector space generated by all k-simplicies called the **singular chain** complex of M. Denote by $C^Y(M, \mathbb{R}) := C_k(M, \mathbb{R})^* = Hom(C_k(M, \mathbb{R}), \mathbb{R})$ called the **singular cochain complex.**

5. There is a map $\partial_k : C_k(M, \mathbb{R}) \to C_{k-1}(M, \mathbb{R})$ given by

$$\partial_k(\sigma) := \sum_{i=1}^{k} (-1)^i \sigma \circ f_j^{k-1}$$

and for

$$c = \sum_{i=1}^{\ell} c_i \sigma_i,$$

$$\partial_k \left(\sum_{i=1}^{\ell} c_i \sigma_i \right) = \sum_{i=1}^{\ell} c_i \cdot \partial_k(\sigma_i)$$

Claim. $\partial_{k-1} \circ \partial_k = 0$.

Proof. I've done this before, so I won't copy this one down. Basically, the signs all cancel out—it's a fun exercise to do. □

Dualizing ∂_k gives us $\delta_{k-1} := \partial_k^* : C^{k-1}(M,\mathbb{R}) \to C^k(M,\mathbb{R})$. This gives us two chain complexes, one which uses the boundary map, and the other uses the coboundary map.

6. Let $Z^k(M,\mathbb{R}) = Ker(\delta_k)$, $B^k(M,\mathbb{R}) = Im(\delta_{k-1})$. Define $H^k(M,\mathbb{R}) := Z^k(M)/B^k(M,\mathbb{R})$, called the k^{th} **singular homology.**

7. If $F \in \mathcal{C}^\infty(M,N)$, then for a k-simplex $\sigma : \Delta^k \to M$, $F \circ \sigma : \Delta^k \to N$, and $F \circ \sigma$ is a k-simplex in N. This gives us an induced map $F_* : C_k(M,\mathbb{R}) \to C_k(N,\mathbb{R})$, and so it's dual gives us $F^* : C^k(N,\mathbb{R}) \to C^k(M,\mathbb{R})$.

11.4 Definition and Proposition

Definition. Let M be a smooth manifold. A k-simplex $\sigma : \Delta^k \to M$ is called **smooth** if for all $x \in \Delta^k$, there exists $U \subset_{open} \mathbb{R}^k$, $x \in U$ such that σ has a smooth extension $\sigma_U : U \to M$, i.e., $\sigma|_{U\cap\Delta^k} = \sigma_U|_{U\cap\Delta^k}$. Denote by $C_k^\infty(M,\mathbb{R})$ the $\mathbb{R}$-vector space generated by smooth $k-simplicies$, i.e.,

$$c \in C_k^\infty(M,\mathbb{R}) \quad if \quad c = \sum_{i=1}^{\infty} c_i\sigma_i$$

where σ_i is a smooth k-simplex. Note that ∂_k preserves smoothness, i.e., if σ is smooth then $\partial\sigma$ is smooth. Denote by $\partial_k^\infty : C_k^\infty(M,\mathbb{R}) \to C_{k-1}^\infty(M,\mathbb{R})$ the induced boundary map $\partial_k^\infty(c) = \partial_k(c)$. More precisely, if

$$i : C_k^\infty(M,\mathbb{R}) \hookrightarrow C_k(M,\mathbb{R})$$

then $i \circ \partial_k^\infty(c) = \partial_k \circ i(c)$, for all $c \in C_k^\infty(M,\mathbb{R})$. This implies that i is a chain map. Define $C_\infty^k(M,\mathbb{R}) := (C_k^\infty(M,\mathbb{R}))^*$ with induced differential $\delta^\infty := (\delta_\infty)^*$, then there is an induced map

$$\rho := i^* : C_\infty^k(M,\mathbb{R}) \to C_\infty^k(M,\mathbb{R})$$

Since i is a chain map, so is ρ. Define,

$$H_\infty^k(M,\mathbb{R}) := \frac{Z_\infty^K(M,\mathbb{R})}{B_\infty^k(M,\mathbb{R})}$$

to be the k^{th} **smooth singular cohomology of** M**.** This implies the existence of an induced map $\rho^* : H^k(M,\mathbb{R}) \to H_\infty^k(M,\mathbb{R})$.

Claim. ρ^* is an isomorphism.

