APPLICATIONS OF THE $\mathcal{L}_2$-TRANSFORM TO PDE'S

The goal of this article is to illustrate the applicability of the $\mathcal{L}_2$-transform to solving certain partial differential equations. Recall that a function $f : [0, \infty) \to \mathbb{R}$ is called *exponential squared order* if $\lim_{x\to\infty} f(x)e^{-x^2} = 0$.

Definition 1. *For any exponential squared order function $f(t)$, the $\mathcal{L}_2$ transform of f is defined as:*

$$\mathcal{L}_2\{f(x); s\} = \int_0^\infty xe^{-x^2s^2} f(x)dx$$

Here is a useful example. For $n \geq 0$ we have:

$$\mathcal{L}_2\{x^{2n}; s\} = \frac{n!}{2s^{2n+2}} \tag{1}$$

In the case $n = 0$, we obtain

$$\mathcal{L}_2\{1; s\} = \int_0^\infty xe^{-x^2s^2} dx = \frac{1}{2s^2}$$

and for $n = 1$, through integration by parts, we have

$$\mathcal{L}_2\{x^{2n}; s\} = \left(\frac{-x^2}{2s^2}e^{-x^2s^2}\right)\Big|_0^\infty + \frac{1}{s^2}\int_0^\infty e^{-x^2s^2}xdx = \frac{1}{s^2}\left(\frac{1}{2s^2}\right)$$

Property (1) then follows by a simple induction.

There is a differential operator δ_x, defined by

$$\delta_x = \frac{1}{x} \cdot \frac{d}{dx}$$

which has some nice properties with respect to the $\mathcal{L}_2$-transform, which we now recall. See also

Proposition 1. *Let f be a function of exponential squared order. Then*

$$\mathcal{L}_2\{\delta_x f(x); s\} = 2s^2\mathcal{L}_2\{f(x); s\} - f(0^+) \tag{2}$$

and for all $n \geq 0$

$$\mathcal{L}_2\{x^{2n} f(x); s\} = \frac{(-1)^n}{2^n}\delta_s^n \mathcal{L}_2\{f(x); s\} \tag{3}$$

Proof. For the first claim, we calculate

$$\int_0^\infty xe^{-x^2s^2}\delta_x f(x)dx = \int_0^\infty xe^{-x^2s^2} \cdot \frac{1}{x}\frac{d}{dx}f(x)(dx) = \int_0^\infty e^{-x^2s^2} f'(x)dx$$

and, integrating by parts, we get:

$$= f(x)(e^{-x^2s^2})\Big|_0^\infty + \int_0^\infty 2xs^2xe^{-x^2s^2} f(x)dx$$

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Evaluating from 0 to ∞, using the fact that f is exponential squared order, we can write this expression in terms of the $\mathcal{L}_2$ transform:

$$2s^2\mathcal{L}_2\{f(x);s\} - f(0^+).$$

For property (3) , taking the case in which n=1, we get:

$$\delta_s\mathcal{L}_2\{f(x);s\} = \delta_s\int_0^\infty xe^{-x^2s^2}f(x)dx$$

We can take the differential operator δ_s into the integral, yielding:

$$\int_0^\infty \frac{1}{s}\frac{d}{ds}\cdot xe^{-x^2s^2}f(x)dx \Rightarrow \int_0^\infty \frac{1}{s}\cdot -2x^2s\cdot xe^{-x^2s^2}f(x)dx = -2\int_0^\infty x^2\cdot xe^{-x^2s^2}f(x)dx$$

We can notice here that the last term of this equation is really equal to:

$$\mathcal{L}_2\{-2x^2f(x);s\}$$

Which is an alternative representation of property (3). Applying the application of the differential operator with respect to s to this result, we get:

$$\delta_s\mathcal{L}_2\{-2x^2f(x);s\} = \delta_s - 2\int_0^\infty x^2xe^{-x^2s^2}f(x)dx$$

As we did before, we re-write the differential operator and apply it within the integeral:

$$-2\int_0^\infty \frac{1}{s}\frac{d}{ds}\cdot x^2xe^{-x^2s^2}f(x)dx = -2\int_0^\infty \frac{1}{s}x^2x^2xe^{-x^2s^2}f(x)-2x^2sdx = 4\int_0^\infty x^2\cdot x^2xe^{-x^2s^2}f(x)dx$$

Which we can see is realy equal to :

$$4\mathcal{L}_2\{x^4f(x);s\}$$

As we can see through induction, for a general $n \geq 1$, we have:

$$\delta_s^n\mathcal{L}_2\{f(x);s\} = -2^n\mathcal{L}_2\{x^{2n}f(x);s\}$$

Or, written alternatively,

$$\mathcal{L}_2\{x^{2n}f(x);s\} = \frac{(-1)^n}{2^n}\delta_s^n\mathcal{L}_2\{f(x);s\}$$

