Spring 2012

Theory of the Integral

Author:
Todd Gaugler

Professor:
Dr. Dragomir Saric

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Contents

Chapter 1

Introduction

The book we will be using for this class is **Real Analysis** by H.L. Royden .We are using the 3rd edition, or the 4th, depending on your interests.

1.0.1 Office Hours and Contact Information

Dr. Saric's office is in KY 405, and office hours will be Tuesday/Thursday from 2-3 p.m. My email is Dragomir.Saric@qc.cuny.edu. In case you need it, my office phone is 718-997-5824.

The grading policy is as follows:

1. Homework (30 %)
2. One in-class Exam (30 %)
3. A (cumulative) Final Exam (40 %)

The general plan of our course is to cover part I sections 2-5 in our textbook. Also we will jump to chapters 11-12 or so, to go through some measure theory.

1.1 Set Theory: Intersections, Unions, and Complements

Suppose we have a set, X. All other sets will be subsets of X. $\mathcal{P}(X)$ is 'the set of all subsets of X', and is called **the power set** of X.
Example.

$$X = \{1, 2, 3\}$$

So,

$$\mathcal{P}(x) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}\{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Notice that if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.
Definition. If $A, B \subset X$ (or $A, B \in \mathcal{P}(X)$) Then we have:

$$A \bigcap B = \{x : x \in A \text{ and } x \in B\}$$

We have the following facts:

1. $A \cap B = B \cap A$
2. $A \cap B \subset A$
3. $A \cap B = A \iff A \subset B$
4. $(A \cap B) \cap C = A \cap (B \cap C)$

Proof of **3**.

- Let us assume that $A \cap B = A$. By property two, we see that $A \cap B \subset B$. Since $A \cap B = A$, we know that $A \subset B$.
- Let us assume that $A \subset B$. We wish to show that $A \cap B = A$, which we can do by showing that $A \cap B \subset A$ and that $A \subset A \cap B$. By 2, we have that $A \cap B \subset A$. Now let $x \in A$. By $A \subset B$, we have that $x \in B$ as well. Then, $x \in A$ and $x\ inB$, which equivalent to saying that $x \in A \cap B$. This shows that $A \subset A \cap B$, so we have that $A \cap B = A$.

□

Definition. We have the following definition for **union**:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

We have the following facts about union:

1. $A \cup B = B \cup A$
2. $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
3. $A \subset A \cup B$
4. $A = A \cup B \iff B \subset A$

The following facts are known about both union and intersection:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof of **(2)**. First let us try to show the following:

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$$

Take an element $x \in A \cup (B \cap C)$. By definition, this means that $x \in A$ or $X \in (B \cap C)$, which is equivalent to saying that that $x \in B$ and $x \in C$. We have two cases to deal with now:

Notice that if $x \in A$, then clearly x must be in $A \cup B$ and $A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$.

Notice also that if $x \in B$ and $x \in C$, then $x \in x \in A \cup B$ and $A \cup C$, from which it follows that $x \in (A \cup B) \cap (A \cup C)$.

So, we are done. The proof of (1) will be left to you. □

Definition. The empty set, denoted $\emptyset$, is the set that has no elements.

We have the basic facts about the empty set:

1. $A \cup \emptyset = A$
2. $A \cap \emptyset = \emptyset$

Definition. If $A \subset X$, then the complement A^C of A (relative to X) is as follows:

$$A^C = \{x \in X : x \notin A\}$$

Example. If $X = \{1, 2, 3\}$ and $A = \{1\}$, then $A^C = \{2, 3\}$. Notice that $\emptyset^C = X$ and that $X^C = \emptyset$.

We have the following facts:

1. $(A^C)^C = A$
2. $A \cup A^C = X$
3. $A \cap A^C = \emptyset$
4. $A \subset B \iff B^C \subset A^C$

Definition. DeMorgan's Laws:

1. $(A \cup B)^C = A^C \cap B^C$
2. $(A \cap B)^C = A^C \cup B^C$

(1) follows from noticing that $x \in (A \cup B)^C$ is the same as saying that $x \notin (A \cup B)$ and this says that $x \notin A$ and $x \notin B$. From this it follows that $x \in A^C \cap B^c$. Going the other way, saying that $x \in A^C \cap B^C$, this tells us that $x \in A^C$ and $X \in B^C$, from which we see that $x \notin A$ and $x \notin B$, so $x \notin (A \cup B)$, telling us that $x \in (A \cup B)^C$.

Definition. We have the following definition of the **difference**, or **relative complement** of two sets A and B: let $A, B \subset X$. Then

$$B - A = \{x \in X : x \in B \text{ and } x \notin A\}$$

notice that the following is true:

$$B - A = B \cap A^C$$

Definition. $A \triangle B$ is called the **symmetric difference** of A and B, and is defined as follows:

$$A \triangle B = (B - A) \cup (A - B) = A \cup B - (A \cap B)$$

Definition. If $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Definition. A collection $\mathcal{C}$ of sets is a *disjoint collection* or a *collection of pairwise disjoint sets* if any two sets in $\mathcal{C}$ are disjoint.