Proof. Not done. □

11.5 Definition and Proposition

Let M be a smooth manifold. Let $\rho_k : \Omega^k(M) \to C_\infty^k(M,\mathbb{R})$ be given by: for $\omega \in \Omega^k(M)$, σ a smooth k-simplex, let

$$(\rho(\omega))(\sigma) := \int_\sigma \omega := \int_{\Delta^k} \sigma^*(\omega)$$

For $c = \sum_{i=1}^{\ell} c_i\sigma_i \in C_k^\infty(M,\mathbb{R})$ where σ_i is a smooth k-simplex, let

$$(I_k(\omega))(c) := \int_c \omega := \sum_{i=1}^{\ell} c_i \cdot \int_{\sigma_i} \omega = \sum_{i=1}^{\ell} c_i \cdot \int_{\Delta^k} \sigma^*\omega$$

Claim. (a) I_k is well-defined.

(b) I_k is a cochain map:

$$I_k \circ d_k = \delta_k^\infty \circ I_k$$

(c) If $F \in \mathcal{C}^\infty(M,N)$, then:

$$F^* \circ I_k = I_k \circ F^*$$

Proof. (a) Note that for a smooth k-simplex $\sigma(\Delta^k)$ is not a regular domain as we defined it. However, each point $x \in \Delta^k$ has a 'nice neighborhood' U, containing x, and without loss of generality assume that $U = B^o$ (open ball), such that $\sigma_U : U \to M$ is smooth. Then $sigma_U^* \omega|_{\sigma_U(U)}$ is defined, and since Δ^k is compact, there are finitely many $U_1, ..., U_\ell$ which cover Δ^k in that way. Then for a partition of unity $\chi_1, ..., \chi_\ell$ subordinate to $U_1, ..., U_\ell$, define

$$\int_\sigma \omega := \sum_{j=1}^{\ell} \int_{\Delta^k \cap U_j} \chi_j \cdot \sigma_{U_j}^*(\omega)$$

This is independent of the chosen cover, ad partition of unity (exercise).

(b) We need to show that

$$\int_C d\omega = I(d\omega)(c) \stackrel{?}{=} \delta^\infty(I\omega)(c) = I(\omega)(\partial^\infty c) = \int_{\partial c} \omega$$

It is enough to show this for smooth k-simplicies, since both sides are $\mathbb{R}$-linear in c. We need to show the following:

$$\int_\sigma d\omega = \int_{d\sigma} \omega$$

for a smooth k-simplex $\sigma : \Delta^k \to M$. We have

$$\int_\sigma d\omega = \int_{\Delta^k} \sigma^*(d\omega) = \sigma_{\Delta^k} d\sigma^* \omega = \int_{\partial\Delta^k} \sigma^* \omega$$

Now, $\partial\Delta^k = \cup_{j=0}^k f_j^{k-1}(\Delta^{k-1})$, and $\Delta^k \subset \mathbb{R}^k$ has standard orientation. This implies that the orientation induced on $f_j^{k-1}(\Delta^{k-1})$ is

$$((-1)^j e_1, ..., \hat{e^j}, ..., e_k), \quad \text{for } j > 0$$

(because $(-e_j, (-1)^j e_1, ..., \hat{e_j}, ..., e_k) \sim (e_1, ..., e_j, ..., e_k)$). This implies that $f_j^{k-1} : \Delta^{k-1} \to f_j^{k-1}(\Delta^{k-1})$ is positively oriented if and only if j is even ($j > 0$) and $f_0^{k-1} : \Delta^{k-1} \to f_0^{k-1}(\Delta^{k-1})$ is also positively oriented - because

$$(\underbrace{e_1 + ... + e_{k-1}}_{outward\ pointing}, e_2 - e_1, .e_3 - e_2, ..., e_k - e_{k-1}) \sim (e_1, ..., e_k).$$

This implies that

$$\begin{aligned} \int_{\partial\Delta^k} \sigma^*\omega = \sum_{j=0}^k \int_{f_j^{k-1}(\Delta^{k-1})} \sigma^*\omega = \sum_{j=0}^k (-1)^j \int_{\Delta^{k-1}} \left(f_j^{k-1}\right)^* \sigma^*(\omega) = \sum_{j=0}^k (-1)^j \int_{\Delta^{k-1}} (\sigma \circ f_j^{k+1})^* \omega \\ = \sum_{j=0}^k (-1)^j \int_{\sigma \circ f_j^{k-1}} \omega = \int_{\sum_{j=0}^k (-1)^j \sigma \circ f_j^{k-1}} \omega = \int_{\partial\sigma} \omega. \end{aligned}$$