□

1. The $\mathcal{L}_2$ Convolution

A binary operation ($\star$) called the convolution of two functions f, g is defined as follows:

$$(f \star g)(t) = \int_0^t xf(\sqrt{(}t^2 - x^2)g(x)dx$$

It can be shown that this operation is associative and commutative. Most importantly for our purposes, the following are true:

$$\mathcal{L}_2\{f \star g;s\} = \mathcal{L}_2(f)\cdot\mathcal{L}_2(g)$$

and:

$$\mathcal{L}_2\{f_1 \star f_2 \star ... \star f_n;s\} = \mathcal{L}_2(f_1)\cdot\mathcal{L}_2(f_2)\cdot ...\cdot\mathcal{L}_2(f_n) \tag{4}$$

From which it follows:

$$\mathcal{L}^{-1}\{\hat{f}_1\cdot\hat{f}_2\cdot ...\cdot\hat{f}_n\} = f_1 \star f_2 \star ... \star f_n \tag{5}$$

2. Applications

Consider the following partial differential equation :

$$t^3 u_{tx} + 2xu = 0 \quad \text{and} \quad u(0^+, t) = 0 \tag{6}$$

where $u = u(x,t)$ for $x, t > 0$ and $u(0^+, t) = \lim_{x \to 0^+} u(x,t)$.

Writing the PDE in Equation 6 in terms of the differential operator, we have:

$$t^3 \frac{1}{x} \frac{d}{dx} u_t + 2u = 0$$

or equivalently,

$$t^3 \delta_x u_t = -2u.$$

Taking the $\mathcal{L}_2$ transform of both sides and using property (3) we obtain

$$2s^2 t^3 \hat{u}_t - u(0^+, t) = -2\hat{u}$$

where $\hat{u} = \hat{u}(s,t) = \mathcal{L}_2\{u(x,t); s\}$. Equivalently,

$$\hat{u}_t = \frac{-1}{s^2 t^3} \hat{u} + \frac{u(0^+, t)}{2s^2 t^3}.$$

By the initial condition in (6), this last term is zero. One solution to this differential equation is

$$\hat{u}(s,t) = \frac{1}{2s^2} e^{\frac{t^{-2} s^{-2}}{2}}$$

We now write $\hat{u}(s,t)$ as a series to obtain:

$$\hat{u}(s,t) = \frac{1}{2s^2} \sum_{n=0}^{\infty} \frac{(\frac{s^2 t^2}{2})^{-2n}}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{2s^{2n+2} t^{2n} n!}.$$

Using property (1), we can calculate that $u(x,t) = \mathcal{L}_2^{-1}\{(\hat{u}(s,t); x\}$ is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2^n x^{2n}}{t^{2n} (n!)^2}.$$

3. Further Generalizations

Similarly, we can solve all partial differential equations of the form:

$$0 = f(t)u + f(t)\frac{1}{x} u_x + g(t)\frac{1}{x} u_{xt}$$

Taking the $\mathcal{L}_2$ transform, we compute:

$$0 = f(t)\hat{u} + f(t) 2s^2 \hat{u} + g(t) 2s^2 \hat{u}_t$$

Rearranging:

$$\frac{\hat{u}_t}{\hat{u}} = -\frac{(1+2s^2)}{2s^2} \cdot M(t)$$

Where $M(t) = \frac{f(t)}{g(t)}$. Claiming that L(s) is any function of s, a solution for $\hat{u}$ is as follows:

$$\hat{u}(s,t) = e^{\frac{-(1+2s^2)}{2s^2} \cdot \int_0^t M(w)dw} \cdot L(s) = e^{\int_0^t M(w)dw} \cdot e^{\frac{-\int_0^t M(w)dw}{2s^2}} \cdot L(s)$$

Writing this as a series, denoting $\int_0^t M(w)dw$ as $\mathcal{M}$,and letting $L(s) = \frac{1}{-s^2}$, we get:

$$\hat{u} = \frac{1}{-s^2}\sum_{n=0}^{\infty}\frac{-\mathcal{M}^n}{n!\cdot(2s^2)^n}\cdot\sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{n!} = \sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{n!\cdot(2^n s^{n+2})}\cdot\sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{n!\cdot 2^{(n-1)}\cdot(2s^{n+2})}\cdot\sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{n!}$$

using identity (2), and taking the $\mathcal{L}_2$ inverse transform, we claim that:

$$u(x,t) = \sum_{n=0}^{\infty}\frac{\mathcal{M}^n\cdot x^{2n}}{(n!)^2\cdot 2^{n-1}}\cdot\sum_{n=0}^{\infty}\frac{\mathcal{M}^n}{2n!}$$

