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Point-Set Topology

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CHAPTER 1

INTRODUCTION

Office hours are Tuesday, Thursday from 3:30 - 4:30.

1.1 Overview

The idea of topology revolves around the study of "spaces" and "functions" between them. We can think of interesting examples of what we could call "spaces", for example spheres circles, tori, etc. The thing worth noticing is that each of these "spaces" is a set of points. The question to ask then, is what differentiates the different surfaces I just named from each other if all they are is a set of points? Thus, we have to add there there exists some additional structure for each of these spaces, which we will call their "topological " structure. For example, we could take some notion of distance and add that notion to these sets.

To compare objects, we need to discuss a notion of "sameness".
Example. If we take sets with the additional structure of the notion of distance, same-ness is a bijection preserving distances. A fancy name for this is an **isometry**. This line of thinking leads to a subject of metric geometry.
Example. If you took no additional structure, we're really just studying sets, and "sameness" is defined by having a bijection between two sets. This line of thinking leads to the subject of Combinatorics.

The subject of topology is somewhere in-between these two notions. Our additional structure on a set will be called "a topology", and a set with a topology will be called a space- more specifically, a topological space. Then, our notion of "sameness" will be given by bijections between sets that preserve the additional structure, i.e., the topology. This will lead to more familiar notions:

1. Connected-ness
2. Compact-ness, or things "being bounded" . For example, the real line $\mathbb{R}$ and $\mathbb{R}^2$ extend forever without bound, but a figure like a circle has a bound.

1.2 Discussions from Set Theory

Definition. A **set** is a collection of objects, with the following operations:

1. Subsets
2. The **union** of an arbitrary number of sets
3. The **intersection** of an arbitrary number of sets.
4. The **Cartesian product** of sets. For example, if A, B are sets, then $A \times B = \{(a, b)|a \in A, b \in B\}$.
5. Functions between sets, where such a function $f : A \to B$ is a subset $X \subseteq A \times B$ such that $(a_1, b_1), (a_2, b_2) \in X$ if $a_1 = a_2$ then $b_1 = b_2$. You can think of this as saying $f(a) = b$ when $(a, b) \in X$.
6. Relations and Equivalence Relations. An **Equivalence Relation** on a set S is a subset X of $S \times S$ satisfying the following:

   (a) $a \leq a$

   (b) If $a \leq b$ then $b \leq a$

   (c) If $a \leq b$, $b \leq c$, then $a \leq c$.

   Where we write $a \leq b$ if and only if $(a, b) \in X$.

Definition. A set S is finite if there exists a bijection $f : \{1, 2, ...n\} \to S$ for some $n \in \mathbb{N}$. We say S has n elements. If S is not finite, then we say that S is infinite.
Fact. If a set is infinite, then every injection, or "one-to-one" function, is also a surjection.
Example. Given $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$, this map is $1 - 1$ and is not onto. This follows from 1 not being in image of f. This tells us that $\mathbb{N}$ is **not** finite, but is an infinite set.
Definition. We say that a set S is **countable** if there is a bijection $f : \mathbb{N} \to S$.
Fact. The integers $\mathbb{Z} = \{\pm 1, \pm 2, \pm 3, ...\}$ are countable. It is also true that $\mathbb{Q}$ are countable. The more surprising fact is that $\mathbb{R}$ are not countable.

A finite union of finite sets is always finite. Similarly, a finite intersection of a finite number of sets is finite, and the same is true of a finite product of finite sets, and a subset of a finite set is finite. However, a subset of a countable set is not necessarily countable- for example, the subset $\{2, 3\}$ of $\mathbb{N}$ is not countable since there exists no bijection between this set and $\mathbb{N}$. On the other hand, a finite union of countable sets is countable, and the countable product product of countable sets is not countable- and even worse, the finite product of countable sets is not countable.

1.3 The Definition of A Topology

Definition. Let X be a set. A **Topology** on X is a collection of subsets of X, denoted $\mathcal{T}$, satisfying the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. An arbitrary union of elements of $\mathcal{T}$ belongs to $\mathcal{T}$. Another way to say this is that $\mathcal{T}$ is closed under arbitrary unions.
3. A finite intersection of elements of $\mathcal{T}$ belongs to $\mathcal{T}$. Another way to say this is that $\mathcal{T}$ is closed under finite intersections. If $U_1 ... U_n \in \mathcal{T}$, then $\cap_{i=1}^{n} U_i \in \mathcal{T}$

A pair $(X, \mathcal{T})$ is called a topological space. Elements of $\mathcal{T}$ (which are in fact subsets of X) are called **open sets**.

Example. Let X be a set. The **trivial topology** on X, $\mathcal{T}$ =$\{\emptyset, X\}$ is one example of a topology on X. Alternatively, the **discrete topology** on X, $\mathcal{T}$ consists of all subsets of X, $\mathcal{T}$ =$\mathcal{P}(X)$. In this topology, all subsets of X are open.

Example. Another example is the **finite complement topology**, whose definition is $\mathcal{T}$ =$\{A \subseteq X | X - A$ is finite or equals all of X $\}$. This is a topology because:

1. $\emptyset \in \mathcal{T}$ since $X - \emptyset = X$
2. suppose $\{U_\alpha\} \subseteq \mathcal{T}$. Looking at their union, $\bigcup_\alpha U_\alpha \in \mathcal{T}$. We know that $X - U_\alpha$ is finite or equals X for all α from the following:

   $$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha) = \text{ is either finite or all of } X$$

   which tells us that $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
3. Suppose that $U_1, U_2 \in \mathcal{T}$. From this we know that $X - U_1$& $X - U_2$ are either finite or all of X. Now looking at $X - (U_1 \cap U_2) = (X - U_1) \cap (X - U_2)$ which is either finite or all of X. Proceeding similarly, you can show the same is true

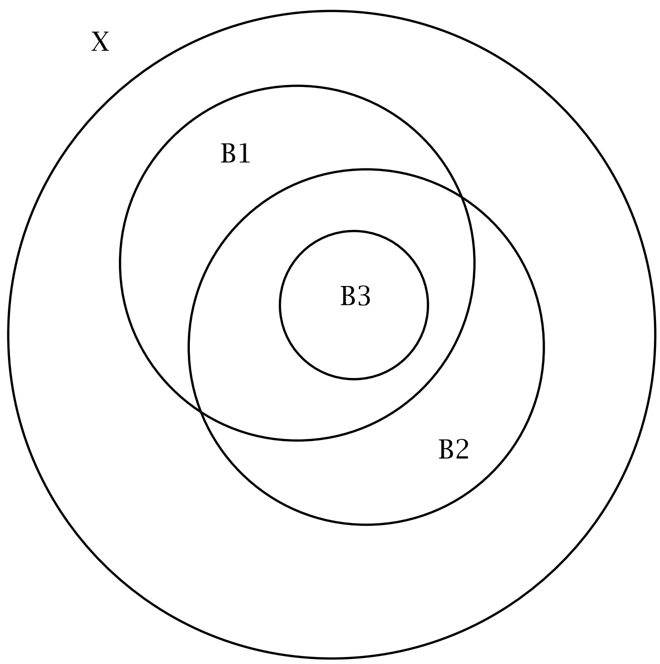
for the intersection of a finite number of elements, and the same is true for finite union.

Definition. Let X be a set. Suppose that you have two different topologies ,$\mathcal{T}, \mathcal{T}'$ on X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subseteq \mathcal{T}$. In this case we say that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$.

Going back to our first two topologies, it is clear that the trivial topology is the coarsest topology, and the discrete topology is the finest topology. Notice that two topologies need not be comparable.

Definition. A **basis** for a topology $\mathcal{T}$ on X is a collection of subsets of X, called $\mathcal{B}$ such that:

1. For all $x \in X$, there is an element $B \in \mathcal{B}$ such that $x \in B$.
2. If you took $B_1, B_2 \in \mathcal{B}$, nand $x \in B_1 \cap B_2$, then there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3$



We'll show a basis for a topology indeed induce a topology.

Example. Let $X = \mathbb{R}$. Let $\mathcal{B} = \{(a, b) | a < b\}$. Let's check the claim that this is a basis for a topology

1. Given some point on the real like $\mathbb{R}$, there is always an interval that contains that point.
2. Give a point contained in two intervals that overlap, there is always a smaller one that exists in between those two contained in their intersection that contains that point.

Example. Given $X = \mathbb{R}^2$, you can let $\mathcal{B}$ be all open disks in the plane. Is this a basis?

1. Given some point, you can construct an open disk around that point.
2. Given a point contained in the intersection of two disks, you can create a smaller disk contained in that intersection that contains that point.

Example. Let $X = \mathbb{R}^2$, and let $\mathcal{B}$ be all open rectangles in the plane. Is this a basis?

1. Any point can be contained in an open rectangle
2. Any point contained in the intersection of two rectangles can be contained in an appropriate smaller rectangle.

The basis we discussed with intervals is called the "standard basis for $\mathbb{R}$".

Claim. If $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}_{\mathcal{B}} = \{..\}$ is a topology on X. This follows from:

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X | \quad \forall x \in U, \exists\, B \in \mathcal{B} \quad s.t. \quad x \in B \subseteq \mathcal{B}\}$$

Proof. Fix a basis $\mathcal{B}$ and let $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

1. First, $\emptyset \in \mathcal{T}$ since there's nothing to check.
2. $X \in \mathcal{T}$ since if you take any $x \in X$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq X$, which follows from the fist condition in the definition of a basis.
3. Given some $\{U_\alpha\} \subseteq \mathcal{T}$ We need to show that $\bigcup_\alpha U_\alpha \in \mathcal{T}$. This follows from noticing that for each α, if $x \in U_\alpha$, then $\exists\, B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha$. Given some $x \in \bigcup_\alpha U_\alpha$, we know that $x \in U_\alpha$ for some α. So, then $x \in B \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha$. This tells us that the B that works for U_α also works for $\cup U_\alpha$.
4. Given $U_1, ...U_n \in \mathcal{T}$, suppose that $n = 2$ and that $x \in U_2 \cap U_2$. Since $\mathcal{T}$ is a basis, we know that open sets B_1, B_2 contained respectively in U_1, U_2 and that there exists a B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. This shows that $U_1 \cap U_2 \in \mathcal{T}$. For $n > 2$, the result follows from induction.

□

We moved in to a discussion of elements of a topology being `open' sets.

Example. In the standard topology on $\mathbb{R}$, a single point is not open- this is true since if $x \in \mathbb{R}$, there does not exist an open interval (a, b) such that $(a, b) \subset \{x\}$.

1.4 Constructing Bases from $\mathcal{T}$

Example. Let X be some set and let $\mathcal{T}$ be the discrete topology. A basis for $\mathcal{T}$ is given by:

$$\mathcal{B} = \{x \in X\}$$

Claim.

1. $\mathcal{B}$ is a basis.
2. The topology that $\mathcal{B}$ generates is the discrete topology.

Proof. For (2), it is enough to show that every set in X is open. Given some subset $U \subseteq X$, and $x \in U$, $x \in \{x\} \subseteq U$ and $\{x\} \in \mathcal{B}$. This is the discrete topology. This follows from allowing U to be any arbitrary set in X.

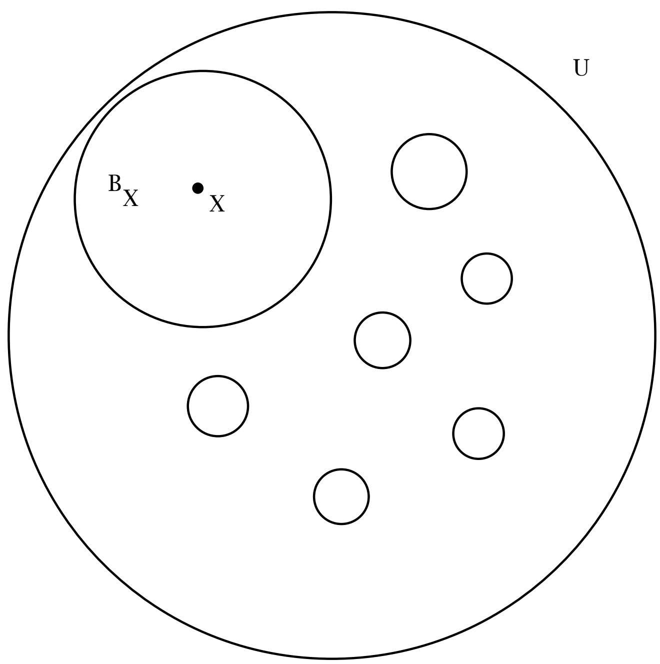
□

More formally,

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X | \forall x \in U, \ \exists B \in \mathcal{B} \quad s.t. x \in B \subseteq U\}$$

is a topology, and we have the claim that $\mathcal{T}_{\mathcal{B}} =$ all unions of elements of $\mathcal{B}$.

Proof. Lets show that $\mathcal{T}_{\mathcal{B}} \subseteq$ all unions of elements of $\mathcal{B}$. So, if $U \in \mathcal{T}_{\mathcal{B}}$, for all $x \in U$, there exits $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then we can just say that $U = \bigcup_{x \in U} B_x$, which follows what we claimed.



Now, in showing that $\mathcal{T}_{\mathcal{B}} \supseteq$ all unions of $\mathcal{B}$. First, $\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}}$, and on the other hand, $\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions, so we know that arbitrary unions of $\mathcal{B}$ are in $\mathcal{T}_{\mathcal{B}}$. □

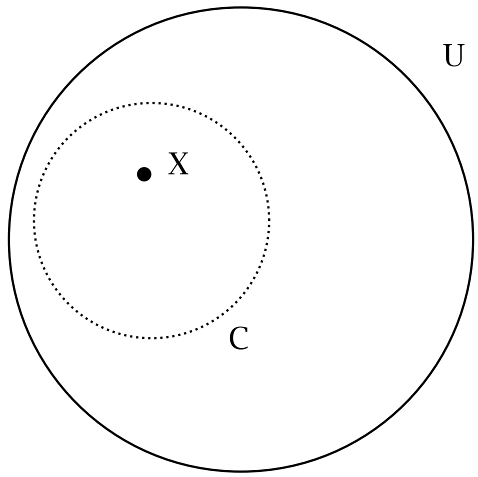
Example. The standard topology on $\mathbb{R}$ has the basis

$$\{(a, b) | a < b\}$$

So, the open sets in $\mathbb{R}$ are precisely the union of open intervals.