Definition. The intersection of the collection $\mathcal{C}$ is the set of all elements of X that belong to each member of $\mathcal{C}$. This can be denoted as follows:

$$\bigcap_{A \in \mathcal{C}} A \quad \text{or} \quad \bigcap\{A : A \in \mathcal{C}\}$$

We have that:

$$\bigcap_{A\in\mathcal{C}} A = \{x \in X : (\forall A)(A \in \mathcal{C} \Rightarrow x \in A\}$$

We also have the union of an arbitrary collection of a set,

$$\bigcup_{A\in\mathcal{C}} = \{x \in X : (\exists A)(A \in \mathcal{C} \text{ and } x \in A\}$$

Again, we have DeMorgan's laws, telling us that:

$$\left(\bigcap_{A\in\mathcal{C}} A\right)^C = \bigcup_{A\in\mathcal{C}} A^C \qquad \left(\bigcup_{A\in\mathcal{C}} A\right)^C = \bigcap_{A\in\mathcal{C}} A^C$$

In trying to prove the right-most identity, we have the following:

Proof. Let $x \in (\bigcup_{A\in\mathcal{C}} A)^C$. This tells us that $x \notin \bigcup_{A\in\mathcal{C}A}$, which tells us that $(\forall A)(A \in \mathcal{C} \Rightarrow x \notin A)$ which tells us that $(\forall A)(A \in \mathcal{C} \Rightarrow x \in A^C) \Rightarrow x \in \bigcap_{A\in\mathcal{C}} A^C$. □

We have the following distributive laws:

1. $B \cap [\bigcup_{A\in\mathcal{C}} A] = \bigcup_{A\in\mathcal{C}}[B \cap A]$
2. $B \cup [\bigcap_{A\in\mathcal{C}} A] = \bigcap_{A\in\mathcal{C}}[B \cup A]$

Also, the union of the empty collection of sets is empty, and the intersection of an empty collection of set is X, since by definition since the collection is empty there is nothing to check, and we wind up with all $x \in X$.

If a collection is given in a sequence, $\mathcal{C} = \{A_i\}_{i=1}^{\infty}$ then

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{A\in\mathcal{C}} A, \qquad \bigcup_{i=1}^{\infty} A_i = \bigcup_{A\in\mathcal{C}} A$$

Definition. $\{A_i\}_{i=1}$, $i \in \mathbb{N}$ is called a **index set**. We can also talk about $\{A_\lambda : \lambda \in \lambda\}$, and we can talk about their unions and intersections:

$$\bigcup_{\lambda\in\mathcal{A}} A_{\mathbb{A}} = \{x : \exists\lambda \in \mathcal{A} \text{ and } x \in A_\lambda\}$$

$$\bigcap_{\lambda\in\mathcal{A}} A_\lambda = \{x : (\forall\lambda)(\lambda \in \mathcal{A}) \Rightarrow x \in A_\lambda\}$$

Let

$$f : X \to X$$

be a function. Then let $\{A_\lambda\}_{\lambda\in\mathcal{A}}$ be a collection of subsets of X, in which case:

$$f[\bigcup_{\lambda\in\mathcal{A}} A - \lambda] = \bigcup_{\lambda\in\mathcal{A}} f(A_\lambda)$$

$$f[\bigcap_{\lambda\in\mathcal{A}} A - \lambda] \subset \bigcap_{\lambda\in\mathcal{A}} f(A_\lambda)$$

Now we assume that $\{B_\lambda\}_{\lambda\in\mathcal{A}}$ is an indexed collection of subsets of Y, then we have that

$$f^{-1}[\bigcup_\lambda B_\lambda] = \bigcup_\lambda f^{-1}(B_\lambda)$$

and that

$$f^{-1}\left[\bigcap_\lambda B_\lambda\right] = \bigcap_\lambda f^{-1}(B_\lambda)$$

Notice that we have the following chain of events:

$$x \in \bigcap_\lambda f^{-1}(B_\lambda) \Rightarrow \forall\lambda \in \mathcal{A}, f(x) \in B_\lambda \Rightarrow f(x) \in \bigcap_\lambda B_\lambda \Rightarrow x \in f^{-1}(\bigcap_\lambda B_\lambda)$$

We also have that

$$f^{-1}[B^C] \subset (f^{-1}[B])^C, \qquad f(f^{-1}[B]) = B \qquad f^{-1}(f[A]) \supset A$$

1.2 Homework

Page 17 questions (16, 17) and Page 19 question (19).

1.3 Algebras of Sets

Definition. A collection $\mathcal{A}$ of subsets is called an algebra of sets (or boolean algebra), if:

1. For all $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$
2. For ever $A \in \mathcal{A}$, $A^C \in \mathcal{A}$.

Given $A, B \in \mathcal{A}$, we would like to show that $A \cap B$ is in $\mathcal{A}$. Based on what we know (along with DeMorgan's laws) we have:

$$(A \cap B)^C = A^C \cup B^C \Rightarrow A^C \cup B^C \in \mathcal{A} \Rightarrow (A^C \cup B^C)^C \in \mathcal{A} \Rightarrow A \cap B = (A^C \cup B^C)^C \in \mathcal{A}$$

Finite unions and finite intersections are also in the algebra.
Proposition 1. *Given any collection $\mathcal{C}$ of subsets of X, there is a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$.*

Proof. $\mathcal{P}(X)$ is an algebra, and it contains $\mathcal{C}$ ☐

Consider the family $\mathcal{F}$ of all algebras that contain $\mathcal{C}$. Then,

$$\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\} \supset \mathcal{C}$$

Now we just need to verify that $\mathcal{A}$ is an algebra. This follows from seeing the following: let $A, B \in \mathcal{A}$. Since both A, B are in $\mathcal{A}$, this implies that $A, B \in \mathcal{B}$, for all $\mathcal{B} \in \mathcal{F}$. Since $\mathcal{B}$ is an algebra, this implies that $A \cup B \in \mathcal{B}$ for all $\mathcal{B} \in \mathcal{F}$, from which we see that $A \cup B \in \mathcal{A}$.
Definition. The smallest algebra containing $\mathcal{C}$ is called an algebra generated by $\mathcal{C}$.