(c)

$$(I \circ F^*(\omega))(\sigma) = \int_\sigma F^*\omega$$

(where $\omega \in \Omega^k(N)$, $F^*\omega \in \Omega^k(M)$, σ is a smooth k-simplex in M)

$$= \int_{\Delta^k} \sigma^* F^*\omega = \int_{\Delta^k} (F \circ \sigma)^*\omega = \int_{F\circ\sigma} \omega = I(\omega)(F \circ \sigma) = (F^* \circ I)(\omega)(\sigma)$$

□

11.6 The DeRham theorem

Theorem 41. *By 11.5, we have an induced map $J^* : H^*_{DR}(M) \to H^k_\infty(M, \mathbb{R})$.*

Claim. J^* is an isomorphism, thus $H^*_{DR}(M) \cong H^k_\infty(M, \mathbb{R}) \cong H^k(M, \mathbb{R})$.

Proof. Cite: Lee, Theorem 18.14. □

11.7 Corollaries of the DeRham theorem

Corollary 42. *$H^*_{DR}(M)$ has properties similar to that of singular cohomology:*

1. ***Homotopy equivalence:** if M and N are homotopy equivalent, then $H^*_{DR}(M) \cong H^*_{DR}(N)$ are isomorphic.*

2. ***Dimension:** $H^k(\{*\}) = \{0\}$, for all $k \neq 0$ (I'm not entirely sure what he means by this, this is not a typo - it almost appears that he's saying that the cohomology groups of EVERYTHING are zero, which is of course, not true). Then, (1) and (2) imply that if M is contractible, that $H^k(M) = \{0\}$ by the Poincaré Lemma. Thus: every closed form is locally exact; more precisely, if $d\omega = 0$ then for all $p \in M$, there exists a contraction $p \in M$: $\omega|_U = d\lambda, \lambda \in \Omega^*(U)$.*

3. ***Additivity:** $H^k(\sqcup_{i\in I} M_i) = \oplus_{i \in I} H^k(M_i)$*

4. ***Mayer-Vietoris Property:** if M is a manifold, $U, B \subset M$ are open, $U \cup V = M$, then there exists a long exact sequence*

$$\ldots \to H^k_{DR}(M) \xrightarrow{f^* \oplus g^*} H^k_{DR}(U) \oplus H^k_{DR}(V) \xrightarrow{p^* - q^*} H^k_{DR}(U \cap V) \xrightarrow{\tau} H^{k+1}_{DR}(M) \to \ldots$$

   where f, g, p, q are the inclusions $f : U \hookrightarrow M, g : V \hookrightarrow M, p : U \cap V \hookrightarrow U, q : U \cap V \hookrightarrow V$, and τ is some map.

11.8 Examples

1. Since $\mathbb{R}^n$ is contractible,

$$H^k_{DR}(\mathbb{R}^n) = \begin{cases} 0 & k \neq 0 \\ \mathbb{R} & k = 0 \end{cases}$$

2. When ℓ is the number of connected components of a manifold M,

$$H^0_{DR}(M) \cong \mathbb{R}^\ell$$

   Taking $[f] \in H^0_{DR}(M)$, we have in local coordinates that $df = 0 \to 0 = \sum \frac{\partial f}{\partial x^i} dx^i \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j} \Rightarrow f$ is locally constant, and so $f = c_i$, where $c_i \in \mathbb{R}$ is chosen for each connected component of M. One can show that if M is simply connected, that this implies $H^1_{DR}(M) = \{0\}$.