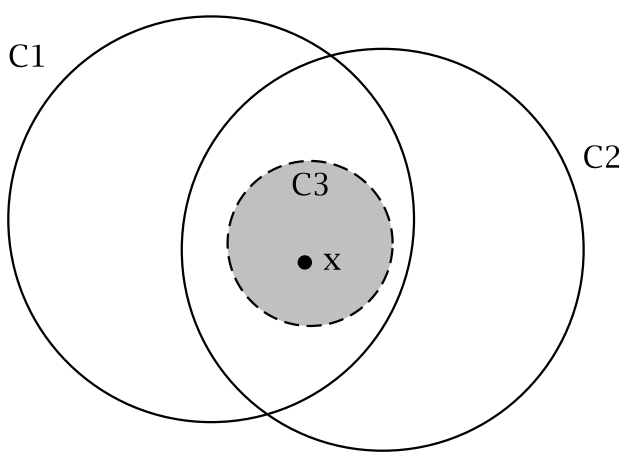
1.5 Constructing $\mathcal{T}$ from a Basis

Let $\mathcal{T}$ be a topology on some set X. Any collection $\mathcal{C}$ of subsets of X satisfying some particular condition is a basis. The condition is as follows: for any $U \in\mathcal{T}$, an $x \in I$, there exits $C \in \mathcal{C}$ such that $x \in C \subseteq U$ is a basis for $\mathcal{T}$.



Why is $\mathcal{C}$ a basis? Well, we need to recognize the following:

1. Given $x \in X$, since $X \in \mathcal{T}$, by our condition we know that there exists $C \in \mathcal{C}$ such that $x \in C \subseteq X$. Thus, every x is in some C.
2. Let $C_1, C_2 \in \mathcal{C}$ and let $x \in C_1 \cap C_2$. We need to find some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$ By letting $C_3 = C_1 \cap C_2$, since $\mathcal{C} \subseteq \mathcal{T}$ and since $\mathcal{T}$ is closed under finite intersection, then $C_3 \in \mathcal{C}$. Thus, $x \in C_3 \subseteq C_1 \cap C_2$. This looks like the following picture:



CHAPTER 2

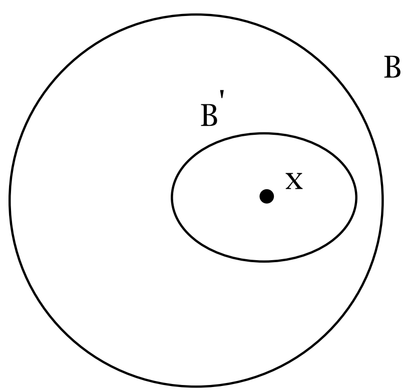
NOTIONS OF TOPOLOGY

2.1 Equality

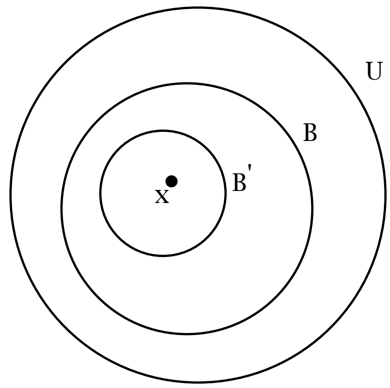
A fair question to ask is `when are two topologies the same?' We have the following theorem:

Theorem 1. *Let $\mathcal{T}$ be generated by $\mathcal{B}$, and let $\mathcal{T}'$ be generated by $\mathcal{B}'$. Then,*

1. *$\mathcal{T}' \supseteq \mathcal{T}$ ($\mathcal{T}'$ is finite, more open sets)*
2. *if and only if for all $B \in \mathcal{B}$ an $x \in B$ there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$.*

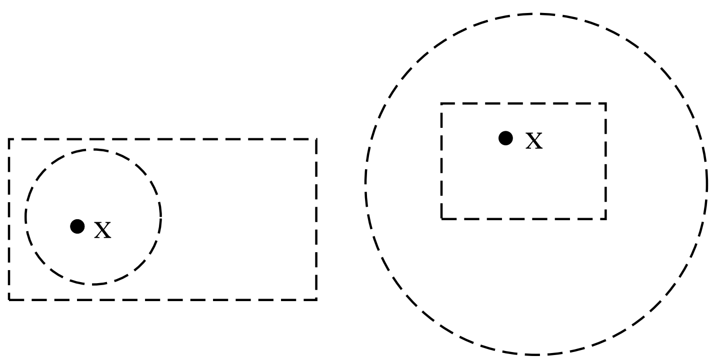


Proof. In showing that $(1) \Rightarrow (2)$, given some $B \in \mathcal{B}$ and $x \in B$, we know that then $B \in \mathcal{T}$ since $\mathcal{B} \subseteq \mathcal{T}$ so $B \in \mathcal{T}'$ since $\mathcal{T}' \supseteq \mathcal{T}$. We also know that $\mathcal{T}'$ is generated by $\mathcal{B}$, which tells us that there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.



Switching to show that $(2) \Rightarrow (1)$, Let $U \in \mathcal{T}$. We want to show that $U \in \mathcal{T}'$. It's enough to show that $\forall x \in U,\ \exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$. Since U is open in $\mathcal{T}$, we know that for all $x \in U$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By (2), we have that $x \in B' \subseteq B \subseteq U$ □

Example. Look at the basis for $\mathbb{R}^2$ given by all open disks. Similarly, we can look at the topology on $\mathbb{R}^2$ given by all open rectangles. Are these basis the same? It turns out, the answer is yes, and we can show this using our theorem. We need to show that the topology from (1) is contained in the topology from (2). This follows from our theorem, by looking at our picture: Similarly, the topology from $(2) \supseteq$



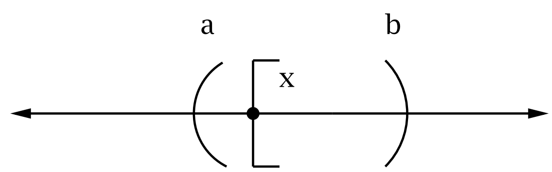
the topology from (1), which can also be seen from our picture. One interesting consequence from this is that the interior of a circle can be represented by open unions of rectangles, and vice versa. Thus these topologies are the same.
Example. We will now define the following new topology on $\mathbb{R}$, given by the following basis:

$$\{[a,b)|a,b \in \mathbb{R}, a < b\}$$

This is a basis, since

1. If $x \in \mathbb{R}, x \in [x, x+1)$.
2. And $x \in [c,b) \subseteq [a,b) \cap [c,d)$.

$\mathcal{B}_l$ is a basis, and generates a topology called the lower limit topology on $\mathbb{R}$, denoted $\mathbb{R}_l$. So, how are $\mathbb{R}_l$ and the standard topology on $\mathbb{R}$ related? Given some $x \in (a, b)$, $x \in [x, b) \subseteq (a, b)$ so (2) holds. By our theorem, $\mathbb{R}_l \supseteq$ the standard topology on $\mathbb{R}$, demonstrated by the picture on the following page.. On the other hand, if



$x \in [a, b), x \neq a$ then $x \in (a, b) \subseteq (a, b]$. But, if $x = a$, there does not exist (c, d) such that $x \in (c, d) \subseteq [a, b)$. Thus, (2) fails, and the standard topology on $\mathbb{R}$ is not finer than the lower limit topology.

From this we have the following Corollary:
Corollary 2. *The interval* $[a, b)$ *cannot be written as an arbitrary union of open intervals.*

Proof. If this were true, than the basis of open intervals (a, b) would be the same as the basis of all $[a, b)$, which we know to not be true. □

Example. The topology $\mathbb{R}_k$ is generated by the basis $\mathcal{B} = \{(a, b)|a, b \in \mathbb{R}, a < b\}$ or $\{(a, b) - K|a, b \in \mathbb{R}, a < b\}$ where $K = \bigcup_{n \geq 1} \frac{1}{n}$. It can be shown that $\mathbb{R}_l \not\supseteq \mathbb{R}_k$ and $\mathbb{R}_k \not\supseteq \mathbb{R}_l$. This won't be worked out in class, but can be done at home.

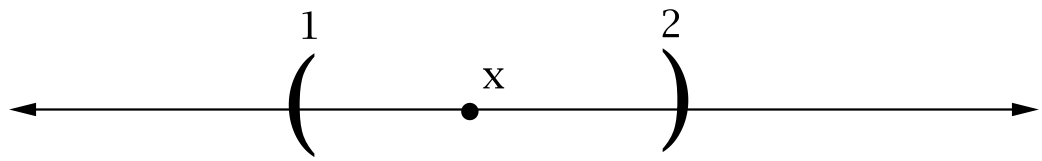
2.2 Subbases

Definition. A **Subbasis** of X is a collection of subsets of X whose union is X.
Example. Let $X = \mathbb{R}$, $\mathcal{C} = \{(-n, n)|n \in \mathbb{N}\}$ is a subbasis since the union $\bigcup_{S \in \mathcal{C}} S = \bigcup_{n \in \mathbb{N}} (-n, n) = \mathbb{R}$. Thus, $\mathcal{C}$ is a subbasis of X. Thanks to this fact, we know that $\mathcal{B}_{\mathcal{C}}$ is a basis for a topology on X.

Proof. We say that $\mathcal{B}_{\mathcal{C}}$ is the set of all finite intersections of elements of $\mathcal{C} = \{c_1 \cap ... \cap c_n | c_1, ... c_n \in \mathcal{C}, n \in \mathbb{N}\}$. We need to now show that each $x \in X$ is contained in some $B \in \mathcal{B}$. To see this, first note that $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{C}}$. Secondly, since , since $\bigcup_{S \in \mathcal{C}} S = X$, we know that for any $x \in X$, $x \in S$ for some $S \in \mathcal{C}$. Thus, $x \in S \in \mathcal{C} \in \mathcal{B}_{\mathcal{C}}$, so $x \in S$ for some $S \in \mathcal{B}_{\mathcal{C}}$.

The second thing to check is if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists some B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. By definition, $B_1 = c_1 \cap ... \cap c_n$ and $B_2 = d_1 \cap ... \cap d_m$ where $c_i, d_j \in \mathcal{C}$. So, let $x \in c_1 \cap ... \cap c_n \cap d_1 \cap ... \cap d_m \subseteq B_1 \cap B_2$, i.e., let $B_3 = B_1 \cap B_2$. □

it turns out that this subbasis is actually a basis. Does this subbasis generate the standard topology on $\mathbb{R}$? It turns out that if $x \in (-n, n)$, there exists (a, b) such



that $x \in (a,b) \subseteq (-n,n)$. But, if $x \in (a,b)$, there may not exist $n \in \mathbb{N}$ such that $x \in (-n,n) \subseteq (a,b)$!

This tells us that the standard topology is finer than the topology generated by this basis, and is not finer than the standard topology (this sounds redundant, but it can happen when two basis are the same). Thus the standard topology is `strictly finer' than this topology.
Definition. A **simply ordered set** is a set X with a binary relation `<' such that :

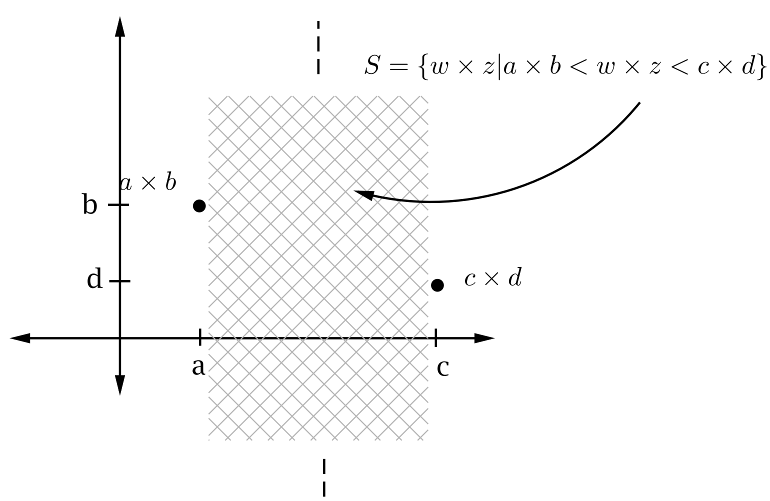
1. Either $x < y$ or $y < x$ For all x, y
2. It is never the case that $x < x$ (it is non-reflexive)
3. If $x < y, y < z, x < z$.

One example is $(\mathbb{R}, <)$, and a non-example is $(\mathbb{R}, \leq)$.
Example. Looking at $\mathbb{R} \times \mathbb{R}$, $a \times b < c \times b$ if $a = c$, or if $b < d$.

For any simply ordered set $(X, <)$ we can define

$(a,b) = \{x \in X | a < x < b\}$ $\quad (a,b] = \{x \in X | a < x < b$ or $x = b\}$ $\quad [a,b] = \{x \in X | a \leq x \leq b\}$

Looking back at our example $\mathbb{R} \times \mathbb{R}$, if we take a point $a \times b$ and take a point $c \times d$, the set $S = \{w \times x | a \times b < w \times z < c \times d\}$, we have:



And we can show that the collection of open intervals in X need not be a basis or a subbasis. For example, $X = \{0,1\}, 0 < 1$ but $(0,1) = \emptyset$. However, if you allow your

intervals to be closed, where $(X, <)$ is a simply ordered set, then $\mathcal{B} = \{(a,b)|a,b \in X\} \cup \{(a,b]|a,b \in X\} \cup \{[a,b)|a,b \in X\}$ is a basis for a topology on X. This basis generates what is called **the order topology**.
Example. We can consider the order topology $[0,1]$ with $<$. The open sets by our theorem are arbitrary unions of intervals of the form $(a,b),(c,d],[e,f)$. Another example would be to take $\mathbb{R}$ using the relation $<$ which has an order topology (since $\mathbb{R}$ is an ordered set). In this case, it turns out that this topology is strictly finer than the standard topology, since the standard topology does not have half-closed intervals as open sets. This follows from noticing that given $a \in [a,b)$, there does not exist some interval (c,d) such that $a \in (c,d) \subseteq [a,b)$. So, the standard topology is not finer than the order topology.
Definition. A **Ray** $(a,\infty) = \{x \in X|a < x\}$. Similarly, we could have defined this to have been $(\infty, a) = \{x \in X|x < a\}$ We also have:

$$[a,\infty) = \{x \in X|x \leq x\} \quad (\infty, a] = \{x \in X|x \leq a\}$$

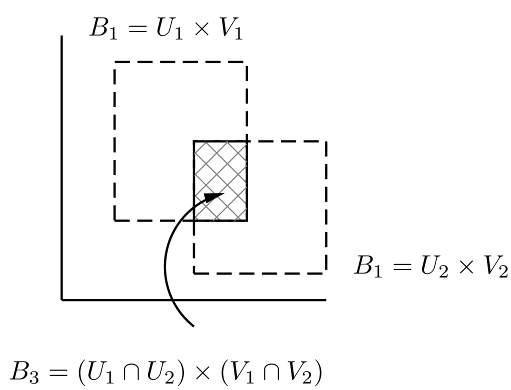
And these are a subbasis. Does this generate the order topology? This is an open question for the class.