Proposition 2. *Let $\mathcal{A}$ be an algebra of subsets of X, and $\{A_i\}_{i=1}^{\infty}$ is a sequence of sets in $\mathcal{A}$.Then, there exists a sequence $\{B_i\}_{i=1}^{\infty}$ of sets in $\mathcal{A}$ such that $B_n \cap B_m = \emptyset$ for $n \neq m$, and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$.*

Proof. Define the following: $B_1 = A_1$. $B_2 = A_2 - A_1 = A_2 \cap A_1^C$. Generally, for $n > 1$, we define:

$$B_n = A_n - (A_1 \cup A_2 \cup ... \cup A_{n-1}) = A_n \cap A_1^C \cap A_2^c \cap ... \cap A_{n-1}^C$$

So, $B_n \subset \mathcal{A}$, and $B_n \subset A_n$. Given some $m < n$, we can look at $B_m \cap B_n$. Well, $B_m \cap B_n \subset A_m \cap B_n = A_m \cap (A_n \cap A_1^C \cap A_2^C \cap ... \cap A_{n-1}^C) = \emptyset$ since $m < n$, and since you'll have m some where in that chain of complements, so you'll hit the empty set somewhere when you take the intersection of A_m with A_m^C.

We also know that $B_i \supset A_i \Rightarrow \cup_{i=1}^{\infty} \mathcal{B}_i \supset \cup_{i=1}^{\infty} A_i$. So, let

$$x \in \cup_{i=1}^{\infty} A_i.$$

Then, x belongs to at least one A_i. Let i_0 be the smallest such index such that $x \in A_{i_0}$. Then,

$$x \in B_{i_0} = A_{i_0} - (A_1 \cup A_2 \cup ... \cup A_{i_0 - 1})$$

So this implies that

$$\cup_{i=1}^{\infty} A_i \supset \cup_{i=1}^{\infty} B_i \Rightarrow \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$$

□

Definition. An algebra $\mathcal{A}$ of sets is called a σ-algebra or a Borel field, if every union of a countable collection of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

Any σ-algebra contains countable intersections, which follows again from union.

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A} \Rightarrow (\bigcup_{i=1}^{\infty} A_i)^C = \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$$

Proposition 3. *Given any collection $\mathcal{C}$ of subsets of X, there is a smallest σ-algebra that contains $\mathcal{C}$ (this is a homework question)*

1.4 The Axiom of Choice

Let $\mathcal{C}$ be any collection of non-empty sets. Then, there is a function F defined on $\mathcal{C}$ which assigns to each set $A \in \mathcal{C}$ an element $F(A)$ in A. The function F is called a *choice-function.* If our collection

$$\mathcal{C} = \{X_\lambda\}_{\lambda \in \Lambda}$$

Notice that

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

How to we talk about the direct product of an infinite collection, $\mathcal{C} = \{X_\lambda\}_{\lambda \in \Lambda}$? Well, it is a collection of all sets $\{x_\lambda\}_{\lambda \in \Lambda}$ indexed by Λ such that $x_\lambda \in X_\lambda$.

Remark. If one X_λ is the empty set, then

$$\prod_{\lambda \in \Lambda} X_\lambda$$

is empty. If all $X_\lambda \neq \emptyset$, then

$$\prod_{\lambda \in \Lambda} X_\lambda \neq \emptyset$$

1.5 Countable Sets

Definition. A set is countable if it is the range of some sequence.
Definition. A set is **finite** if it is either empty or the range of a finite sequence.

A set is countably infinite if it can be put in a one-one correspondence with the natural numbers $\mathbb{N}$.
Proposition 4. *Every subset of a countable set is countable.*

Proof. Let $E = \{x_n\}$ be a countable set, and $A \subset E$. If A is empty, then we are done by definition. Otherwise, let $x \in A$. Define a sequence $\{y_n\}_{n=1}^{\infty}$ as follows: if $x_n \in A$, then $y_n = x_n$. If $x_n \notin A$, then $y_n = x$. We now have a sequence $\{y_n\}$, and its range is A- thus it is countable.

$\square$

Proposition 5. *Let A be a countable set. Then the set of all finite sequences of A is countable.*

Proof. We establish a one-one correspondence between the set of all finite sequences of elements of A and $\mathbb{N}$. Since A is one-one with $\mathbb{N}$, let us look at sequences of the following type:

$$< 2, 3, 5, 7, ..., P_nk, ... >$$

Now,

$$n = 2^{x_1} \cdot 3^{x_2} .... \cdot P_k^{x_k}$$

Notice that

$$f : n \mapsto (x_1, x_2, ..., x_k)$$

which is a finite sequence in $\mathbb{N} \cup \{0\}$. The range of f are all finite sequences in $\mathbb{N} \cup \{0\}$, which contains all finite sequences in $\mathbb{N}$- and thus is countable. $\square$

Proposition 6. *The set of all rational numbers is countable- they map into all sequences of length 2.*
Proposition 7. *The union of a countable collection of countable sets is countable.*

Proof. $\mathcal{C}$, $\{A_n\}_{n=1}^{\infty}$. Each A_n is a set $\{x_{m\,n}\}_{m=1}^{\infty}$, indexed by the natural numbers. Thus,

$$\bigcup_{n=1}^{\infty} A_n \quad \text{is indexed by sequences of length } 2, \{m, n\}$$

$\square$

1.6 Relations and Equivalences

R is a relation on set X is a subset of $X \times X$. We write the following:

$$(x, y) \in R \qquad xRy$$

Some examples of relations on the set $\mathbb{R}$ are $=$, or $\leq$. We denote them in the following way:

$$= \quad \{(x, x) : x \in \mathbb{R} \qquad\qquad \leq \quad \{(x, y) : x, y \in \mathbb{R}, x \leq y\}$$

Definition. A relation R is transitive on X if

$$xRy, \quad xRz \Rightarrow \quad xRz$$

Notice that the examples above are transitive relations.
Definition. A relation R is symmetric if

$$xRy \Rightarrow yRx$$

Notice that $=$ is symmetric, but $\leq$ is not a symmetric relation.
Definition. A relation R on X is reflexive if

$$xRx$$

For all $x \in X$. Again, $=$ is an example of a reflexive relation. Notice that $<$ is not reflexive on $\mathbb{R}$.
Definition. A relation R on X is an **equivalence relation** if it is reflexive, symmetric, and transitive. For example, $=$ is an equivalence relation on $\mathbb{R}$. However, $\leq$ is not an equivalence relation.