3. We have the following equalities, the middle one will then be proven:

$$H^0_{DR}(S^1) \cong \mathbb{R}, \quad H^1_{DR}(S^1) \stackrel{\star}{=} \mathbb{R}, \quad H^k_{DR}(S^1) = 0 \; \forall k \neq 0, 1$$

Proof. We now prove equality $\star$:

$$\ldots \to \{0\} \to \Omega^0(S^1) \to \Omega^1(S^1) \to \{0\} \to \ldots$$

implies that for all $a \in \Omega^1(S^1)$ that $d\alpha = 0 \Rightarrow Z^1(S^1) = \Omega^1(S^1)$ (by the exact-ness, I think). What is $B^1(S^1)$? Well, let $\rho := dxy - ydx \in \Omega^1(\mathbb{R}^2)$. Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map, and let $\omega = i^*(\rho) \in \Omega^1(S^1)$.

Claim. (a) $\omega \notin B^1(S^1)$

(b) For all $\alpha \in \Omega^1(S^1)$, there exists a $c \in \mathbb{R}$, $\zeta \in \Omega^0(S^1)$ such that $\alpha = c \cdot \omega + d\zeta$. Thus, $[\alpha] = [c\omega - d\zeta] = c[\omega] - [d\zeta]$, but as $d\zeta = 0$ in $H^1(S^1)$, $[\alpha] = c \cdot [\omega] \in H^1(S^1)$. This implies that $H^1(S^1) \cong \mathbb{R}$ is generated by $[\omega]$.

Proof. In proving our first claim, we now prove this claim:

(a) Let $\rho : S^1 - \{(-1, 0)\} \to (-\pi, \pi)$ be given by $\varphi^{-1}(t) = (cos(t), sin(t)) \in S^1 \subset \mathbb{R}^2$. This implies that $(\varphi^{-1})(\omega) = (\varphi^{-1})^* i^*(xdy - ydx) = cos^2 t \; dt + s^2 t \; dt = dt$. As such,

$$\int_{S^1} \omega = \int_{-\pi}^{\pi} (\varphi^{-1})^*(\omega) = \int_{-\pi} 6\pi dt = 2\pi$$

but for all $\zeta \in \Omega^0(S^1)$,

$$\int_{S^1} d\zeta = \int_{\partial S^1} \zeta = \int_{\emptyset} \zeta = 0$$

Thus, $\omega \neq d\zeta$.

(b) We have $(\varphi^{-1})^*(\alpha) = f(t)$ for some smooth function $f : (-\pi, \pi) \to \mathbb{R}$. Define

$$c := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \; ds$$

and

$$\zeta \in \mathcal{C}^\infty(S^1, \mathbb{R}); \qquad \zeta(\cos t, \sin t) := \begin{cases} \int_0^t f(s)ds - c \cdot t & t \in (-\pi, \pi) \\ \int_0^\pi f(s)ds - c \cdot \pi = \int_0^{-\pi} f(s)ds - c \cdot (-\pi) & t = \pi \end{cases}$$

Then, ζ is smooth (consider another chart $\psi : S^1 - \{(1, 0)\} \to (0, 2\pi), \psi^{-1}(t) = (\cos t, \sin t)$ and check this). Then,

$$d(\varphi^{-1})^*\zeta = d(\int_0^t f(s)ds - c \cdot t) = f(t)dt - c \cdot dt = (\varphi^{-1})^*\alpha - c \cdot (\varphi^{-1})^*(\omega)$$

Applying φ^* gives us that $\alpha = d\zeta + c \cdot \omega$ on $Im(\varphi)$, and by the continuity of S^1. We have our claim, and we are done with everything now.

□

□

4. We have,

$$H^k_{DR}(S^n) = \begin{cases} \mathbb{R} & k = 0 \; or \; n \\ \{0\} & k \neq 0 \; or \; n \end{cases}$$

Proof. This follows by induction on n. For $n = 1$, this is true by example (3). Going from n to $n+1$, assume that the statement is true for $H^*_{DR}(S^n)$. Then, $H^0_{DR}(S^{n+1}) \cong \mathbb{R}$, $H^1_{DR}(S^{n+1}) \cong \{0\}$ by (2), since S^{n+1} is connected and simply connected. Covering S^{n+1} by two hemispheres, U, V such that $U \cap V \cong \S^n$. Then the Mayer-Vietoris sequences gives us for $k \geq 2$, that

$$\underbrace{H^{k-1}_{DR}(U) \oplus H^{k-1}_{DR}(V)}_{=\{0\}\oplus\{0\}} \to \underbrace{H^{k-1}_{DR}(U \cap V)}_{H^k_{DR}(S^n)} \to H^k_{DR}(S^{n+1}) \to \underbrace{H^k_{DR}(U)}_{=\{0\}} \oplus \underbrace{H^k_{DR}(V)}_{=\{0\}}$$

Implying then, that

$$H^k_{DR}(S^{n+1}) \cong H^{k-1}_{DR}(S^n) = \begin{cases} \mathbb{R} & k = n+1 \\ \{0\} & k \neq n+1 \end{cases}$$

□

12 The Lie Derivative

This chapter comes from Lee's book, chapters 9 and 12 (though more of 9).