2.3 Products of Topological Spaces

Definition. Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be topological spaces. The product topology one the set $X \times Y$ is the topology whose basis is:

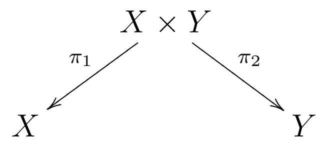
$$\mathcal{B} = \{U \times V \ \mid \ \begin{matrix} U \text{ open in } X \\ V \text{ open in Y} \end{matrix} \ \}$$

The first question to ask is if $x \times b \in X \times Y$. Well, X open in X, Y open in Y, so $a \times b$ is contained in some element on $\mathcal{B}$. Secondly, we can check what the following picture illustrates:



Note however, that $\mathcal{B}$ is not closed under union. This is simply because the union of two rectangles is often not a rectangle. The above picture illustrates that fact plainly.

We have the following maps:



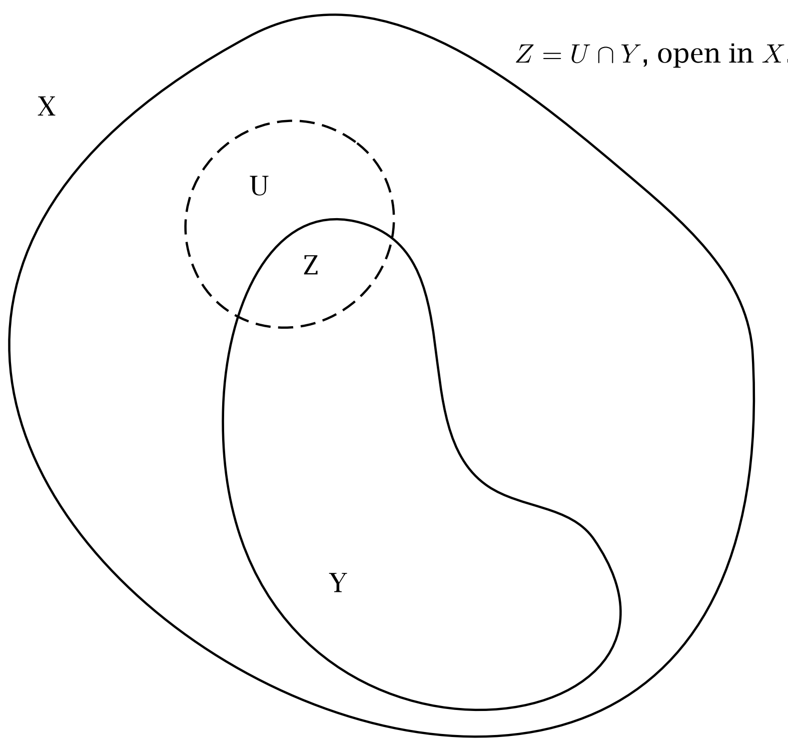
And the set $S = \{\pi_1^{-1}(U) | U$ is open in X$\} \cup \{\pi_2^{-1}(V) | V$ is open in Y $\}$ is a subbasis for the product topology on $X \times Y$. This is because $\pi_1^{-1}(X) \in S$ and $\pi_2^{-1}(X) = X \times Y$, so S is a subbasis. Notice also that

$$\pi_1^{-1}(U)\{(a,b) \in X \times Y | \pi_1(a,b) \in U\}$$

Definition. Let $(X, \mathcal{T})$ be a (topological) space. If Y is a subset of X, then the subspace topology on Y is given by:

$$\mathcal{T}_Y = \{U \cap Y | U \text{ is open in } X\}$$

We have the following picture:



Example. Let $X = \mathbb{R}$ with the standard topology. $Y = [0,1]$. Is $[0, \frac{1}{2}]$ open in Y in the subspace topology? Well, this would have to mean that there is some interval that when we intersect it with Y, we get this set. And we have such an interval:

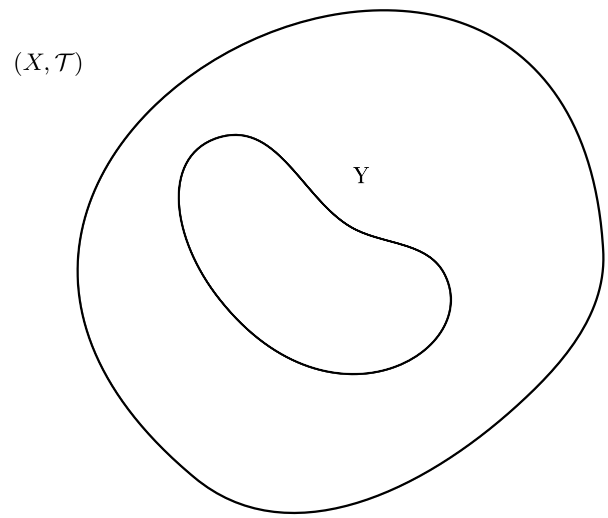
$$[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap Y$$

Notice that if $0 < a < b < 1$, then (a, b) is open in $[0, 1] = Y$, since

$$(a, b) = (a, b) \cap Y$$

Similarly, $(\epsilon, 1]$ is open in Y for all $0 < \epsilon < 1$.

what this essentially says if that if you're given a topological space X with a subspace Y, you can put a topology on Y.



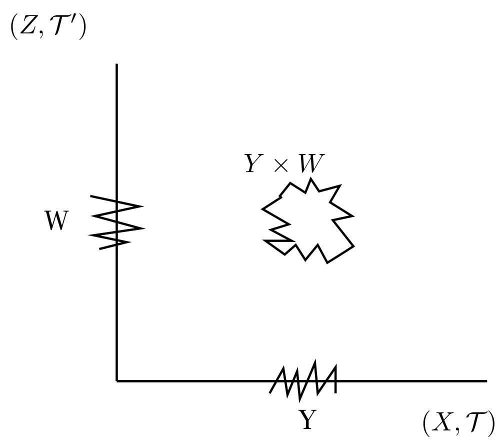
we can use the following interesting formulas to help show this:

$$\emptyset = \emptyset \cap Y$$
$$\bigcup_{\alpha}(U_\alpha \cap Y) = \left(\bigcup_{\alpha} U_\alpha\right) \cap Y$$
$$\bigcap_{i=1}^{n}(U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y$$

And we have the following properties of the subspace topology:

1. If $\mathcal{T}$ is generated by $\mathcal{B}$, then $\mathcal{T}_Y = \{B \cap Y | B \in \mathcal{B}\}$, which is a basis for the subspace topology $\mathcal{T}_Y$.
2. If $V \subseteq Y \subseteq X$, Y is open in X and V is open in Y, then V is V is open in X since $V = U \cap Y$, which are both open in$X$.
3. (See Picture)

Our Picture Can demonstrate what were were interested in talking about. More will follow this in our next class.



2.4 The Subspace Topology, Continued

If $Y \subseteq X$, where you have $(X_{\text{an ordered set}}, \text{order topology})$ Then $(Y, \text{order topology})$ and $(Y, \text{subspace topology})$, which are not the same in general.
Example. Let $Y = [0,1] \cup \{2\}$, and let $X = \mathbb{R}$, an ordered set. Notice that

$$\{2\} = (\frac{3}{2}, \frac{5}{2}) \cap Y$$

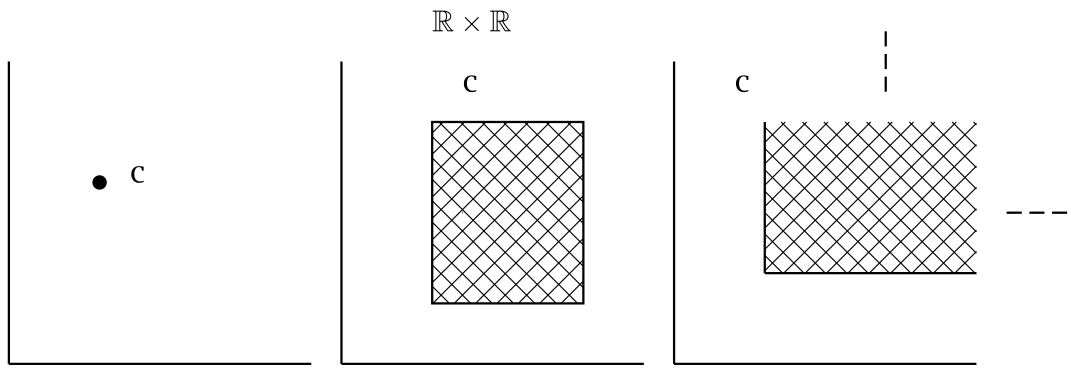
So, $\{2\}$ is open in Y, the subspace topology, and in the order topology. However, if you removed the element $\{1\}$ from Y, then (this example is in the book on page 83-91) somewhere on page 85 i think.
Definition. Let $X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is called closed if and only if $X - C \in \mathcal{T}$, i.e., $X - C$ is open.
Example. Let $X = \mathbb{R}$ be the standard topology. Given $[a.b]$,

$$\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$$

notice that both $(-\infty, a)$ and (b, ∞) are both open sets, so thus their union is also open. This tells us that $[a,b]$ is closed in $\mathbb{R}$.
Example. Take $\mathbb{R}$, and look at $C = \{11\}$. C is closed, since its compliment $\mathbb{R} - C$ is open under the same argument as above.
Example. Take any set X with the discrete topology. Unusually, every subset of X is closed, since the complements are all open.
Example. We also have examples of sets that are both closed and open. For example, take

$$Y \subseteq \mathbb{R}, \quad Y = [2,3)$$

Its not very hard to show that $[2,3)$ is not open, and that $(-\infty, 2) \cup [3, \infty)$ is also not open. This tells us that $[2,3)$ is not closed in $\mathbb{R}$.
Example. Given $\mathbb{R} \times \mathbb{R}$ with the product topology, closed sets look like the following:



Example. Now look at $X = [0,1] \cup (2,3)$ as a subspace of $\mathbb{R}$. Since

$$[0,1] = (-1, \frac{3}{2}) \cap X$$

we know that $[0,1]$ is open in X and is closed in $\mathbb{R}$, Alternatively,

$$(2,3) = (2,3) \cap X$$

So $(2,3)$ is open in X and is open in $\mathbb{R}$. However, notice that

$$[0,1] \text{ is closed in X since : } X - [0,1] = (2,3)$$

is open in X. Similarly, $(2,3)$ must be closed in X since

$$X - (2,3) = [0,1]$$

which is open in X.

We have the following Properties of closed sets, where $(X, \mathcal{T})$:

1. X and $\emptyset$ are closed.
2. An arbitrary intersection of closed sets is closed.
3. A finite union of closed sets is closed.

A few of these follow from DeMorgan's laws,

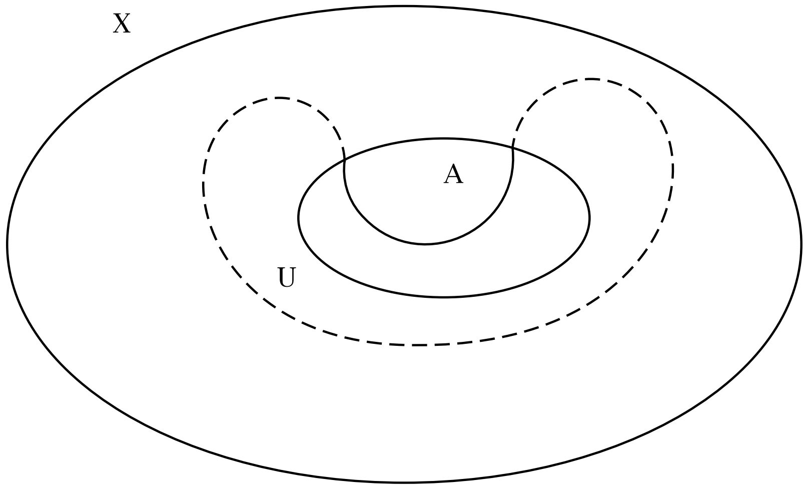
$$\bigcup_{i=1}^{n}(X - U_i) = X = \bigcap_{i=1}^{n} U_i \qquad \bigcap_{\alpha}(X - U_\alpha) = X - \bigcup_{\alpha} U_\alpha$$

(3) uses the one on the left, and the one on the right is used for (2).

Definition. Given some topological space X and a subspace Y of X, and $A \subseteq Y$, we say that A is closed in Y if and only if A is a closed subset of Y in the subspace topology.

Claim. Suppose that $A \subseteq Y \subseteq X$ where Y is a subspace, X is a space. Then, A is closed in Y if and only if $A = C \cap Y$ for some C closed in X.

Proof. A is closed in Y if and only if we can write A as $A = C \cap Y$ for some C closed in X. And well, $A = C \cap Y$ for some C closed in X if and only if $A = (X - U) \cap Y$ for some U open in X if and only if $Y - A = Y \cap Y$ for some U open in X if and only if $Y - A$ is open if and only if A is closed. This looks like the following picture: □



If $A \subseteq X$, where X is a space, we have define the **Interior** and the **Closure** of A.
Definition.

$$Int(A) = \bigcup_{\text{U open } U \subseteq A} \qquad Closure(A) = \overline{A} = \bigcap_{\text{closed } A \subseteq C} C$$

Notice that

$$Int(A) \subseteq A \subseteq \overline{A}$$

Also, if A is open, $Int(A) = A$. Similarly, if A is closed, $A = Closure(A)$. Also notice that the interior of A is always open, and the closure of A is always closed.
Example. Let $A = [0,1] \subseteq \mathbb{R}$. A is closed, so $\overline{A} = A$, and $Int(A) = (0,1)$.
Example. Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \subseteq \mathbb{R}$. A is not closed since $\mathbb{R} - A$ is not open: this is true since for $0 \in \mathbb{R} - A$ and any a, b such that $0 \in (a, b)$, we have $(a, b) \not\subseteq \mathbb{R} - A$. Notice that A is a countable union of closed sets and not closed. Also, notice that if you took

$$\bigcup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

we see that we have a countable intersection of open sets that is not open. Similarly,

$$\bigcup_{n=1}^{\infty}(1, 2 + \frac{1}{n}) = (1, 2]$$

is neither closed or open.
Lemma 3. *Given Y, a subspace of X< the closure of A in Y4 is equal to the closure of A in X.*

Proof. We know that

$$\bigcap_{\text{C closed in Y, } A \subseteq C} C = \bigcap_{\text{D closed in X, } D \subseteq Y \cap Y} (D \cap Y) = \left(\bigcup_{A \subseteq D} D \right) \bigcap Y = \text{closure of } A \text{ in } X \bigcap Y$$

□

Definition. We say that $A, B \subseteq X$ intersect if $A \cap B \neq \emptyset$. We also say that Given a topological space X, and $x \in X$, a **neighborhood** is an open set U in X such that $x \in U$.
Theorem 4. *Let $A \subseteq X$, $x \in \overline{A}$ if and only if every neighborhood of x intersects A.*

Proof. It is enough to show that $x \notin \overline{A}$ if and only if there exists a neighborhood U of x such that $U \cap A = \emptyset$. Firstly, notice that if $x \notin \overline{A}$, then $x \in X - \overline{A}$, and $X - \overline{A}$ is open. Then, let $U = X - \overline{A}$, then $x \in U$ and we can see that

$$U \bigcap A = (X - \overline{A}) \bigcap A$$

Working in the other direction, If there exists a neighborhood U of x such that $U \cap A = \emptyset$, then $X - U$ is closed since U is open and

$$X - U \supseteq A, \qquad \text{so } X - U \supseteq \overline{A} = \bigcap_{\text{C closed, } C \supseteq A} C$$

and since $x \in I$, $x \notin X - U$ so $Xx \notin X - U \supseteq \overline{A}$ so $x \notin \overline{A}$. □

Roughly, if the topology for X has the basis $\mathcal{B}$, $x \in \overline{A}$ if and only if every basis elements $B \in \mathcal{B}$ such that $x \in B$ satisfied $B \cap A \neq \emptyset$.