Suppose that $\equiv$ is an equivalence relation on some set X. Then, we can define:

$$E_x = \{y \in X : x \equiv y\}$$

If $y, z \in E_x$, then $x \equiv y$, and $x \equiv z$.Then, $y \equiv z$. This set E_x is called an equivalence class. Thus, either $E_x = E_y$ or $E_x \cap E_y = \emptyset$. One nice construction of such equivalence relations is a partition on a set, where each pair of sets in your partition are either disjoint or the same. So, we get this:

$$X/ \equiv \; = \{E_x : x \in X\} = \text{ A Collection of Equivalence Classes}$$

Since $x \in E_x$, then $E_x \neq \emptyset$ for all $x \in X$.
Definition. A binary operation on X is a mapping $map : X \times X \to X$. One example is the addition of real numbers. An equivalence relation $\equiv$ is compatible with a binary operation $+$ if

$$x \equiv x' \text{ and } y \equiv y' \Rightarrow x + y \equiv x' + y'$$

If $+$ is compatible with $\equiv$, then $+$ defines a new binary operation on the set $Q = X/ \equiv$. For example,

$$x/ \equiv, \; ; \; y/ \equiv \; \in Q, \qquad (x/ \equiv) + (y/ \equiv) = (x + y/ \equiv)$$

One good example is the addition of rational numbers.

1.7 Homework 2/7/2012

On page 23, do questions 24 and 25.

1.8 Partial Orderings and the Maximal Principle

Definition. A relation R on a set X is **antisymmetric** if

$$xRy \text{ and } yRx \Rightarrow x = y$$

One example would be the $\leq$ relation on $\mathbb{R}$. Similarly, $\subseteq$ on $\mathcal{P}(X)$ is an antisymmetric relation.
Definition. A relation $<$ is a **partial ordering** on X if it is transitive and antisymmetric. Some examples of such a relation include $\leq$ on $\mathbb{R}$, and $\subsetneq$ on $\mathcal{P}(X)$.
Definition. A partial ordering $<$ on a set X is a **linear ordering** if for all $x, y \in X$ either $x < y$ or $y < x$. One such example is $\leq$ on $\mathbb{R}$. However, $\subset$ on $\mathcal{P}(X)$ is not a linear ordering if X has more than one element (for example you could pick two disjoint subsets of X, neither of which will then be contained in each other).

We read

$$a < b$$

As 'a precedes b', or 'a less than b'. Alternatively, we can say that 'b is greater than a'.
Definition. If $E \subseteq X$, an element $a \in E$ is the smallest element in E if for every $x \in E$, if $x \neq a$ then $a < x$. It is very simple to show that this smallest element must be unique.
Definition. A minimal element of E is an element $a \in E$ such that there is no $x \in E$ with $x \neq a$ and $x < a$.
Example.

$$X = \{1, 2, 3, 4\} \qquad C = \{\{1\}, \{1, 2\}, \{3\}\}$$

Notice that with the relation $\subseteq$, C has two minimal elements: $\{1\}, \{3\}$. But, there is **no** smallest element.
Fact. The smallest element is a minimal element. This can be shown from their definitions.
Definition. If for all $x \in X$, $x < x$ then $<$ is a **reflexive** partial ordering. If it is never true that $x < x$, then $<$ is called a **strict** partial ordering.

1.9 Hausdorff Maximum Principle

Let $<$ be a partial ordering on a set X. Then, there exists a maximal linearly ordered subset S of X.
Remark. X is maximal with property M if a any $S' \supset S$ does not have property M.

1.10 Well ordering

A strict linear ordering on a set X is called a **well-ordering** if every nonempty subset of X contains the smallest element.

Example. Take the set $\mathbb{N}$ with the relation $<$. This is well-ordered. Alternatively, if we take $\mathbb{R}$ and the relation $<$, it is not well-ordered. This follows from taking an open interval on $\mathbb{R}$, which does not have a smallest element.

1.10.1 The well-Ordering Principle

Every set X can be well-ordered.
Proposition 8. *There is an uncountable set X that is well-ordered by a relation $<$ in the following way:*

1. *There is the largest element Ω in X.*
2. *If $x \in X$, and $x \neq \Omega$, then*
$$\{x \in X : y < x\}$$
is countable.

Proof. (i) Let Y be an uncountable set. By the well-ordering principle, Y has a well-ordering $<$. If Y does not have the largest element, then we introduce $Z = Y \cup \{\alpha\}$, where $\alpha \notin Y$. Ordering on Z extends the ordering $<$ on Y by saying that for all $y \in Y$, $y < \alpha$. Then, α is the largest element of Z.

(ii) The set of all $y \in Z$ such that $\{x \in Z : x < y\}$ is countable is nonempty, because α is in this set. Let Ω be the smallest element in this set. Let

$$X = \{x \in Z : x < \Omega \text{ or } x = \Omega\}$$

Then, X satisfies (i) and (ii). □

Ω is called the first uncountable ordinal, and anything less that Ω is called a 'countable ordinal'.

1.11 The Real Numbers

Definition. The set $\mathbb{R}$ is a set that consists of two binary operations $+$ and $\cdot$, and can be constructed from the following axioms:

1. $(\mathbb{R}, +, \cdot)$ is a field, or in other words, $(\mathbb{R}, +)$ is a commutative group (for all $x, y \in \mathbb{R}$, $x+y \in \mathbb{R}$, addition is commutative, there exists an identity element (0), and there exists an additive inverse for all elements in $\mathbb{R}$) and $(\mathbb{R}, \times)$ for all $x, y \in \mathbb{R}$ we have that $x \cdot y \in \mathbb{R}$, $\cdot$ is associative, there exists a multiplicative identity (1) in $\mathbb{R}$, and for all the non-zero elements in $\mathbb{R}$ there exists another element in $\mathbb{R}$ that is its multiplicative identity, multiplication is commutative, and multiplication distributes over addition.