12.1 Proposition and Definition

Let M be a manifold, let $X \in \mathbb{X}(M)$ be a smooth vector field. Then, one can do the following:

1. Let $a \in M$. Then there exists an open interval I, $0 \in I$, such that there exists a smooth curve $\alpha_a : I \to M$ such that $\alpha_a(0) = a$, and for all $t \in I$,

$$\alpha_a'(t) = X_{\alpha(t)}. \tag{12.1.1}$$

   For another curve $\tilde{\alpha_a} : \tilde{I} \to M$ as above, it is true that

$$\alpha_a\big|_{I\cap\tilde{I}} = \tilde{\alpha}_a\big|_{I\cap\tilde{I}}.^*$$

   Any such curve α_a is called an **integral curve of X.**

2. For $a \in M$, let I_a be the maximal interval for which α_a can be defined, $\alpha_a : I_a \to M$. Let

$$D := \cup_{a\in M} I_a \times \{a\} \subset \mathbb{R} \times M$$

   Then $D \subset \mathbb{R} \times M$ is open, and the map $\theta : D \to M$, $(t, a) \mapsto \theta(t, a) := \alpha_a(t)$, is smooth. We call θ the **flow of X**.

3. Let $\theta_t(a) := \theta(t, a) = \alpha_a(t)$. Then, $\theta_0 = id_M$, $\theta_s \circ \theta_t = \theta_{s+t}$ for any s, t for which $\theta_t, \theta_s, \theta_{s+t}$ is defined. Let $U \subset M$ be open such that θ_t is defined on U,

$$\theta_t : U \to M$$

   Then $\theta_T : U \to \theta_t(U)$ is a smooth diffeomorphism with inverse given by θ_t^{-1} by $\theta_t^{-1} = \theta_{-t}$.

Proof. 1. This is actually a local problem. Let $\varphi : U \to \mathbb{R}^d$ be a chart, $a \in U$. Then, any curve $\alpha : I \to M$ has a velocity of

$$\alpha'(t) = \sum_{j=1}^{d} \frac{d(\varphi\circ\alpha)^j}{dt} \cdot \frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$$

*This asserts the uniqueness of integral curves (from 5.6(3)) and

$$X_{\alpha(t)} = \sum_{j=1}^{d} X^j(\alpha(t)) \cdot \frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$$

where $X^j \in \mathcal{C}^\infty(U,\mathbb{R})$. Let $\tilde{\alpha} := \varphi \circ \alpha$ and $\tilde{X}^j := X^j \circ \varphi^{-1}$ is also smooth, then the equation 12.1.1 is equivalent to:

$$\begin{cases} \frac{d\tilde{\alpha}^1(t)}{dt} = \tilde{X}^1(\tilde{\alpha}^1(t), ..., \tilde{\alpha}^d(t)) \\ \vdots \\ \frac{d\tilde{\alpha}^d(t)}{dt} = \tilde{X}^d(\tilde{\alpha}'(t), ..., \tilde{\alpha}^d(t) \end{cases}$$

We have the initial condition $\tilde{\alpha}(0) = \varphi(a)$. Now, (1) follows from the existence and uniqueness of solutions of O.D.E.'s (see Lee, Appendix D, Theorem D.1(a,b)).