2.5 Limit Points

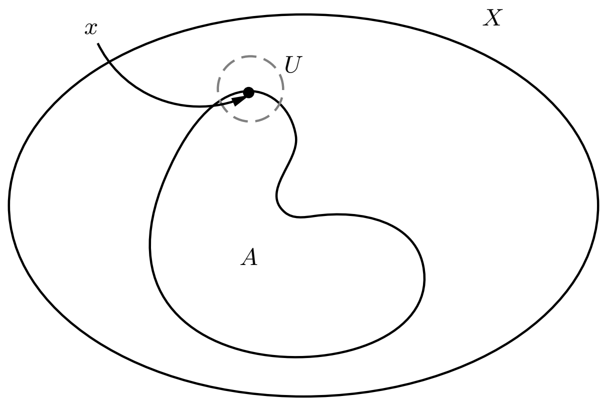
Let $A \subseteq X$. A point $x \in X$ is a **limit point** of A if every neighborhood U of x intersects A in some point other than A.
Definition. If $A \subseteq X$, then

$$A' = \{x \in X | \text{ x is a limit point of } A\}$$

is called the set of limit points of A in X.
Example. Let $X = \mathbb{R}$, and let $A = (0, 1)$. It is clear that $\frac{1}{2}$ is not a limit point, but $0, 1$ are limit points of A in X.

$$A' = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$$

Example. Let $X = \mathbb{R}$ and let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ We can see that $\frac{1}{n}$ is not a limit point, 0 is a limit point, so we say that $A' = \{0\}$.
Claim. $\overline{A} = A \bigcup A'$.



Proof. We know that $A \subseteq \overline{A}$, so we must show that $\overline{A} \supseteq A'$. Let $x \in A'$: then every neighborhood U of x intersects A in a point other than X, so every neighborhood U of X intersects A, so $x \in \overline{A}$ by our last theorem. It remains to show that $\overline{A} \subseteq A \bigcup A'$. So, let $x \in \overline{A}$. If $x \in A$, we're done, so assume that $x \notin A$. We must show that $x \in A'$. Since $x \in \overline{A}$, every neighborhood U of x intersects A, i.e., there exists a $y \in U \cap A$, and $x \in U$. But, since $x \notin A$, we know that $y \neq x$. This tells us that $x \in A'$. □

Corollary 5. *A is closed if and only if $A' \subseteq A$.*

Proof.

$$\overline{A} = A \cup A6\prime$$

If $A' \subseteq A$ then $A' = A$ so A is closed. If A is closed, $A = \overline{A}$ so $A = \overline{A} = A' \cup A$ so $A' \subseteq A$. Sets are closed if they contain their limit points. □

2.6 Hausdorff Spaces

Definition. A space X is Hausdorff if for all $x, y \in X$ there exists neighborhoods U of x and V of y such that $U \cap V = \emptyset$.
Example. $\mathbb{R}^n$ is Hausdorff for all n. A non-example would be $\mathbb{R}$ under the finite-compliment topology.
Definition. A sequence of points in a space $x_n \in X, n \in \mathbb{N}$ is said to converge to x if and only if for every neighborhood U of x, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.
Claim. If X is Hausdorff and $x_n \to x$ and $x_n \to y$, then $x = y$.

Proof. Suppose that $x \neq y$. Then, there exist neighborhoods U of x and V of y such that $U \cap B = \emptyset$. This follows from X being Hausdorff. In particular, if $x_n \to x$, then for n sufficiently large($n \geq N$), x_n is in U. However, since $U \cap V = \emptyset$, so $n \geq N$, $x_n \notin V$. This tells us that x_n does not converge to y, and this is a contradiction. □

Example. Let X have the indiscreet topology, which means $\mathcal{T} = \{\emptyset, X\}$. In this space, every sequence converges to every point. Also notice that this space is highly non-Hausdorff.

We have the following properties:

1. A subspace of a Hausdorff space is Hausdorff.
2. A product of two Hausdorff spaces is Hausdorff

2.7 Continuity

Definition. Let X, Y be spaces. A function $f : X \to Y$ is called **continuous** if and only if the inverse image of an open set is open. More formally, if V is open in Y, then $f^{-1}(V)$ is open in X.

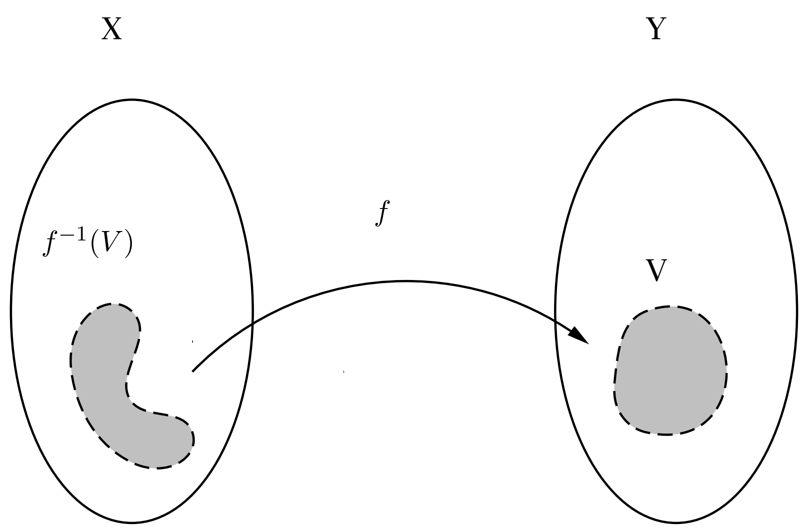


Figure 2.1: The Picture for Continuity

It turns out that this actually agrees with the usual notion of continuity. Recall that if you have a function

$$f : \mathbb{R} \to \mathbb{R}$$

is called $\epsilon - \delta$ continuous if for all $\epsilon > 0$ there exists some $\delta > 0$ such that if

$$|x - x_0| < \delta \qquad \text{then} \qquad |f(x) - f(x_0)| < \epsilon$$

The condition in this definition is true for all open sets V if it is true for all basis elements. This follows from the fact from set theory that the inverse image of a union (or an intersection, for that matter) is the union of the inverse images (or the intersections). We will show that $\epsilon - \delta$ continuity at all points implies topological continuity. To show this, let V be open in $\mathbb{R}^n$, and we'll try to show that $f^{-1}(V)$ is open.

Since V is open, we can chose for each $y \in V$ an interval $I_y \subseteq V$ and with $y \in I_y$. This interval I_x has some radius ϵ. Let $x \in f^{-1}(y)$, so $x \in f^{-1}(V)$, and since f is ϵ-δ continuous at x, we can find $\delta > 0$ so that the interval J_x containing x and or radius δ satisfies the condition that $f(J_x)$ is contained in I_y. This then tells us that $J_x \subseteq f^{-1}(I_y)$. We should notice the following:

$$f^{-1}(V) = f^{-1}\left(\bigcup_{y\in V} I_y\right) = \bigcup_{y\in Y} f^{-1}(I_y) \supseteq \bigcup_{y\in V} J_x$$

also, for every $x \in f^{-1}(V)$, $x \in J_x$ so $x \in \bigcup_{x\in f^{-1}(V)} J_x$ so $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} J_X$ and since J_x are open intervals, we can see that $f^{-1}(V)$ is open.

Fact. The definition of continuity agrees with those you're used to for

$$f:\mathbb{R}\to\mathbb{R}, \quad f:\mathbb{R}^n\to\mathbb{R}^m, \quad \text{etc.}$$

Example. Let X, Y be spaces. Suppose that $f : X \to Y$ is a constant function. In other words, $f(x) = y_0 \in Y$ for all $x \in X$. Let V be open in Y. We now have two cases, either $y \in V$ or it isn't. If $y \in V$, then $f^{-1}(y_0) = X$, so $f^{-1}(V) = X$ which is open in X. Now if $y_0 \notin V$, then $f^{-1}(V) = \emptyset$, which is trivially open in X. This shows that f is continuous.

Example. Let X be a space. Every space X has the canonical identity function, $id : X \to X$ that assigns to every $x \in X$ the element $x \in X$. Thus, $f^{-1}(V) = V$, and if V is open, it is clear that $f^{-1}(V)$ is open. So, the identity function is continuous.

Example. Let X, Y be topological spaces where X has the discrete topology. Any function $f : X \to Y$ must be continuous, since $f^{-1}(V)$ corresponds to some set in X, which by definition is open. Thus, any function from a topological space with the discrete topology to another topological space is continuous A similar situation occurs when you take a function from any topological space to a space with the indiscreet topology.

Theorem 6. *Let $f : X \to Y$. The following are equivelant:*

1. *f is continuous*
2. *If $A \subseteq X$, then $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *If C is closed in Y, then $f^{-1}(C)$ is closed in X.*
4. *For every $x \in X$ and V open in Y such that $f(x) \in V$, there is a U open in X such that $x \in I$ and $f(U) \subseteq V$.*

Proof.

$(1) \Rightarrow (2)$ Let $x \in \overline{A}$, we want to show that $f(x) \in \overline{f(A)}$. Let V be open in Y such that V contains $f(x)$. Our goal is to show that V intersects $f(A)$. Recall that $y \in \overline{B}$

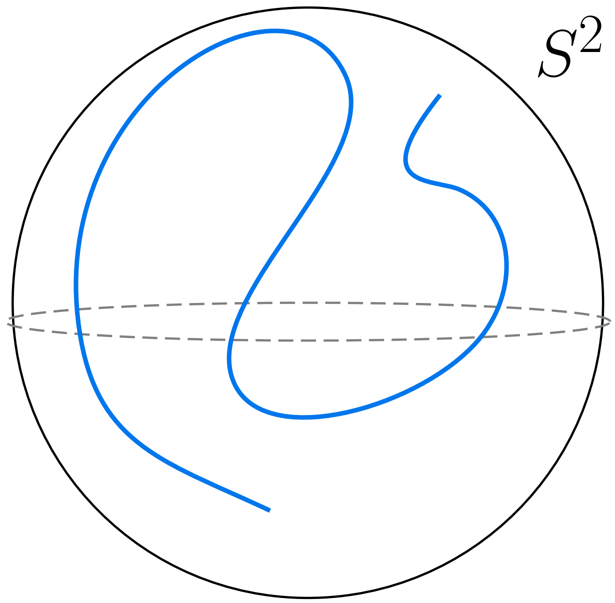


Figure 2.2: $f : \mathbb{R} \to S^2 \subseteq \mathbb{R}^3$ is an example of a continuous path from $\mathbb{R}$ to the 2-Sphere

if and only if every neighborhood of Y intersects B. By (1), $f^{-1}(V)$ is open in X. Since $x \in f^{-1}(V)$, and $x \in \overline{A}$, we know that $F^{-1}(V)$ intersects A. So for example, if $y \in f^{-1}(V) \cap A$, we know that $f(y) \in V \cap f(A)$. This proves our goal.

$(2) \Rightarrow (3)$ Let C be closed in Y. It is enough to show that $f^{-1}(C) = \overline{f^{-1}(C)}$. So it suffices to show that $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$. We have the following fact:

$$f(f^{-1}(C)) \subseteq C$$

taking the closures of both sides and using (2), we have:

$$f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))} \subseteq \overline{C} = C$$

this tells us that

$$\overline{f^{-1}(C)} \subseteq f^{-1}(C)$$

So we're done

$(3) \Rightarrow (1)$ Let V be open in Y. Then, $C = Y - V$ is closed in Y. So, by (3), $f^{-1}(C) = f^{-1}(Y) - f^{-1}(V) = x - f^{-1}(V)$ is closed in X. So, $f^{-1}(V)$ is open.

$(1) \Rightarrow (4)$ Given $x \in X$, V open in Y, with $f(x) \in V$, we must find U open in X such that $f(U) \subseteq V$. Let $U = f^{-1}(V)$, which is open by (1), and $f(U) = f(f^{-1}(V)) \subseteq V$.

$(4) \Rightarrow (1)$ Let V be open in Y. Show that $f^{-1}(V)$ is open in X. From (4), for each $x \in f^{-1}(V)$ there is an open set U_x with $x \in U_x$ such that $f(U_x) \subseteq V$. Notice that if U is open, it is a union of open sets. And, $f(U_x) \subseteq V$ for all x so $U_x \subseteq f^{-1}(V)$ for all x, so

$$U = \bigcup_{x \in f^{-1}(V)} \subseteq f^{-1}$$

But $f^{-1}(V) \subseteq U$ since if $x \in f^{-1}(v)$ then $x \in U_x \subseteq U$. So, $U = f^{-1}(V)$ is open.

□

Definition. Let X and Y be spaces. A function $f : X \to Y$ is called a homeomorphism if and only if the following are fulfilled:

1. f is a bijection
2. f is continuous
3. f^{-1} is continuous

We say X and Y are **homeomorphic** if there exists some function f from $X \to Y$ that satisfies these conditions.
Example. (a, b) is always homeomorphic to (c, d)
Corollary 7. *Let L be a linear homomorphism from (a, b) to (c, d). It turns out that any two open intervals are homeomorphic. It also turns out that any open interval is homeomorphic to $\mathbb{R}$. It turns out that homeomorphisms are composable.*
Definition. And **imbedding**[1] is an injection $(1 - 1)$ $f : X \to Y$ such that

1. f is continuous
2. $f : X \to Im(f) \subseteq Y$ is therefore a bijection, and we require that this function $f : X \to Im(f)$ is a homeomorphism.

So (1) holds and $f^{-1} : Im(f) \to X$ is continous.
Example. Show that $f(t) = (t, sin(t))$ (so naturally, $f : \mathbb{R} \to \mathbb{R}$) is an embedding.

If we have:

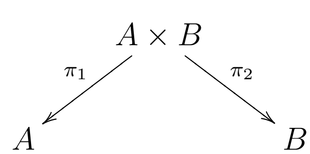
$$f : X \to A$$
$$g : X \to B$$

then

$$(f \times g) : X \to A \times B$$
$$(f \times g)(x) = (f(x), g(x))$$

Claim. $f \times g$ is continuous if and only if f and g are both continuous.