Lastly, we can always solve the following types of partial differential equations:

$$0 = f(t)u + f(t)\frac{1}{x}u_x + g(t)u_t + g(t)u + xt$$

Taking the $\mathcal{L}_2$ transform, we get:

$$0 = \hat{u}(1+2s^s)f(t) + \hat{u}_t(1+2s^2)g(t) \Rightarrow \frac{\hat{u}_t}{\hat{u}} = J(t)$$

Where $J(t) = -\frac{g(t)}{f(t)}$. This implies that a solution for $\hat{u}$ is as follows:

$$\hat{u}(s,t) = e^{\int_0^t J(w)dw}\cdot L(s)$$

Where L(s) is an arbitrary function of s. Representing this expression as a series, and letting $L(s) = \frac{1}{2s^2}$, we get:

$$\hat{u} = \sum_{n=0}^{\infty}\frac{(\int_0^t J(w)dw)^n}{n!\cdot 2y^2}$$

And after applying the $\mathcal{L}_2$ inverse, we get the following solution for u(x,t):

$$u(x,t) = x^0\cdot e^{\int_0^t J(w)dw)}$$

4. An Application to a Partial Differential Equation of Exponential Squared Order

Consider the following partial differential equation, with the following condition:

$$0 = g(t)u - f(t)u_t + \frac{1}{x}f(x)u_{xt} \qquad u(0^+,t) = 0$$

Writing this equation in terms of the differential operator,

$$0 = g(t)u - f(t)u_t + \delta_x f(t)u_t$$

and taking the $\mathcal{L}_2$ transform of it, we get:

$$0 = g(t)\hat{u}_t - f(t)\hat{u}_t + 2s^2 f(t)\hat{u}_t$$

Rearranging, we get:

$$0 = g(t)u + f(t)\hat{f}_t(-1+2s^2)$$

Which leads to:

$$\frac{\hat{u}_t}{\hat{u}} = -\frac{1}{-1+2s^2}\cdot H(t)$$

Where $H(t) = \frac{g(t)}{h(t)}$. This implies that a solution for $\hat{u}$ is as follows:

$$\hat{u}(s,t) = e^{\frac{-1}{-1+2s^2}\cdot\int_0^t H(w)dw}$$

Writing this solution as a series, we get:

$$\hat{u}(s,t) = \sum_{n=0}^{\infty} \frac{(\int_0^t H(w)dw)^n}{n!} \cdot \left(\frac{1}{-1+2s^2}\right)^n$$

from (4), (5) and the following identity:

$$\mathcal{L}_2\{e^{ax^2}; s\} = \frac{1}{-2a+2s^2}$$

We can arrive at the following solution:

$$\mathcal{L}_2^{-1}\{\hat{u}(s,t)\} = \sum_{n=1}^{\infty} \left(\frac{(\int_0^t H(w)dw)^n}{n!} \cdot (e^{(1/2)x^2})^{\star n} \right)$$

Where $e^{(1/2)e^2})^{\star n}$ represents the following:

$$\underbrace{e^{(1/2)e^2} \star e^{(1/2)e^2} \ldots \star e^{(1/2)e^2}}_{n}$$

Which leads the following proposition, which can be shown through induction:

$$(e^{(1/2)e^2})^{\star n} = \frac{1}{2^{n-1}} \cdot \frac{x^{2(n-1)} \cdot e^{(1/2)x^2}}{(n-1)!}$$

After a substitution, we have the following solution for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{(\int_0^t H(w)dw)^n}{n!} \cdot \frac{1}{2^{n-1}} \cdot \frac{x^{2(n-1)} \cdot e^{(1/2)x^2}}{(n-1)!} \right)$$

Notice that the $e^{(1/2)x^2}$ term in this sum prohibits this solution to satisfy the definition of exponential oder. To find the limit of this solution when multiplied with e^{-x^2} as $x \to \infty$, we recognize the following:

$$\sum_{n=0}^{\infty} \frac{x^{2n} \cdot e^{(1/2)x^2}}{(n+1)(n!)^2 2^n} = e^{x^2/2} \sum_{n=0}^{\infty} \frac{(\frac{x^2}{2})^n}{(n+1)(n!)^2} \leq e^{x^2/2} \sum_{n=0}^{\infty} \frac{(\frac{x^2}{4})^n}{n!} = e^{x^2/2} \cdot e^{x^2/4}$$

Which follows from the fact that:

$$(n+1)(n!) \leq 2^n \quad \forall \; n \geq 0$$

And since the following is true:

$$\lim_{x\to\infty} e^{-x^2} \cdot e^{x^2/2} \cdot e^{x^2/4} = 0$$

We see that our solution is in fact, of exponential squared order. Thus, this is a solution that could not have been obtained through some other integral transforms, including the Fourier transform.

References

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