2. Follows from the smoothness condition of O.D.E'.s (Lee, Theorem D.1(c)). We have: for all $a \in M$, there exists an open subset $U \subset M$ with $a \in U$ such that there exists $\epsilon > 0$ such that $\theta|(-\epsilon,\epsilon) \times U \to M$ is defined. This implies that $D \subset \mathbb{R} \times M$ is open (because if $(t,a) \in D$, then for $b := \theta(t,a)$: there exists an open subset $V \subset M$ with $b \in V$ and there exists an $\epsilon > 0$ such that θ is defined on $(-\epsilon,\epsilon) \times V$. Since $\theta(t,-)|_U : U \to M$ is smooth, and therefore continuous, this implies that $\tilde{U} := (\theta(t,-)|U)^{-1}(V) \subset M$ is open, and $a \in \tilde{U}$. Then θ is defined on $(t-\epsilon, t+\epsilon) \times \tilde{U}$, since by uniqueness of the integral curve, the flow of θ and the solution of the O.D.E. near b have to coincide.).

3. $\theta(0,a) = a$, and $\theta_t \circ \theta_s = \theta_{s+t}$ by uniqueness of solutions. This implies that $\theta_t \circ \theta_{-t} = id_M$, and so $(\theta_t)^{-1} = \theta_{-t}$.

□

12.2 12.2 Definition

Definition. A vector field X is called **complete** if the flow of X is defined on all of $\mathbb{R}$, i.e.,

$$\theta : \mathbb{R}\times \to M$$

12.3 Proposition

Proposition 43. *If M is a compact manifold, and X is any smooth vector field on M, then X is complete.*

Proof. For every $a \in M$, there exists a neighborhood U_a and there exists an $\epsilon_a > 0$ such that θ is defined on $\theta|(-\epsilon_a,\epsilon_a) \times U_a \to M$. Since M is compact, there exist finitely many $U_{a^1}, ..., U_{a^k}$ such that M is equal to their union. Let $\epsilon := min(\epsilon_{a^1}, ..., \epsilon_{a^k}) > 0$. Then, θ is defined for $(-\epsilon,\epsilon) \times M \to M$, in particular, $\theta|_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]\times M} \to M$. Now, let $t \in \mathbb{R}$. Write t as a finite sum, $t = \pm\epsilon 2 \pm \frac{\epsilon}{2} \pm ... \pm \frac{\epsilon}{2} + s$ where $s \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$. Then, $\theta_t = \theta_{pm\epsilon 2 \pm \frac{\epsilon}{2} \pm ... \pm \frac{\epsilon}{2} + s} =$

$$\theta_{\pm\epsilon/2} \circ ... \circ \theta_s$$

is defined for $t \in \mathbb{R}$. □

12.4 Definition

Definition. Let M be a smooth manifold. Let X be a smooth vector field on M, and let $\theta : D \to M$ be the flow of X. Let $(t,a) \in D$, then there exists an open neighborhood $U \subset M$, $a \in U$ such that θ_t is defined on U, and is a diffeomorphism

$$\theta_t : U \to \theta_t(U) =: V$$

Therefore, $\theta_{-t} : V \to U$, and in particular, $\theta_{-t}(\theta_t(a)) = a$. This tells us that

$$(d\theta_{-t})_{\theta_t(a)} : T_{\theta_t(a)} \to T_a M$$

Now, let $W \in \mathbb{X}(M)$. Then: $(d\theta_{-t})_{\theta_t(a)}(W_{\theta(a)}) \in T_a M$. Therefore, $(d\theta_{-t})_{\theta_t(a)}(W_{\theta_t(a)}) - W_a \in T_a M$. We define the **Lie derivative of W with respect to X** to be:

$$(\mathcal{L}_X W)_a := \lim_{t \to 0} \frac{(d\theta_{-t})_{\theta_t(a)}(W_{\theta_t(a)}) - W_a}{t}$$

Similarly, let $\alpha \in \mathcal{T}^{(0,\ell)}(M)$ (a tensor field of type $(0,\ell)$), (e.g., $\alpha \in \Omega^\ell(M)$). Note that

$$(d\theta_t^*)_A : T^{(0,\ell)}(T_{\theta_t(a)}M) \to T^{(0,\ell)}(T_a M)$$

Then we define the **Lee derivative of α with respect to X:**

$$(\mathcal{L}_X)_a \alpha := \lim_{t \to 0} \frac{(d\theta_t^*)_a(\alpha_{\theta_t(a)}) - \alpha_a}{t}$$