Proof.



[1] `embedding' and `imbedding' will be used interchangeably, since Professor Wilson uses both, and Wikipedia tells me they are equivalent.

Where $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$. Recall that π_1, π_2 are continuous. Notice that

$$\pi_1 \circ (f \times g)(a,b) = f(a)$$

so,

$$f = \pi_1 \circ (f \times g)$$

thus, if $f \times g$ is continuous, then so is f. Similarly, $g = \pi_2 \circ (f \times g)$ so g is continuous. In the other direction, suppose that f, g are continuous. Let $U \times V$ be a basis element for $A \times B$: it is enough to show that $(f \times g)^{-1}(U \times V)$ is open. We can see that:

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \subset X$$

and since $f^{-1}(U)$ and $g^{-1}(V)$ are open, we know that $(f \times g)^{-1}(U \times V)$ must be open, since it is the intersection of two open sets. □

The `bad news' is that there is no nice condition for checking when a function $f : X \times Y \to Z$ is continuous.
Example. Suppose you have

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & x.y \neq 0 \\ 0 & x = y = 0 \end{cases}$$

Let X_α be a collection of sets, where $\alpha \in J$, where J is some arbitrary set called the **index set**.
Definition.

$$\prod_{\alpha \in J} X_\alpha := \{f : J \to \bigcup X_\alpha \ \mid \ f(\alpha) \in X_\alpha\}$$

Example. Suppose that $J = \{1, 2\}$ and X_1, X_2 are sets, then

$$X_1 \times X_2 = \{f : \{1,2\} \to X_1 \cup X_2 \ \mid \ f_1 \in X_1, f(2) \in X_2\}$$

so, you can think of (a, b) as being shorthand for $f(1) = a, f(2) = b$.
Example. Let $J = \mathbb{N}$. $X_1, X_2, ...$ are sets.

$$\prod X_i = \{f : \mathbb{N} \to \bigcup X_i \ \mid \ f(i) \in X_i\}$$

similarly as the above example, you can think of $(a_1, a_2, ...)$ as being shorthand for

$$f(i) = a_i \quad \text{for all } i$$

There are two natural ways to put a topology on $\prod_{\alpha \in J} X_\alpha$:

1. **The Box Topology**

   The idea is to take the topology generated by the basis consisting of the collection:
   $$\mathcal{B} = \{\prod_{\alpha \in J} U_\alpha \ | \ U_\alpha \text{ is open in } X_\alpha\}$$
   it isn't very difficult to check that this is a basis, and is roughly the same as the proof for the product of two spaces.

2. **The Product Topology**

   This topology is the topology generated by the following sub-basis[2]:
   $$S = \{\pi_\beta^{-1}(U_\beta) \ | \ U_\beta \text{ is open in } X_\beta\}$$
   notice that there exist functions:
   $$\pi_\beta(\prod_{\alpha \in J} X_\alpha) \to X_\beta$$
   So, suppose you fix some β. What is
   $$\pi_\beta^{-1}(X_\beta)?$$
   it turns out,
   $$\pi_\beta^{-1}(X_\beta) = \prod_{\alpha \in J} X_\alpha$$
   So, S is a sub-basis. The basis generated by S is given by all finite intersections of elements of S:
   $$\mathcal{B}_S = \{\pi_{\beta_1}^{-1}(U_{\beta_1}) \bigcap ... \bigcap \pi_{\beta_k}^{-1}(U_{\beta_k}) | \beta_i \in J, \ \ U_{\beta_i} \text{ is open in } X_{\beta_i}\}$$

Example. Let $J = \mathbb{N}$. Calculating the following:
$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \bigcap ... \bigcap \pi{-1}_{\beta_k}(U_{\beta_k}) \subseteq X_1 \times X_2 \times ... \times X_k \times ...$$
$$= (X_1 \times X_2 \times ... U_{\beta_1} \times ... \times X_j \times ..) \bigcap (X_1 \times X_2 \times ... \times U_{\beta_2} \times ... \times X_j \times ...) \bigcap ... \bigcap (X_1 \times X_2 \times ... \times U_{\beta_k} \times ...)$$
where each U_{β_i} is in the β_i^{th} spot,
$$= X_1 \times X_2 \times ... \times U_{\beta_1} \times .. \times X ... \times X \times U_{\beta_k} \times ...$$
So, an element of $\mathcal{B}$ is a subset
$$\prod_{\alpha \in J} Y_\alpha$$
where $Y_\alpha = X_\alpha$ except for finitely many $\alpha \in J$, and for these α, Y_α can be any open set $U_\alpha \subseteq X_\alpha$.

[2]Recall that a subbasis is a collection of subsets whose union is the entire space.

The box topology generates a finer topology than the product topology, and if you have a finite set for J, then notice that the restriction on the product topology is sort of waived, and the topologies are the same.

Both the box and product topology on $\prod_{\alpha \in J} X_\alpha$ have the following properties:

1. If A_α is a subspace of X_α for all α then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$
2. If X_α are Haudsdorff, for all $\alpha \in K$, then $\prod_{\alpha \in J} X_\alpha$ are Hausdorff.
3. If $A_\alpha \subseteq X_\alpha$, then
$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$$

Claim. Let $f : A \to \prod_{\alpha \in J} X_\alpha$, $f(a) = \prod_{\alpha \in J} (f_\alpha(a))$ Our claim is now that giving $\prod_{\alpha \in J}$ the product topology, then f is continuous if and only if f_α is continuous for all $\alpha \in J$.
Remark. It turns out that this is actually false for the box topology, and we have many examples of this. One such examples would be:

$$f : \mathbb{R} \to \prod_{n \in \mathbb{N}} \mathbb{R} \qquad f(x) = (x, x, x, x, ...)$$

Consider the following:

$$U = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times .. = \prod_{n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) \subseteq \prod_{n \in \mathbb{N}} \mathbb{R}$$

Let us calculate f^{-1} of U

$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\} = \{x \in \mathbb{R} \mid (x, x, x, ...) \text{ is contained in } (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times ..\}$$

and it turns out, that this set is $\{0\}$. This is not an open set in $\mathbb{R}$, since it does not contain an interval and is nonempty. Thus, f is not continuous.

Proof. If f is continuous, then
$$f_\beta = \pi_\beta \circ f$$
where
$$\pi_\beta : \prod_{\alpha\ inJ} X_\alpha \to X_\beta \qquad \pi_\beta(\{X_\alpha\}) = X_\beta \quad \text{if } U \text{ is open in } X_\beta$$

Notice that π_β is continuous if U is open in X_β . Then $\pi_\beta^{-1}(U)$ is open in $\prod_{\alpha \in J} X_\alpha$ by definition. Now, if f is continuous, then $f_\beta = \pi_\beta \circ f$ is continuous. Suppose that f_α is continuous for all α. We want to show that

$$f : A \to \prod_{\alpha \in J} X_\alpha \quad \text{is continuous}$$

It is enough to show that f^{-1} of a subbasis element. We know that the elements of the subbasis are the collection:

$$\{\pi_\beta^{-1}(U) \ | \ U \text{ is open in } X\}$$

then, looking at:

$$f^{-1}(\pi_\beta^{-1}(U)) = f_\beta^{-1}(U)$$

which is open by assumption, since we assumed f_β is continuous. □

2.8 Metric Spaces

Definition. A **metric** (or "distance function") on a set X is a function:

$$d : X \times X \to \mathbb{R}$$

such that:

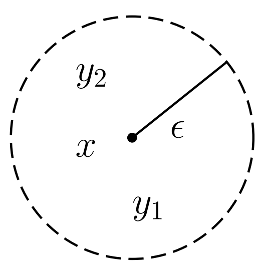
1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z. This is the **triangle inequality**.

Fact.

Definition. If X has metric d, then the "ϵ-ball" centered at some $x \in X$ is by definition:

$$B(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}$$

This looks sort of like the following, where y_1, y_2 are in the ϵ-ball centered around x:



Given some X with a metric d, the set of all ϵ-balls centered at any point form a basis for a topology.

Proof. Given $x \in X$, $x \in B_\alpha(x, .17)$, since $x(x, x) = 0$. Now, given some $z \in B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$ it can be shown that there exists a new ball $B(x, \beta)$ such that it is properly contained in $B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$ □

Example. $\mathbb{R}$ is an example of a metric space, where

$$d(x, y) = |x - y|$$

is its metric. The balls

$$B_\epsilon = \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$$

and this is exactly:

$$= \{y \in \mathbb{R} \mid -\epsilon < x - y < \epsilon\}$$

$$= \{y \in \mathbb{R} \mid x - \epsilon < y < x + \epsilon\}$$

In other words, the ϵ-balls in this space are open intervals.
Definition. A space X is **metrizable** if there exists a metric d on X such that the induced metric topology is the same as the given topology.

Notice that $\mathbb{R}$ with the standard topology is metrizable.

Let

$$\overline{d}(x, y) = \begin{cases} d(x, y) \text{ if } d(x, y) < 1 \\ L \text{ if } d(x, y) \geq 1 \end{cases}$$

so,

$$B_{\overline{d}} = \{y \in X \mid \overline{d}(x, y) < 2\}$$

$$= \{y \in X \mid min\{d(x, y), 1\} < 2\} = \{y \in X\} = X$$

so the ϵ-balls for $\overline{d}$ are either ϵ-balls for d (if $\epsilon \leq 1$) or all of X.

If X has two metrics, d_1, d_2, then the topology for d_1 is finer than the topology for d_2 if and only if for every $x \in X$, $\epsilon > 0$, there is a $\delta > 0$ so that

$$x \in B_{d_1}(x, \delta) \subseteq B_{d_2}(x, \epsilon)$$

So, $\overline{d}$ and d generate the same topology since this condition holds in showing that the topology of $\overline{d}$ is finer than the topology of d, and vice-versa.
Example. On $\mathbb{R}^n$, the metric is:

$$d(x, y) = \text{max}_{1 \leq i \leq 1}\{|x_i - y_i\}$$

this is called the **square metric**. Notice that when $n = 1$, the square metric is just the Euclidean metric. When $n = 2$, we have the following picture:

$$B(X, 1) = \{y \in \mathbb{R}^2 \mid max\{|x_1 - y_1|, |x_2 - y_2| < 1\}$$

and the following picture (on the next page) for when $n = 3$:
Example. Let $\alpha \in \mathbb{R}, \alpha > 0$.

$$d_\alpha = \left(\sum |x_1 - y_i|^\alpha\right)^{1/\alpha}$$

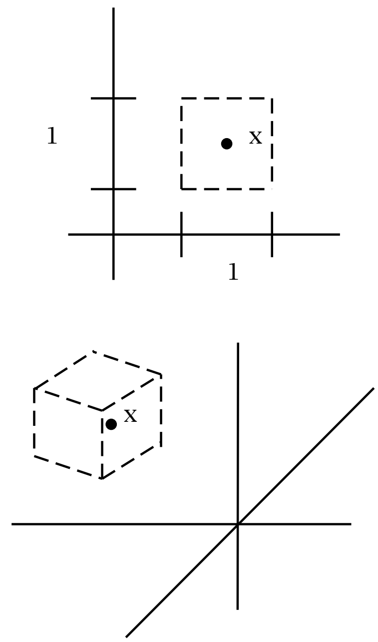


Figure 2.3: The picture for when $n = 3$ of the square metric. The point x is at the center of the box.

we can check this is a metric. Notice that d_2, where $\alpha = 2$, is the Euclidean metric. When $\alpha = 1$, we see that

$$d(x, y) = \sum_{x=1}^{n} |x_i - y_i|$$

and when $n = 2$, we have the following picture:

we can put these in a family, can draw them all when $n = 2$ as alpha ranges, and you eventually get a square with rounded edges as $alpha$ gets really large, like the following: The limit as $\alpha \to \infty$, the ball $B_\alpha(0, 1)$ becomes the unit ball for the square metric. The claim is that the vector $(1^\alpha + 1^\alpha)^{1/\alpha} = 2^{1/\alpha} \to 1$ as $\alpha \to \infty$.

While this may seem confusing, it turns out that these all actually induce the same topology on $\mathbb{R}^n$.

Proof. Recall that the topology induced by d' is finer than the topology induced by d if and only if for all $B_d(x, \epsilon)$ and $y \in B_d(x, \epsilon)$ there exists some ϵ' such that

$$y \in B_{d'}(y, \epsilon') \subseteq B_d(x, \epsilon)$$

Let us first show that the square is finer than the Euclidean metric. WE can always pick $\epsilon' = \epsilon - d(x, y)$, and allows our square to be inside of any euclidean ϵ-ball. Alternatively, given some ϵ for a ϵ-ball in the square metric, we can let

$$\epsilon' = \epsilon - d'(d, y)$$

□

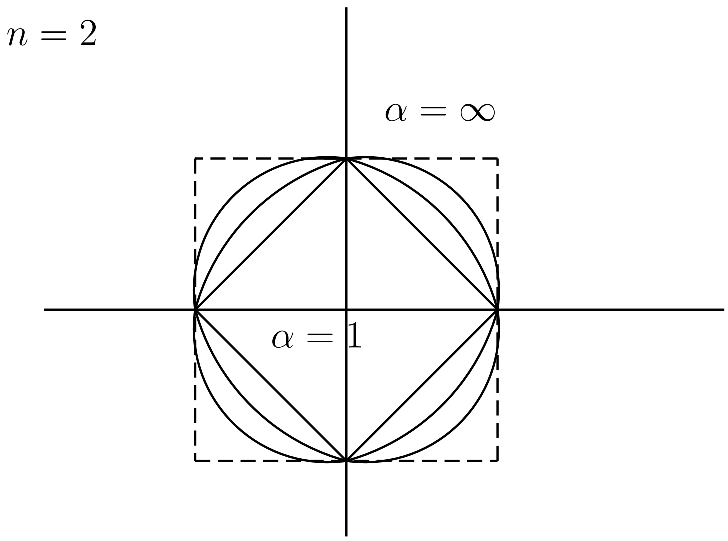


Figure 2.4: The following picture is an illustration of the ϵ-balls for the square metric when $n = 2$ and α is getting very large. Starting with $\alpha = 1$, we have a rotated square, and as α gets larger, the square eventually becomes what we call the `unit ϵ-ball for the square metric'.

Fact. The square metric and the Euclidean metric induce the same topology on $\mathbb{R}^n$ as the product topology. the idea follows from given some ϵ-ball in the square metric, first off, notice that it is a square. So, it is open in the product topology. Conversely, given a basis element in the product topology, notice that we essentially have a rectangular box containing some point y. From what we know about the square metric, we can find a basis element in the square metric topology containing y and contained in the box.