We have the axioms of order: there is a set P of positive real numbers that satisfies:

1. For all $x, y \in P$, $x + y \in P$
2. For all $x, y \in P$, $x \cdot y \in P$

3. For all $x \in P$, $-x \notin P$
4. For every $x \in \mathbb{R}$, $x = 0$ or $x \in P$ or $-x \in P$

Thus, our set is called an 'ordered field'. Such examples of an ordered field include $\mathbb{Q}$, and of course $\mathbb{R}$.

We can introduce the following relation by definition:

$$x < y \iff y - x \in P$$

Similarly, by definition

$$x \leq y \iff y < x \text{ or } y = x$$

Definition. Let $S \subseteq \mathbb{R}$. The number b is an upper bound for S if for every $x \in S$, $x \leq b$.
Definition. The least upper bound for S is a number b such that if c is an upper bound of S, then $b \leq c$. Sometimes this is denoted:

$$sup(S) = b$$

And called 'the supremum' of S
Definition. A lower bound for S is a if for all $x \in S$, $a \leq x$. Similarly, the greatest lower bound for $Sis\ a$ if for any lower bound d for S, we have that $d \leq a$. Sometimes this is denoted

$$imf(S) = a$$

Pronounced 'infimum'.
Example. Let

$$S = \{\frac{1}{n} : n \in \mathbb{N}\}$$

Notice that 2 is an upper bound for S. However, also notice that $sup(S) = 1$. Similarly, - 6 is a lower bound for S, however the $imf(S) = 0$.

1.11.1 The Completeness Axiom

Every non-empty set of real numbers which has an upper bound has a least upper bound.

1.11.2 The Natural Numbers

We have the following definition:

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

Notice that $\mathbb{N} \subseteq \mathbb{R}$.

1.11.3 Axiom of Archimetes

For all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$. This axiom isn't particularly helpful, but it does give us the following:
Corollary 9. *Between any two real numbers, there is a rational number.*

1.11.4 The Extended Real numbers

The extended real numbers are the set $\mathbb{R}$, along with the two numbers $+\infty, -\infty$. Notice that

$$-\infty < x < \infty$$

And that

$$x + \infty = \infty \quad x - \infty = -\infty \quad x \cdot \infty = \infty \text{ if } x \text{ is positive}, \infty + \infty = \infty \quad -\infty - \infty = -\infty$$

$$\infty(\pm\infty) = (\pm\infty) \quad -\infty(\pm\infty) = \mp\infty$$

$$\infty - \infty = \text{ undefined} \quad 0 \cdot \infty = 0 \text{ by convention}$$

Definition. If S has no upper bound, then $sup(S) = \infty$. If S has no lower bound then $imf(S) = -\infty$.

1.11.5 Sequences of Real Numbers

A sequence is a map $f : \mathbb{N} \to \mathbb{R}$, $f(n) = x_n$, denoted

$$\{x_n\}_{n=1}^{\infty}, \qquad \text{or} \qquad \{x_n\}$$

and we have that

$$\lim_{n\to\infty} x_n = x$$

if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - x| < \epsilon$$

Definition. $\{x_n\}$ is a Cauchy sequence if for all $\epsilon > 0$ there exists N such that for all $n, m \geq N$ we have that

$$|x_n - x_m| < \epsilon$$

Fact. If a sequence has a limit, then it is unique.
Fact. If a sequence of real numbers is Cauchy, then it has a limit. This follows from the Completeness axiom.
Definition. x is a **cluster point** of a sequence $\{x_n\}$ if every neighborhood of x contains infinitely many elements of the sequence.
Definition.

$$\lim_{n\to\infty} x_n = \infty$$

If for all $C > 0$, there exists $N(C) \in \mathbb{C}$ such that if $m \geq N(C)$ then $x_n > C$. Similarly,

$$\lim_{n\to\infty} x_n = -\infty$$

If for all $D \in \mathbb{R}$, there exists $N(D) \in \mathbb{N}$ such that if $n \geq N(D)$, then $x_n < D$.
Example. $x_n = n!$. Then

$$\lim_{n\to\infty} x_n = \infty$$

Also, if $x_n = -n$ then

$$\lim_{n\to\infty} x_n = -\infty$$

Definition. Let $\{x_n\}$ be a sequence of real numbers. The **limit superior** of $\{x_n\}$ is defined by

$$\overline{\lim_{n\to\infty}} x_n = inf_n sup_{k\geq n} x_k = \lim_{n\to\infty} sup_{k\geq n} x_k$$

Example. We can show that the limit superior of the following sequence is 0:

$$x_n = \frac{(-1)^n n}{n^2+1} = 0$$

Definition. The **limit inferior** of $\{x_n\}$ is:

$$\underline{\lim_{n\to\infty}} x_n = sup_n\ inf_{k\geq n} x_k$$

Notice that unlike wit the limit superior, $inf_{k\geq n} x_k$ is monotonically increasing.
Fact. $x = \overline{\lim}_{n\to\infty} x_n$ if and only if it satisfies:

1. For all $\epsilon > 0$, there exists n such that $x_k < l + \epsilon$ for all $k \geq n$
2. For all $\epsilon > 0$ and for all n, there exists $k \geq n$ such that $x_k > l - \epsilon$.

Example. $x_n = sin(\frac{n\pi}{2}) + 1$. Notice that

$$\overline{\lim_{n\to\infty}} x_n = 2, \quad \underline{\lim_{n\to\infty}} x_n = 0$$

Fact.

$$\overline{\lim_{n\to\infty}}(-x_n) = -\underline{\lim_{n\to\infty}} x_n$$

and

$$\underline{\lim_{n\to\infty}} x_n \leq \overline{\lim_{x\to\infty}} x_n$$

$$\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim(x_n+y_n)} \leq \overline{\lim}\, x_n + \underline{\lim}\, y_n \leq \lim(x_n+y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$$

Let us prove that $\overline{\lim(-x_n)} = -\underline{\lim}\, x_n$.