12.5 Proposition

Let M be a manifold, let $X, W \in \mathbb{X}(M)$. Then we have:

$$(\mathcal{L}_X W)_a = [X, W]_a \tag{12.5.1}$$

where the right hand side is the Lie bracket from Lemma 6.7.e. In particular,

1. $\mathcal{L}_X W \in \mathbb{X}(M)$ is smooth (by 6.7e)
2. $\mathcal{L}_X W = -\mathcal{L}_W X$ (by 6.8c)
3. For $Z \in \mathbb{X}(M)$, $\mathcal{L}_X([W, Z]) = [\mathcal{L}_X W, Z] + [W, \mathcal{L}_X Z]$
4. $\mathcal{L}_{[X,Z]} W = \mathcal{L}_X(\mathcal{L}_Z W) + \mathcal{L}_Z(\mathcal{L}_X W)$, ((4) and (3) are from 6.8(b))
5. $\mathcal{L}_X(f \cdot W) = X(f) \cdot W + f \cdot \mathcal{L}_X(W)$ (by 6.8(e))

We now prove 12.5.1:

Proof. Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$, let a be contained in an open subset $U \subset M$. Then,

$$(\mathcal{L}_X W)_a(f) = \lim_{t \to 0} \frac{((d\theta_{-t})_{\theta_t(a)}(W_{\theta_t(a)}))(f) - W_a(f)}{t}$$

Let $\epsilon > 0$ be such that $g : (-\epsilon, \epsilon) \times U \to \mathbb{R}$ given by

$$g(t, x) := f(\theta_t(x)) - f(x)$$

is defined for $t \in (-\epsilon, \epsilon), x \in U$, and g is smooth. Let φ be a chart at a, without loss of generality assume also that $\varphi : U \to \mathbb{R}^d$ (or otherwise, take the intersection $U \cap Domain(\varphi)$). Let $h : (-\epsilon, \epsilon) \times \varphi(U) \to \mathbb{R}$ be given by

$$h(t, y) = g(t, \varphi^{-1}(y)).$$

Note that h is a smooth function, and $h(0, y) = 0$. By the fundamental theorem of calculus,

$$h(t, y) = \int_0^t \frac{\partial h(s, y)}{\partial t} ds \overset{substitute}{=} t \cdot \int_0^1 \frac{\partial h(t \cdot u, y)}{\partial t} du = t \cdot k(t, y)$$

(the substitution is $u = \frac{1}{t} \cdot s$, $du = \frac{1}{t} ds$) where k is a smooth function, $k : (-\epsilon, \epsilon) \times U \to \mathbb{R}$. Now, let $\ell(t, x) := k(t, \varphi(x))$ which is also a smooth map $\ell : (-\epsilon, \epsilon) \times U \to \mathbb{R}$. We have:

$$f(\theta_t(x)) - f(x) = g(t, x) = h(t, \varphi(x)) = t \cdot k(t, \varphi(x)) = t \cdot \ell(t, x) \tag{12.5.2}$$

Thus:

$$\ell(0, x) = \lim_{t \to 0} \frac{f(\theta_t(x)) - f(x)}{t} = \lim_{t \to 0} \frac{f(\alpha_x(t)) - f(x)}{t} = (\alpha_x'(0))(f) = X_X(f)$$

(the equality above is because α is an integral curve of X). Therefore,

$$(\mathcal{L}_X W)_a(f) = \lim_{t \to 0} \frac{((d\theta_{-t})_{\theta_t(a)}(W_{\theta_t(a)}))(f) - W_a(f)}{t} = T \lim_{t \to 0} \frac{W_{\theta_t(a)}(f \circ \theta_{-t}) - W_a(f)}{t}$$

$$\overset{12.5.2}{=} \lim_{t \to 0} \frac{W_{\theta_t(a)}(f) + W_{\theta_t(a)}(-t \cdot \ell(-t, x)) - W_a(f)}{t}$$

(and because $W_{\theta_t(a)}$ is $\mathbb{R}$-linear);

$$- \lim_{t \to 0} \frac{W_{\theta_t(a)}(f) - W_a(f)}{t} - \lim_{t \to 0} \frac{t \cdot W_{\theta_t(a)}(\ell(-t, x))}{t} = \frac{\partial W(f)(\theta_t(a))}{\partial t}\Big|_{t=0} - W_{\theta_0(a)}(\ell(0, x))$$

$$= (X_a(W(f)) - W_a(X(f)) = X(W(f)) - W(X(f)))_a = [X, W]_a(f)$$

$\square$