Corollary 8. *$\mathbb{R}^n$ is metrizable for all n.*

Consider $\prod_{\alpha \in J} \mathbb{R}$. Recall that this has the box topology **and** the product topology. Try now to make the definition of $d(x, y)$ for $x, y \in \prod_{\alpha \in J} \mathbb{R}$. It turns out this is quite hard, since our lack of restrictions on the set J make this difficult- for example J might be come uncountable set, and we may accidentally define our metric to be an infinite set of non-zero numbers that doesn't converge. Suppose we tried to say:

$$d(x, y) = max\{|x_\alpha - y_\alpha\}$$

this **almost** works, but since it may not converge to a real number as α varies, it doesn't always work.

Another candidate is the following:

$$d(x, y) = sup_{\alpha \in J}\{min\{\|x_\alpha - y_\alpha|\}, 1\}$$

where the `sup' represents the **supremum**, the least upper bound. In our case,

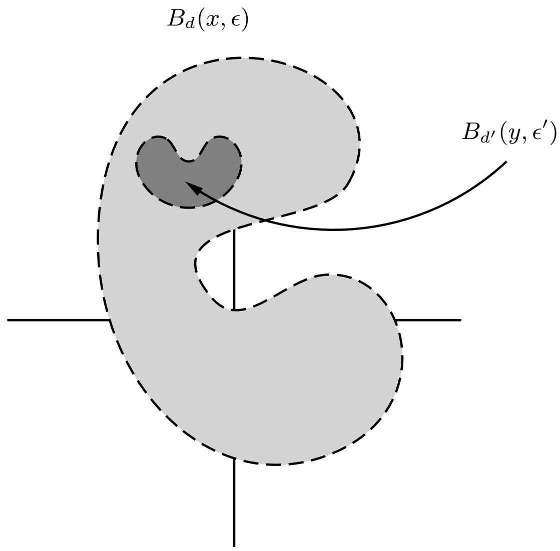


Figure 2.5: Here, d' is finer than d.

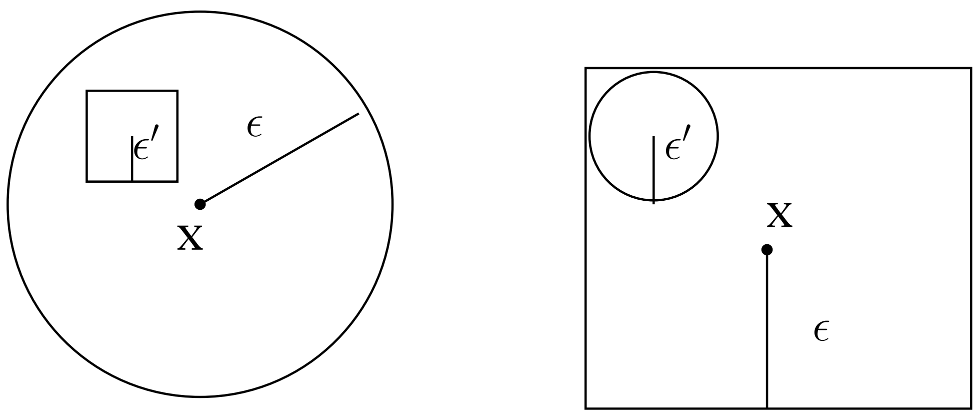


Figure 2.6: This picture illustrates that the Square metric is finer than the Euclidean metric, and that the Euclidean metric is finer than the Square metric.

since the numbers are all bounded by the 1, this definition holds. However, is this a metric? Well, we have the following:

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ is true, since $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, z) = sup_{\alpha \in J}\{min\{|x_\alpha - z_\alpha|, 1\}\}$ and since we know that

$$|x_\alpha - z_\alpha| = |x_\alpha - y_\alpha + y_\alpha - z_\alpha| \leq |x_\alpha - y_\alpha| + |y_\alpha - z_\alpha|$$

so, the $min\{|x_\alpha - y_\alpha|, 1\} \leq min\{|x_\alpha - z_\alpha|, 1\} + min\{|y_\alpha - z_\alpha, 1\}$, so taking the supremum,

$$d(x, z) \leq sup_{\alpha \in J}\{min\{|x_\alpha - y_\alpha|, 1\}\} + sup_{\alpha \in J}\{min\{|y_\alpha - z_\alpha|, 1\}\}$$

$$= d(x, y) + d(y, z)$$

so,

$$min\{|x_\alpha - z_\alpha|, 1\} \leq min\{|x_\alpha - y_\alpha|, 1\} + min\{|y_\alpha - z_\alpha|, 1\}$$

and

$$sup_{\alpha\in J}(R_\alpha + T_\alpha) \leq sup_{\alpha\in J}R_\alpha + sup_{\alpha\in J}T_\alpha$$

Which leads nicely into the following question: What is $B(x, 2)$ in $\prod_{\alpha\in J}\mathbb{R}$ in the uniform metric? It turns out that:

$$B(x, 2) = \{y \in \prod_{\alpha\in J}\mathbb{R} \ | \ sup_{\alpha\in J}\{min\{|x_\alpha - y_\alpha|, 1\} < 2\} = \prod_{\alpha\in K}\mathbb{R}$$

Also notice that $B(x, \epsilon), l\epsilon < 1$ in the uniform metric is all $y \in \prod_{\alpha\in J}\mathbb{R}$ such that

$$|x_\alpha - y_\alpha| < \epsilon$$

so,

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...) \in B(0, \frac{1}{2})$$

Fact. The Box topology is finer than than the Uniform topology (where the Uniform topology is the topology generated by the Uniform metric) and the Uniform topology is finer than the Product topology. In other words,

$$\text{Box Topology} \supsetneq \text{ Uniform Topology } \supsetneq \text{Product Topology}$$

for J infinite. There is a nice proof of this fact in our book.

Fact. We have the following nice statements about metric spaces:

1. A subspace of a metric space is a metric space. For example $\mathbb{Z} \subseteq \mathbb{R}$ where $\mathbb{R}$ has the usual metric, is a metric space.
2. All metric spaces are Hausdorff.

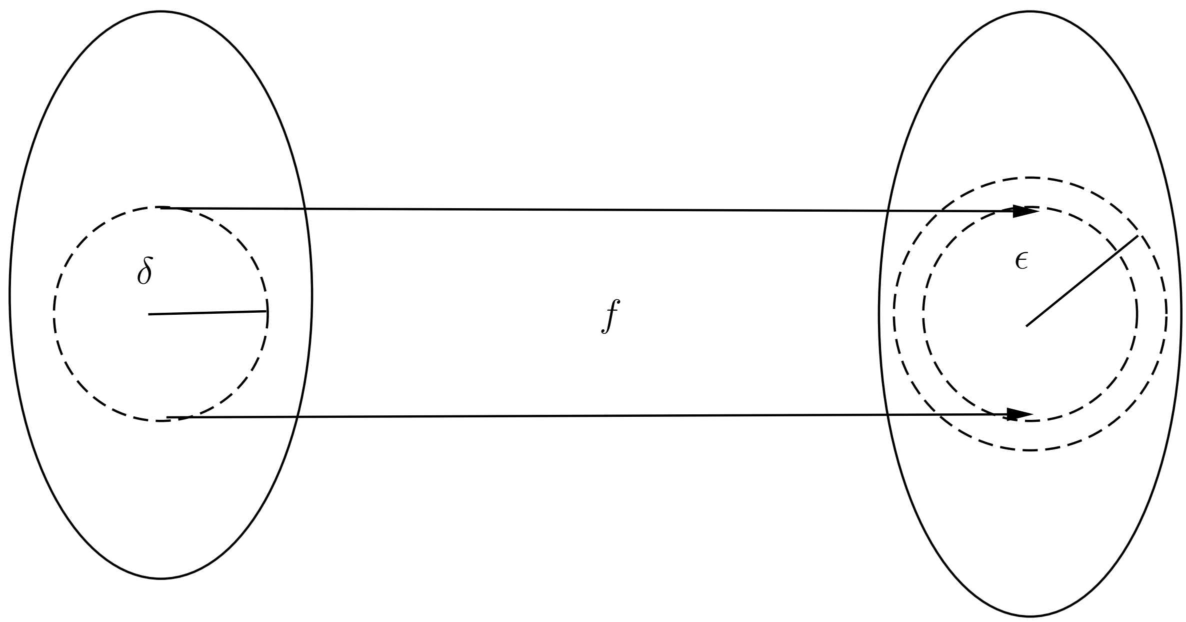
Definition. We say $f : X \to Y$ is **sequentially continuous** if and only if for all sequences $x_n \to x$, we have that $f(x_n) \to f(x)$. Saying that $x_n \to x$ means that for every neighborhood U of x, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in U$. Our claim is that if we left $f : X \to Y$, if f is continuous, then f is sequentially continuous. If X, Y are metrizable, the converse is also true.

Theorem 9. *If $f : X \to Y$ is continuous, then f is sequentially continuous. The converse is true if X, Y are metric spaces.*

Proof.

Proposition 10. *If X, Y are metric spaces with d_x, d_y and $f : X \to Y$, then f is continuous if and only if for all $x \in X, \epsilon > 0$ there exists $\delta > 0$ such that*

$$y \in B_{d_x}(x, \delta) \Rightarrow f(y) \in B_{d_y}(f(x), \epsilon)$$



The proof of this is that: if f is continuous, then

$$f^{-1}(B_{d_y}(f(x), \epsilon))$$

is open in X. Since this is open in X, for any

$$f \in f^{-1}(B_{d_y}(f(x)\epsilon)$$

there exists a basis element $B_{d_x}(x, \delta)$ for some δ such that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(B_{d_y}(f(x), \epsilon))$$

so $y \in B_{d_x}(x, \delta)$, then

$$f(y) \in B_{d_y}(f(x), \epsilon)$$

working the other way, suppose that f satisfies the condition in our proposition. Let V be open in Y. We need to show that $f^{-1}(V)$ is open in X. For each $f(X) \in V$, pick some $\epsilon < 0$ so that $B_{d_Y}(f(x), \epsilon) \subseteq V$, since V is open. By our proposition, there exists some $\delta > 0$ so that $B_{d_y}(x, \delta)$ satisfies

$$f(B_{d_Y}(x, \delta)) \subseteq V$$

So, $B_{d_x}(x, \delta) \subseteq f^{-1}(V)$. So for any $x \in f^{-1}(V)$, there is a δ so that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(V)$$

so $f^{-1}(V)$ is open. An equivalent way to state our proposition is that

$$f(B_{d_x}(x,\delta)) \subseteq B_{x_y}(f(x),\epsilon)$$

□

Lemma 11. *Let $x \in X$, $A \subseteq X$. If there is a sequence of points $x_n \in A$ such that $x_n \to x$, then the point x is in the closure of A, $x \in \overline{A}$. The converse of this statement holds if X is a metric space, or if X satisfies a different and weaker condition.*

Proof. We can say that $x - n \to x$ means that for every neighborhood U of x, there exists some natural number $N \in \mathbb{N}$ such that $x_n \in U$ for $n > N$. But, $x_n \in A$ for all n. So, for any neighborhood U of x, $x_n \in U \cap A$ for $n \geq N$, so U intersects A, so $x \in \overline{A}$. Now suppose that X is a metric space. The converse of our statement is to allow $A \subseteq X$, and to take $x \in \overline{A}$. To say that x is in the closure of A is to say that each neighborhood of x intersects x. For each $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \emptyset$. So, we can chose for each $n \in \mathbb{N}$ some

$$x_n \in B(x, \frac{1}{n} \subseteq U$$

Our claim is that $x_n \to x$, and its proof is that given a neighborhood U of x, chose N so that

$$B(x, \frac{1}{m}) \subseteq B(x, \frac{1}{N}) \subseteq U$$

for $M \geq N$. So, $x_n \in B(X, \frac{1}{n})$ for all $n \geq M$. This shows that $x_n \to x$. □

It turns out that we can add the 1^{st} **countability condition: For every $x \in X$ there is a countable collection of basis elements U_n for $n \in \mathbb{N}$ so that for any neighborhood U of x, there exists some $N \in \mathbb{N}$ such that**

$$U_n \subseteq U \qquad \text{for all } n > N$$

Theorem 12. *Let $f : X \to Y$, if f is continuous then f is sequentially continuous. The converse is true if X, Y are metric spaces.*

Proof. Suppose that $x_n \to x$. We need to show that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. We know that $f^{-1}(V)$ is open, and contains x. Since $x_n \to x$, we know $x_n \in f^{-1}(V)$ for all $n > N$ for some $N \in \mathbb{N}$, then $f(x_n) \in V$ for all $n > N$. This shows that $f(x_n) \to f(x)$. □

Suppose that $x_n \to x$ implies that $f(x_n) \to f(x)$, where X, Y are metric spaces. We want to show that f is continuous. We can do this by showing:

$$f(\overline{A}) \subseteq \overline{f(A)} \quad \text{for all } A \subseteq X$$

Suppose that $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. Since x is in the closure of A, by our lemma, we know that there exists some sequence x_n in A such that

$x_n \to x$. Then, since f is sequentially continuous, you know that $f(x_n) \to f(x)$. Again by our lemma,

$$x_n \in A \quad \text{so} \quad f(x_n) \in A$$

and by our lemma

$$f(x_n) \to f(x) \Rightarrow f(x) \in \overline{f(A)}$$

So f is continuous. So, is the converse true if X is 1^{st} countable, Y is 1^{st} countable, or both?

Definition. Let $f_n : X \to Y$ be a sequence of functions, where Y is a metric space. We say that $f_n \to f$ uniformly if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_Y(f_n(x), f(x)) < \epsilon$$

for all $n > N$ **and** all $x \in X$.

Theorem 13. *Let $f_n : X \to Y$ where Y is a metric space. If each f_n is continuous, and $f_n \to f$ uniformly, then f is continuous. I.e., `the uniform limit of continuous functions is continuous'.*

Remark. $f_n(x) = x^n$ on the interval $(0, 1]$ but the limit is not continuous- it has a jump discontinuity.

The proof of this theorem is in our textbook.

2.9 Review of Sets

Recall that if X is a set, then the following are equivalent:

1. There exists a partition of X into disjoint subsets,

$$X = X_1 \sqcup X_2 \sqcup X_2 \sqcup ... \sqcup X_n \sqcup ... \quad {}^3$$

2. There exists an equivalence relation on X, i.e. a relation, says x_1 x_2 satisfying reflexive, symmetric, and transitive properties.

Definition. If X has an equivalence relation $\star$, then for any $x \in X$ there exists a set:

$$[x] = \{y \in X | y \star x\}$$

called the equivalence class of x

The proof of our `the following are equivalent' claim can be done by showing that the collection of equivalence classes for any equivalence relation define a partition of X. Conversely, given a partition, you can construct an equivalence relation that mimics the structure of your partition.