Proof. We have that

$$\overline{\lim_{n\to\infty}}(-x_n) = inf_n sup_{k\geq n}(-x_k) = inf_n(-inf_{k\geq n} x_k) = -sup_n(inf_{k\geq n} x_k)$$

We have the following fact:

$$-A = \{-a : a \in A\}$$

So,

$$sup(-A) = -inf A$$

This follows from seeing that

$$-sup(-A) = inf A$$

So, letting $b = sup(-A)$ we notice that b is an upper bound for $-A$, namely for all $a \in A$ we have that $b \geq -a$. From this, $-b \leq a$, for all $a \in A$. Thus, $-b$ is a lower bound for A. Let c be a lower bound for A, so $c \leq a$ for all $a \in A$. In this case, $-a \leq -c$ for all $a]inA$, so $-c$ is an upper bound for the set $-A$. Since b is the least upper bound, we have that $b \leq -c$, or $sup(-A) = -b \geq C$. This implies that $-sup(-A)$ is the greatest lower bound, and is the infinitum of A. ∎

1.11.6 Homework

Prove the last two of the above claims that aren't finished.

1.12 Open and Closed Set of Real Numbers

We have the standard open interval on $\mathbb{R}$,

$$(a, b) = \{x : a < x < b\}$$

and we have

$$(a, \infty)\{x; x > a\}$$

which is an infinite open interval, similarly

$$(-\infty, b) = \{x : x < b\}$$

which is also an infinite open interval. Alternatively,

$$[a, b] = \{x : a \leq x \leq b\}$$

which is a closed interval. Notice that $[a, b)$ and $(a, b]$ are half-open and half-closed intervals.
Definition. A set O of real numbers is called 'open' if for every $x \in O$, there exists some $\delta > 0$ such that each y with $|y - x| < \delta$ belongs to O. Another way to say this, is to say that

$$(x - \delta, x + \delta) \subset O$$

Open sets, for example, include things like open intervals, $\mathbb{R}$, $\emptyset$
Proposition 10. *The intersection of two open sets O_1 and O_2 is an open set.*

Proof. Let $x \in O_1 \cap O_2$. We know that there exists some $\delta_1 > 0$ such that $(x - \delta_1, x + \delta_1) \subset O$, because O_1 is open. Similarly, O_2 has some δ_2, so defining $\delta = min\{\delta_1, \delta_2\}$, we see that

$$(x - \delta, x + \delta) \subset O_1 \cap O_2$$

So, $O_1 \cap O_2$ must be open. $\square$

As a corollary, this implies that the finite intersection of open sets is open.
Proposition 11. *The union of any collection $\mathcal{C}$ of open sets is open.*

Proof. Suppose we took $U = \bigcup_{O \in \mathcal{C}} O$. If $x \in U$, this implies that there exists some O_1 in our collection such that $x \in O_1$, and since O_1 is open there must exist some $\delta > 0$ such that the interval around x of radius δ is open. $\square$

Example.

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

Notice that this infinite intersection of open sets is $\{0\}$- which isn't open.

Proposition 12. *Every open set of real numbers is the union of a countable collection of disjoint open intervals.*

Proof. Let O be an open set. For all $x \in O$, there exists $y > x$ such that $(x, y) \subset O$. There also exists some $z < x$ such that $(z, x) \subset O$. Let

$$b = \sup\{y : (x, y) \subset O\}$$

and

$$a = \inf\{z : (z, x) \subset O\}$$

This implies that $a < x < b$, also, $I_x = (a, b)$. Our first claim is that

$$I_x \subset O.$$

Let $w \in I_x$. We can say that $x < w < b$. Then, there exists some y such that $y > w$ and $(x, y) \subset O$. This implies that $w \in O$.

Also, $b \notin O$. If $b \in O$, then you know that there would have to exist some $\delta > 0$ such that $(b - \delta, b + \delta$ is a subset of O. But if this were true, then b would not be the supremum of all $\{y : (x, y) \subset O\}$, since $(x, b + \frac{\delta}{2}) \subset O$. This would imply that $(x, b + \delta) \subset O)$, and then b is not the supremum. There is a similar proof that $a \notin O$. It can follows that $\cup_{x \in O} I_x = O$.

If we have to intervals (a, b) and (c, d) that are elements of $\{I_x\{_{x \in O}$ and $(a, b) \cap (c, d) \neq \emptyset$, then we want to show that $(a, b) = (c, d)$. If their intersection was nonempty, then $c < b$, and also, $a < d$. Since $c \notin O$, then $c \leq a$. Since $a \notin O$, then we know also that $a \notin (c, d)$, which implies that $a \leq c$. Putting these two together, we have that $a = c$. Similarly, we get that $b = d$. Thus, $(a, b) = (c, d)$. Thus, $\{I_x\}_{x \in O}$ is a disjoint family of open sets. Each open interval contains a rational number, since the intervals are disjoint this gives a well-defined function from $\mathbb{Q} \to \{I_x\}_{x \in O}$, thus $\{I_x\}_{x \in O}$ is countable. □

Proposition 13. *Let $\mathcal{C}$ be a collection of open sets of real numbers. Then there exists a countable sub-collection $\{O_i\}_{u=1}^{\infty}$ such that $\bigcup_{O \in \mathcal{C}} O = \bigcup_{i=1}^{\infty} O_i$*