[3] notice that $\sqcup$ denotes `disjoint union'

Example. What are **all** of the equivalence classes on $\mathbb{R}$, where the relation is $mod\ 1$? Well, it looks like the following:

$$\{[x] \mid 0 \leq x < 1\}$$

Suppose that $(X, \mathcal{T})$ is a space. Suppose the set X has an equivalence relation $\star$ on it. Let $X/\star$ be the set of equivalence classes on X. Well, we have the map:

$$q : X \to X/\star \quad x \to [x]$$

Is there a topology on this set $X/\star$ such that this map q is continuous? The answer is yes: we can define a topology on $X/\star$ so that $U \subseteq X/\star$ is open if and only if $q^{-1}(U)$ is open in X. This is not the unique topology on $X/\star$ making q continuous.
Claim. There is a unique topology on $X/\star$ such that q satisfies the following: U is open in $X/\star$ if and only if $q^{-1}(U)$ is open. Such maps q that satisfy this property are called quotient maps.

Proof.

Definition. More formally, $q : X \to Y$ is called a quotient map if V is open in Y if and only if $q^{-1}(V)$ is open in X.

Let $\mathcal{T}' = \{U \subseteq X/\star \mid q^{-1}(U) \text{ is open in } X\}$. This is a topology on $X/\star$, since we have the following:

1. $q^{-1}(\emptyset) = \emptyset,\ q^{-1}(X/\star) = X$, so $\emptyset, X/\star \in \mathcal{T}'$.
2. $q^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha q^{-1}(U_\alpha)$
3. $q^{-1}(\bigcap_{i=1}^n U_i) = \bigcap_{i=1}^n q^{-1}(U_i)$

These can all be proven, though I have omitted them. □

2.10 Connectedness

Definition. A space X is called disconnected if there exist non-empty, disjoint open sets U, V such that $X = U \cup V$.
Example. Consider $X = [0,1] \cup [2,3) \subseteq \mathbb{R}$. Notice that $[0,1]$ is open in X, and non-empty. Similarly, $[2,3)$ is open in X and open. These two sets are disjoint, and since X is the union of these two sets, X is disconnected.

So, is the fact that $[0,1] = [0,\frac{1}{2}) \cup [\frac{1}{2}, 2]$ a proof that $[0,1]$ is disconnected? Well no, since $[\frac{1}{2}, 1]$ is not open.
Lemma 14. *If $f : X \to Y$ and f is continuous where X is connected, then*

$$f(X) = Im(f)$$

is connected.

Corollary 15. *If X and Y are homeomorphic, then X is connected if and only if Y is connected.*
Fact. If there exists a homeomorphism $f : X \to Y$ then

$$f : \{X\} - x \to Y - \{f(x)\}$$

is a homeomorphism.

So, if $X -$ { some point } is connected, and $Y -$ {any point} is not connected, then X and Y are **not** homeomorphic.

We have the following properties:

1. $\cup A_\alpha$ is connected if there exists $p \in \cap A_\alpha$ and each A_α is connected.
2. If A is connected and $A \subseteq B \subseteq \overline{A}$ then B is connected
3. If $f : X \to Y$ is connected and X is connected then $f(X)$ is connected.
4. If X and Y are connected, then $X \times Y$ is connected.

Proof.

1. The idea is to suppose that you have some overlapping connected sets A_α, then considering a point p in all their intersections, using the fact that each set A_α is connected, you can show that $\cup A_\alpha$ is connected.
2. Left for Homework
3. Already discussed
4. Take the Cartesian product $X \times Y$. Notice that $a \times Y,\ a \in X$ is connected since it is homeomorphic to Y. Similarly, $X \times b,\ b \in Y$ is connected. Now taking $y \in Y$,

$$(X \times y) \bigcup (a \times Y)$$

it is clear that this space is connected, since they have a point in common, and they are both connected. Thus,

$$X \times Y = \bigcup_{y \in Y} [(X \times y) \cup (a \times Y)]$$

is connected, since all of these spaces are connected and contain the point $a \times b$. Thus by (1), $X \times Y$ is connected. By induction and (4), we know that a finite Cartesian product $X_1 \times ... \times X_n$ is connected if each X_i is connected.

□

2.10.1 The Connected-ness of $\mathbb{R}$

Definition. A simply-ordered set $(X, <)$ is called a **linear continuum** if and only if:

1. Every non-empty bounded subset of X has a least upper bound
2. For all $x, y \in X$ such that $x < y$ there exists $z \in X$ such that $x < z < y$.

Our prototypical example is $(\mathbb{R}, <)$. A non example is $(\mathbb{Z}, <)$. Similarly, a non example is $(\mathbb{Q}, <)$, which does not satisfy (2).
Claim. Let $(X, <)$ be a linear continuum. Every convex subset of X is connected.
Convex tells us that if you take $a, b \in Y$, then $[a, b] \subseteq Y$.

Proof. Suppose that $Y = A \cup B$, where A, B are open, nonempty, and disjoint. So there exists $a \in A, b \in B$, and we can consider:

$$A_0 = [a, b] \cap A \quad B_0 = A[a, b] \cap B$$

A_0 is nonempty since $a \in A_0$, and is bounded by b. So, A_0 has a least upper bound, which we will call $c \leq b$. We'll show that $c \notin A_0$, but $c \in [a, b]$ implies that $c \in Y$, so $Y \neq A \bigcup B$.
Again by contradiction, if $c \in B_0$ then since B_0 is open, we can find an interval $(d, c]$ such that $c \in (d, c] \subseteq B_0$. So, c is no the least upper bound, any $x \in (d, c]$ is an upper bound for A_0 and $x < c$. This shows that $c \notin B_0$, and similarly if $c \in A_0$, then since A_0 is open we can find $[c, e)$ such that $c \in [c, e) \subseteq A_0$. Then from the second property of being a linear continuum, there exists y such that $c < y < e$. But then, $y \in A_0$, since $[c, e)$ is contained in A_0. Then $c < y \in A_0$, so c is not an upper bound. Thus we again arrive at a contradiction. □

Corollary 16. *$\mathbb{R}$ is connected, so $[a, b]$ is connected, and $[a, \infty)$ is connected. So is (a, b) connected? Well,*

$$(a, b) = \bigcup_n [a + \frac{\epsilon}{n}, b - \frac{\epsilon}{n}]$$

So by a previous proposition, (a, b) is connected.
Theorem 17. *Suppose that X is connected and Y is a simply ordered set. If $f : X \to Y$ is continuous and $f(X) \leq r \leq f(b)$, $a, b \in X$ then there exists $c \in X$ such that $f(c) = r$.*

Proof. We know that $f(X)$ is connected. If there does not exist a c such that $f(c) = r$, then $r \notin Im(f)$, i.e.,

$$Im(F) = \left[(-\infty, r) \bigcap Im(f)\right] \bigcup \left[(r, \infty) \bigcap Im(f)\right]$$

Notice that this is a union of disjoint subsets of $Im(f)$.

□

2.11 Path Connectedness

Definition. A path in X is a continuous function

$$f : [a, b] \to X$$

We say that a space X is path-connected if for all $x, y \in X$ there exists a path f such that $f(a) = x, \ \ f(b) = y$.

If X is path-connected, then X is connected.

Proof. If X is not connected, then you can write X as:

$$X = A \bigcup B$$

where A, B are open, disjoint and non empty. Now picking some pint $x \in A, \ y \in B$, if there were a function $f : [a, b] \to X$ such that $f(a) = x, \ f(b) = y$, then

$$[a, b] = f^{-1}(X) = f^{-1}(A) \bigcup f^{-1}(B)$$

where A, B are nonempty and disjoint. If f were continuous, than $f^{-1}(A)$ and $f^{-1}(B)$ would be open, in which case $[a, b]$ is not connected. So, f is not continuous. □

2.12 Compactness

Definition. Let X be a space. An open covering of X is a collection $\{U_\alpha\}$ of open sets in X such that

$$X = \bigcup_\alpha U_\alpha$$

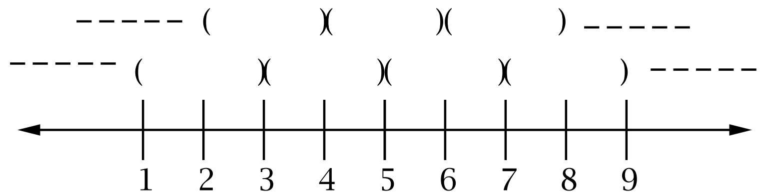
Definition. Let X be a space. We say X is compact if every open cover $\{U_\alpha\}$ of X has a finite sub-cover, i.e., there is some collection $U_1, ..., U_n \in \{U_\alpha\}$ such that

$$X = U_1 \bigcup U_2 \bigcup ... \bigcup U_n$$

Example. Suppose that X is a space consisting of finitely many points. Then, X is compact.
Example. Take $\mathbb{R}$. It turns out that by this definition, $\mathbb{R}$ is **not** compact. Let

$$U_n = (n, n+2) \quad n \in \mathbb{Z}$$

Then $\{U_n\}$ is an open covering of $\mathbb{R}$. However, the thing to notice is that $n + 1 \in U_k$ if and only if $k = n$. Thus, if we throw out any one of these sets, we will not cover the real line- implying that $\{U_n\}$ has no finite sub-cover.



Example.

$$A = \{0\} \bigcup \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

A is compact. If $\{U_\alpha\}$ is an open covering, then $\{0,\} \in U_\beta$ for some β. Then, $\frac{1}{k} \in U_\beta$ for all $k > M$, for some M. For each $\frac{1}{j} \notin U_\beta$, pick U_j such that $\frac{1}{j} \in U_j$, so

$$\{U_\beta\} \bigcup \{U_j\} \text{ is a finite sub-cover of } A$$

Example.

$$(0,1)$$

Is not compact. Let $U_n = \left(\frac{1}{n}, 1\right)$ where $n \in \mathbb{N}$. then U_n is open in A. Notice that

$$\bigcup_{n=1}^{\infty} U_n = A$$

so thus, we have an open cover of A. We would like to now show that there is no finite subcover for A. Given

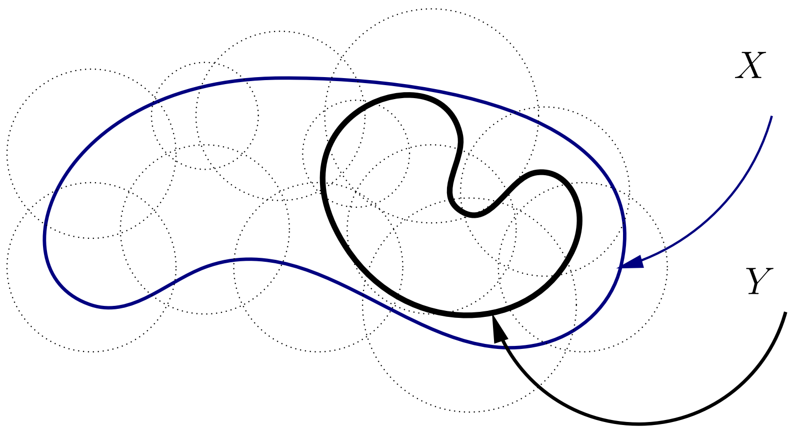
$$U_{k_1}, U_{k_1}, ..., U_{i_j}$$

there exists some maximal integer $k_m = max\{k_1, ...k_j\}$ such that

$$\bigcup U_{k_i} = \bigcup U_{k_m} = (\frac{1}{k}, 1) \neq (0,1)$$

so thus, we have no finite subcover for A. Similarly, $(0,1]$ and $[0,1)$ are not compact. However, $[0,1]$ **is** compact, and this will be shown in a future class.

2.13 Compactness of a Subspace



Definition. A collection of open set $\{U_\alpha\}$ each open in X, is said to cover a subspace Y if and only if $Y \subseteq \bigcup_\alpha U_\alpha$.

Proposition 18. *Let X be a space. A subspace $Y \subseteq X$ is compact if and only if every collection of open sets in X that covers Y has a finite subcover.*

Proof. V is open in Y if and only if $V = U \bigcap Y$ for some open set U in X. Suppose that Y is compact. Let $\{U_\alpha\}$ be a collection of open sets in X that covers Y. Then,

$$V_\alpha = U_\alpha \bigcap Y$$

is an open cover of Y. Y is then compact, and there then exist $V_1, V_2, ...V_n$ that cover Y. Then, $U_1, U_2, ..U_n$ open in X must cover Y. The other direction of this proof was left as an exercise. □

Theorem 19. *Every closed subset of a compact space is compact.*

Proof. Let $C \subseteq X$, where C is closed and X is compact. Let $\{U_\alpha\}$ be an open cover of C.

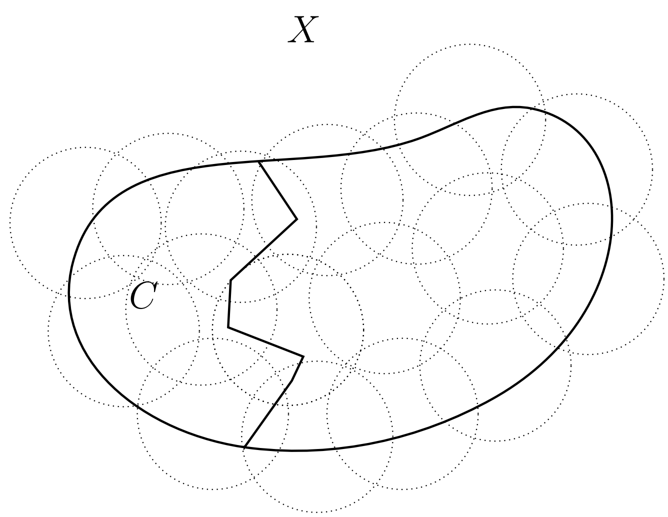
Assume that U_α are open in X. Since C is closed, $X - C$ is open in X, and

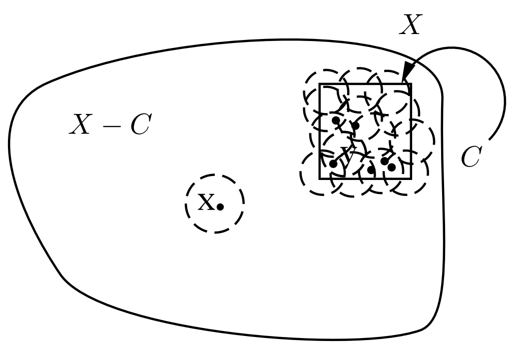
$$\{U_\alpha\} \bigcup \{X - C\}$$

is an open cover of X. X is compact, so there is a finite subcover of X. X is compact, so there is a finite subcover which may not use $X - C$. If it does throw it out, what remains is an open covering of C. Thus, C is compact. □

Remark. $(1, 2) \subseteq [0, 3]$, and notice that $(1, 2)$ is open but not compact, and $[0, 3]$ is compact.