Proof. Let $U = \bigcup_{O \in \mathcal{C}} O$. Let $x \in U$. This implies that there exists some $O \in \mathcal{C}$ such that $x \in O$. Since O is open, there exists I_x such that I_x (an open interval) such that $x \in I_x \subset O$. Suppose that $I_x = (a, b)$. This implies that there exists J_x, an open interval with rational endpoints such that $x \in J_x$ and $J_x \subseteq I_x$. The collection of all intervals with rational endpoints is a countable set. Thus $\{J_x\}_{x \in U}$ is a countable collection, and $\bigcup_{x \in U} J_x = U$, because $x \in J_x$. For each J_x, choose one $O \in \mathcal{C}$ such that $J_X \subset O$–we get a countable collection. □

Definition. A real number x is a point of closure of E if for all $\delta > 0$, there exists some $y \in E$ such that $|x - y| < \delta$.
Remark. Every point of E is a point of closure of E.
Definition. $\overline{E}$ is the set of points of closure of E. Notice that $E \subset \overline{E}$.
Proposition 14. *If $A \subset B$, then $\overline{A} \subset \overline{B}$, and also $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof. Since $A \cup A \cup B$, then $\overline{A}, \overline{B} \subset (\overline{A \cup B})$, so

$$\overline{A} \cup \overline{B} \subset (\overline{A \cup B})$$

Let $x \notin \overline{A} \cup \overline{B}$. Then there exists $\delta_1 > 0$ such that no $y \in A$ satisfies $|x-y| < \delta_1$. Similarly there exists $\delta_2 > 0$ such that no $y \in B$ satisfies $|x-y| < \delta_2$. Simply, take $\delta = min(\delta_1, \delta_2) > 0$. If $|x-y| < \delta$, then $y \notin A$, and $y \notin B$, which implies that $y \notin A \cup B$. This means that $x \notin \overline{A \cup B}$. $\square$

Definition. A set F is closed if $F = \overline{F}$.
Example. Some examples of closed sets include closed intervals, $\mathbb{R}$, and $\emptyset$. Also, $[a, \infty)$ and $[-\infty, b)$ are closed.
Proposition 15. *For any set E, the set $\overline{E}$ is closed. It is also true that $\overline{\overline{E}} = \overline{E}$.*

Proof. Let $x \in \overline{\overline{E}}$. Given $\delta > 0$, there exists $y \in \overline{E}$ such that $|x-y| < \frac{\delta}{2}$. Since $y \in \overline{E}$, there then exists some $z \in E$ that $|y-z| < \frac{\delta}{2}$. Finally, we get that $|x-z| \leq |x-y| + |y-z| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, which also implies that $x \in \overline{E}$. $\square$

Proposition 16. *If F_1 and F_2 are closed, then $F_1 \cup F_2$ is closed.*

Proof. Based on what we proved before,

$$\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} = F_1 \cup F_2$$

which must be closed. $\square$

Corollary 17. *The union of finitely many closed sets is closed.*
Example. Look at $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2 - \frac{1}{n}]$. Taking their union, we'll get $(0, 2)$, which is open- despite the fact that each one of these sets is closed.
Proposition 18. *The intersection of any collection $\mathcal{F}$ of closed sets is closed.*

Proof. Let $x \in \overline{(\bigcap_{F \in \mathcal{F}} F)}$. Then for any $\delta > 0$ there exists some $y \in \bigcap_{F \in F} F$ such that $|x-y| < \delta$. Then y is in each $F \in \mathcal{F}$. Then, $x \in \overline{F}$. But since $F = \overline{F}$(remember, F is closed) we thus have that $x \in \bigcup_{F \in \mathcal{F}} F$. So, $\bigcap_{F \in \mathcal{F}} F$ is closed, since "any point in the closure of this intersection is in the intersection". $\square$

Proposition 19. *The complement of any open set is closed, and the complement of any closed set is open.*

Proof. Let O be open. If $x \in O$, then there exists $\delta > 0$ such that if $|x-y| < \delta$, then $y \in O$. Then, $x \notin \overline{O^C}$. Thus, $\overline{O^C} = O^C$, from which we gather that O^C is closed.

Let F be closed, and say that $x \in F^C$. Then x is not a point of closure of F^C, which implies that there exists $\delta > 0$ such that if $|x-y| < \delta$, then $y \in F^C$. Thus, F^C is open. $\square$

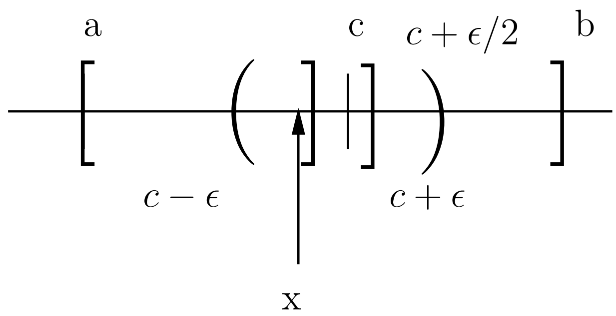
Definition. A collection $\mathcal{C}$ of sets **covers** a set F is $F \subset \cup\{O : O \in \mathcal{C}\}$. If each O is open, we say that $\mathcal{C}$ is an **open cover**. If $\mathcal{C}$ is finite, then $\mathcal{C}$ is a **finite cover**.
Theorem 20. Heine-Borel*: Let F be a closed and bounded set of real numbers. Then each open cover of F has a finite subcover.*

Proof. Assume a covering $\mathcal{C}$ of F is given. First, we assume that $F = [a, b]$. We define

$$E = \{x : x \leq b \text{ such that for } [a, x] \text{ there is a choice of finite subcover of } \mathcal{C}\ \}$$

First, we observe that $a \in E$. In particular, this tells us that $E \neq \emptyset$. E is bounded, let $c = \sup(E) \in \mathbb{R}$. Since b is an upper bound for c, we have that $c \leq b$. We need to show that

$c = b$. Since $c \in [a, b]$, there exists some open set $O \in \mathcal{C}$ such that there exists some ϵ such that $(c-\epsilon, c+\epsilon) \subset O$. Since $c = \sup(E)$, this implies that there exists some $x \in E$ and $x > c - \epsilon$. This implies that there exists some finite sub-collection $\{O_1, O_2, .., O_n\}$ of $\mathcal{C}$ that covers $[a, x]$. Then $\{O_1, ..., O_n\}$ covers $[a, c + \frac{1}{2}\epsilon]$.