Theorem 20. *A compact subset of a Hausdorff space is closed.*





Proof. Let $C \subseteq X$, where C is compact and X is Hausdorff. For each $y \in C$, pick U_y, V_y, disjoint neighborhoods of x, y. C is compact, so there is a finite subcover $V_{y_1} ... V_{y_k}$ of C. Then, consider neighborhoods U_{y_1} up to U_{y_k}, which are open, and contain x. We know that

$$x \in \bigcap_{i=1}^{k} Y_{y_i}$$

is a finite intersection of open sets, and is open. Then see that

$$U = \bigcap_{i=1}^{k} U_{y_i} \subseteq X - C$$

but

$$C \subseteq \bigcup_{i=1}^{k} V_{y_i}$$

and

$$U_{y_i} \bigcap V_{y_i} = \emptyset \quad \text{for } i = 1, ..., k$$

So if

$$z \in \bigcap_{i=1}^{k} U_{y_i} \quad \text{then } z \notin V_{y_i} \quad \text{for } i = 1, ..., k$$

so

$$z \notin \bigcup_{i=1}^{k} V_{y_i}$$

and so, $z \notin C$, i.e., $z \in X - C$.

□

Corollary 21. *Non-closed subsets of Hausdorff spaces are not compact.*
Proposition 22. *If $F : X \to Y$ is continuous and X is compact then $Im(f)$ is compact. In other words, the continuous image of a compact set is compact.*

Proof. Suppose that $\{U_\alpha\}$ is an open cover of $Im(f)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X. X is compact, so it has a finite subcover $\{f^{-1}(U_1), ..., f^{-1}(U_k)\}$, so the corresponding set $\{U_1, .., U_k\}$ is a finite subcover of $Im(f)$. This tells us that the image of f is compact. □

Corollary 23. *If $f : X \to Y$ is a homeomorphism and X is compact, then Y is compact.*
Proposition 24. *Let X be an ordered set with the least upper bound property. Then, every closed interval $[a, b]$ is compact in the order topology.*

Proof. Let $[a, b]$ be an interval, $a < b$. Let $\mathcal{C} = \{U_\alpha\}$ be an open cover of $[a, b]$. We now aim to show that there is a finite subcover. If $[a, b] = \{a, b\}$, then we're done- there exists a finite subcover. To continue this proof, we would like to use the following lemma:

Lemma 25. *Let $x \in [a, b]$, $x \neq b$, then there exists y such that*

$$x < y \leq b$$

and $[x, y]$ can be covered by finitely many elements of $\mathcal{C}$.

Proof.

Definition. The **immediate successor** of x (if it exists) is a element p such that $p > x$ and there is no q such that $x < q < p$.

We break this proof into two cases:

1. If x has an immediate successor: Let y be the immediate successor. Then, $[x, y] = \{x, y\}$, and we can chose two sets covering this.
2. If x does not have an immediate successor: We can pick $U \in \mathcal{C}$ such that $x \in U$. Since U is open, we can find:

$$[x, c) \subseteq (d, c) \subseteq U$$

Pick $y \in [x, c)$, then $[x, y] \subseteq [x, c) \subseteq U$. So, we can cover $[x, y]$ by one element of $\mathcal{C}$.

□

Proceeding, let $S = \{y > x \mid y \in [a, b] \text{ and } [a, y] \text{ can be covered by finitely many elements of } \mathcal{C}\}$. Our first goal is to show that S is non-empty. Secondly, we would like to show that the least upper bound of S is in S, and thirdly we would like to show that this least upper bound is equal to b. In proving our first goal, we apply the lemma where $x = a$, then there exists $y \in [a, b]$ such that $y \in S$. In proving the second, call the least upper bound of S `L', we know that it exists by what we just proved. We want to show that $L \in S$, i.e., we need to show that $[a, L]$ can be covered by finitely many elements of $\mathcal{C}$. Pick $U \in \mathcal{C}$ such that $L \in U$. Then there exists a s such that

$$(d, L] \subseteq (d, e) \subseteq U \subseteq [a, b]$$

If $L \notin S$, then there exists some element $z \in (d, L]$ such that $z \in S$ (since if not, then d is a smaller upper bound for S). Since $z \in S$, $[a, z]$ can be covered by finitely many open sets. But, $[z, L] \subseteq U$ so:

$$[a, L] = [a, z] \bigcap [z, L]$$

can be covered by finitely many sets, so $L \in S$. Thirdly, we want to show that $L = b$. Suppose that $L < b$. Applying the lemma to $x = L$ gives us $y > L$ such that $[L, y]$ can be covered by finitely many sets:

$$[a, y] = [a, L] \bigcup [L, y]$$

$L \in S$, so we can cover the left most interval by finitely many sets, and $[L, y]$ can be covered by finitely many sets by our lemma. But $y > L$ and $y \in S$, so L is not the least upper bound, and we arrive at a contradiction. □

Corollary 26. *A closed interval $[a, b]$ is compact in $\mathbb{R}$*

Fact. A product of compact space is compact in the product topology.

Corollary 27. *A closed box in $\mathbb{R}^n$ is compact, i.e.,*

$$[a_1, b_1] \times .. \times [a_n, b_n]$$

is compact.

Corollary 28. *$[a, b]$ and (a, b) are not homeomorphic.*

Theorem 29. *Take $\mathbb{R}^n$ with the usual topology fromt he Euclidean metric. A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Definition. A subset $A \subseteq \mathbb{R}^n$ is called bounded if there exists $M > 0$ such that $d(x, y) \leq < M$ for all $x, y \in A$.

Proof. If A is compact then A is closed since $\mathbb{R}^n$ is Hausdorff.

$$\mathbb{R}^n = \bigcup_{N \in \mathbb{N}} B(0, N)$$

Notice that $\{B_d(0,N)\}$ is an open cover of A, and since A is compact there exists a finite subcover of A. So,

$$A \subseteq B_d(0,M)$$

for some M, so A is bounded.

Now suppose that A is closed and bounded. Then

$$A \subseteq B_d(0,M) \subseteq [-M,M]^n \quad {}^4$$

A is a closed subset of a compact space, so A is compact. □

Definition. We say X is limit-point compact if every infinite subset of X has a limit point.

Proposition 30. *We have the following for spaces:*

Compactness $\Rightarrow$ *Limit Point Compactness*

Proof. Let $A \subseteq X$, where X is a compact space. Suppose that A has no limit point. Then for each $a \in A$, chose U_a such that $U_a \bigcap A = \{a\}$. Then,

$$X = (X - A) \bigcup \left(\bigcup_{a \in A} U_a \right)$$

is an open cover of X. So since X is compact, there must exist a finite subcover of X. I.e., we can cover A with finitely man $\{U_a\}$ $a \in A$, in which case A is finite. The converse of this statement is false. □

Definition. We say that X is sequentially compact if and only if every sequence in X has a subsequence that converges.

Theorem 31. *For metric spaces,*

Compactness $\Rightarrow$ *Limit Point Compactness* $\Rightarrow$ *Sequential Compactness* $\Rightarrow$ *Compactness*

Proof. Suppose that X is limit point compact. Let x_n be a sequence in X. Consider $A = \{x_n\}$. If A is finite, then x_n is eventually constant, and has a convergent subsequence:

$$x_1, x_2, ..., \underbrace{x_M = x_{M+1} = x_{M+2} = ..}_{\text{this is a convergent subsequence}}$$

but if A is infinite, pick x to be a limit point of A. For each $x \in \mathbb{N}$, chose

$$x_{n_k} \in B\left(x, \frac{1}{k}\right)$$

then x_{n_k} is a subsequence that converges to x. This is shown by taking $\epsilon > 0$, and choosing N so that $\frac{1}{N} < \epsilon$. Then $x_{n_k} \in B(x,\epsilon)$ for all $k > N$. □

[4] the right hand side is the n-dimensional cube of side lengths $2M$.

CHAPTER 3

ALGEBRAIC TOPOLOGY

3.1 The Fundamental Group

Given a space X, we would like to assign to it a piece of algebraic structured called the **fundamental group** of that space. This has nice consequences: for example, if we have a continuous map $f : X \to Y$ and we have their associated fundamental groups, we end up with a group homomorphism $\hat{f}$ from $X \to Y$.
Definition. A **homotopy** between two spaces:

$$f_1 : X \to Y \qquad f_1 : X \to Y$$

is a function $H : X \to I \to Y$ where I is $[0, 1]$ [1] such that:

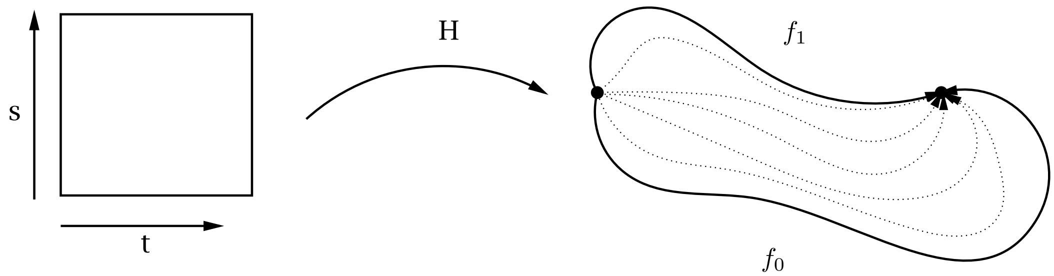
$$H(x, 0) = f_1(x)$$
$$H(x, 1) = f_2(x)$$

Definition. A **path** in X is a continuous function $f : [0, 1] \to X$.
Definition. We say that two paths, $f_0,\ f_1 : I \to X$ are **path homotopic** if and only if there exists a continuous function $H : I \times I \to X$ such that:

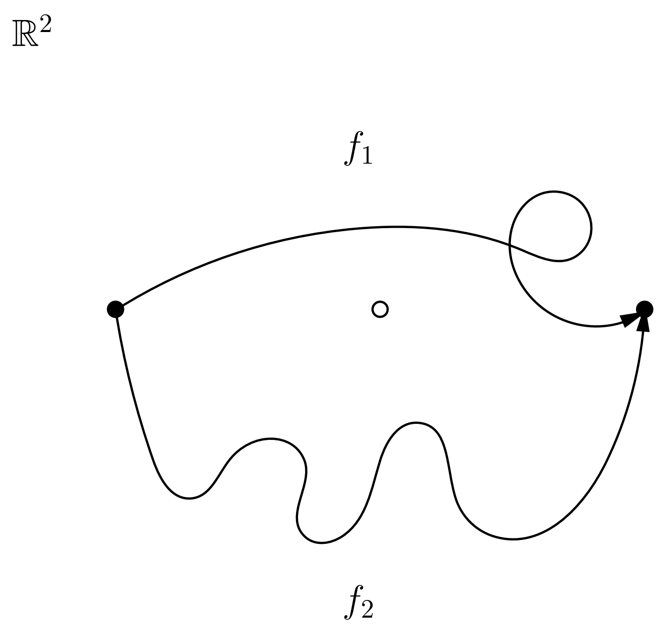
$$H(t, 0) = f_0(t)$$
$$H(t, 1) = f_1(t)$$
$$H(0, s) = f_0(0) = f_1(0)$$
$$H(1, s) = f_0(1) = f_1(1)$$

The idea is sort of as follows:

[1] It turns out, this is somewhat of a formality- I can really be any closed interval, since they're all homeomorphic.



Example. Take the space $\mathbb{R}^2 - \{0\}$. Intuitively, we can see that it is **not** true that



these two paths are path homotopic, since any continuous map would have had to `pass through' the point that was omitted from $\mathbb{R}^2$. We will show that this is true.

3.2 Path Composition

Let $f,\ g : I \to X$. If $f(1) = g(0)$, then define the **composition** of these paths as:

$$(f \star g)(s) = \begin{cases} f(2s), & s \in [0, \frac{1}{2}] \\ g(2s) & s \in [\frac{1}{2}, 1] \end{cases}$$

Path homotopy, denoted "$\simeq_p$" is an equivalence relation:

1. $f_0 \simeq_p f_0$
2. $f_0 \simeq_p f_1 \Rightarrow f_1 \simeq_p f_0$

3. $f_0 \simeq_p f_1,\ f_1 \simeq_p f_2 \Rightarrow f_0 \simeq_p f_2$

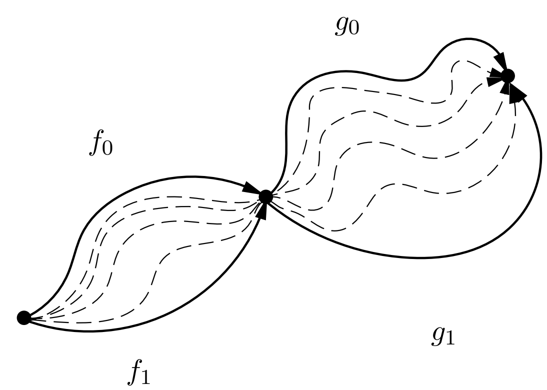
So Given $f : I \to X$, let us define the equivalence classes of f:

$$[f] = \{g : I \to X \mid f \simeq_p g\}$$

Claim. Composition induces a well-defined operation on equivalence classes of paths:

$$[f] \star [g] = [f \star g]$$

This looks like the following: Then



$$H(t,s) = \begin{cases} H_f(2t,s) & t \in [0,1/2] \\ H_g(2t-1,s) & t \in [1/2,1] \end{cases}$$

This is a homotopy from $f_0 \star g_0$ to $f_1 \star g_1$. So if f_0, f_1 are in some equivalence class, and g_0, g_1 are also in the same equivalence class, then $f_0 \star g_0$ and $f_1 \star g_1$ are in the same equivalence class.

We have the following properties:

1. Associativity:

$$[f] \star ([g] \star [h]) = ([f] \star [g]) \star [h]$$

   Just as a remark, notice that $(f \star g) \star h \neq f \star (g \star h)$.

2. For any $[f]$ there is an equivalence class $[f(1)$ and $[f(0)$., which are the equivalence classes under $\simeq_p$ of constant paths at $f(0)$ and $f(1)$ respectively.

3. Given f let $\hat{f}(t) = f(1-t)$, then

$$f \star \hat{f} \simeq_p f(1)$$
$$\hat{f} \star f \simeq_p f(0)$$

   so,

$$[f] \star [\hat{f}] = [f(1)] \qquad [\hat{f}] \star [f] = [f(0)]$$

Just to be clear with notation, notice that we have the following:

$$(f \star g)(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

Definition. Given $x_0 \in X$, let:

$$\pi_1(x, x_0) = \{ \ [f] \mid f : I \to X \quad \text{such that} \quad f(0) = f(1) = x_0\}$$

Proposition 32. *Let X be a space. For any $x_0 \in X$, $G = \pi_1(X, x_0)$ is a group under composition $\star$.*

Proof. First notice that $[f] \star [g]$ is always defined, since $f(1) = x_0 = g(0)$. Also, $[f] \star [g] \in \pi_1(X, x_0)$. □

Theorem 33. *If $f : X \to Y$ is continuous, then for any $x_0 \in X$, there is a group homomorphism:*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

defined by:

$$h_*([f]) = ([h \circ f])$$

Corollary 34. *If $h : X \to Y$ is a homeomorphism, then:*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, f(x))$$

is an isomorphism for all $x_0 \in X$.