Thus c is not the supremum of E unless $c = b$. This finishes our first case, when $F = [a, b]$.

Now let F be any closed bounded set of the reals. Then, there exists some $[a, b] \supset F$. Then we define the following:

$$\mathcal{C}^* := \mathcal{C} \cup F^C$$

which is also an open cover of $[a, b]$. By the first case, there exists some $\{O_1, ..., O_n, F^C\}$, a sub-collection of $\mathcal{C}^*$ which covers $[a, b]$. Restricting this set to $\{O_1, ..., O_n\}$, we have a sub-collection of $\mathcal{C}$, and covers F. $\square$

Proposition 21. *Let $\mathcal{C}$ be a collection of closed sets with the property that every finite subcollection of $\mathcal{C}$ has a non-empty intersection and suppose that one of the sets in $\mathcal{C}$ is bounded. Then,*

$$\bigcap_{F \in \mathcal{C}} F \neq \emptyset$$

This will be left as an exercise, the idea is to apply the Heine-Borel Theorem.

1.13 Continuous Functions

Suppose we have $E \subseteq \mathbb{R}$ and we have $f : E \to \mathbb{R}$. f is continuous at $x \in E$ if for all $\epsilon > 0$ there exists some $\delta > 0$ such that for all $y \in E$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Definition. f is continuous on $A \subset E$ if f is continuous at each point of A.

Proposition 22. *Let $f : F \to \mathbb{R}$ be a continuous function, and $F \subseteq \mathbb{R}$ is closed and bounded. Then f is bounded on F, and it assumes it is maximal and minimal on F.*

Proof. For all $x \in F$. there exists an interval I_x such that $x \in I_x$ where if $y \in I_x \cap F$ then $|f(x) - f(y)| < 1$. This implies that $|f(y)| < |f(x)| + 1$ for all $y \in I_X \cap F$. Then, $\{I_x\}_{x \in F}$ is an open cover of F. By the Heine-Borel theorem, we have that there exists a finite subcover $\{I_{x_1} ... I_{x_k}\}$ of F. Let

$$M = max\{|f(x_1) + 1|, ..., |f(x_k)| + 1\}$$

Let $y \in F$. Then there exists some interval I_{x_i} that contains y. Then $|f(y)| < |f(x_i)| + 1 \leq M$. Thus f is bounded on F by the constant M.

Since f is bounded, $m = \sup_{x \in F} f(x)$ is finite. We need t o show that there exists some $x_1 \in F$. such that $f(x_1) = m$. Suppose such $x_!$ does not exist. Then $f(x) < m$ for each $x \in F$. Then by the continuity of f, for each $x \in F$ there exists an interval I_x containing x such that for all $y \in I_x$ we have

$$\frac{1}{2}f(y) + \frac{1}{2}f(y) = f(y) < \frac{1}{2}(f(x) + m)$$

(we abandoned the above proof to do the following) There exists an interval I_x such that $x \in I_x$. For all $x \in I_x \cap F$, we have

$$|f(y) - f(x)| < \frac{1}{2}(m - f(x))$$

so

$$f(y) - f(x) < \frac{1}{2}(m - f(x))$$

$$f(y) < \frac{1}{2}(m + f(x))$$

Well, $\{I_x\}_{x \in F}$ is an open cover of F. By the Heine-Borel theorem again, there exits a finite subcover $\{I_{x_1}, ..., I_{x_n}\}$ of F. Let $a = max\{f(x_1), ..., f(x_n)\} < m$. For $y \in F$, there exists I_x containing y, and we have

$$f(y) < \frac{1}{2}(f(x_i) + m) \leq \frac{1}{2}(a + m) < \frac{1}{2}(m + m) = m$$

This implies that m is not the least upper bound because $\frac{1}{2}(a + m)$ is an upper bound which is less than m. This is a contradiction, thus f must achieve tis maximum. □

Proposition 23. *Let*

$$f : (-\infty, \infty) \to \mathbb{R}$$

Then f is continuous if and only if $f^{-1}[O]$ is open for each open $O \subset \mathbb{R}$.

Proof. For all O open subsets of $\mathbb{R}$, we want to show that $f^{-1}[O]$ is open. Given $x \in \mathbb{R}$, for all $\epsilon > 0$ we have $I = (f(x) - \epsilon, f(x) + \epsilon) \subset \mathbb{R}$ and is open, and $x \in f^{-1}(I)$. This implies that there exists some $\delta > 0$ such that

$$(x - \delta, x + \delta) \subset f^{-1}(I)$$

so

$$f((x - \delta, x + \delta)) \subset (f(x) - \epsilon, f(x) + \epsilon)$$

Working in the other direction, let f be a continuous function and let $O \subset \mathbb{R}$ that is open. Let $x \in f^{-1}(O)$. Then $f(x) \in O$, O is open implies that there exists some $\epsilon > 0$ such that

$$(f(x) - \epsilon, f(x) + \epsilon) \subset O$$

There then exists some $\delta > 0$ such that

$$f((x - \delta, x + \delta)) \subset (f(x) - \epsilon, f(x) + \epsilon)$$

which in turn implies that

$$(x - \delta, x +]delta) \subset f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \subset f^{-1}(O)$$

which implies that $f^{-1}(O)$ is open. □

Definition. A real-valued function f on a set E is uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
Example. Uniformly continuous functions are also continuous. However, it is not true that continuous functions are uniformly continuous. For example,

$$f(x) = \frac{1}{x}, \quad x > 0$$

is continuous, but not uniformly continuous. This follows from the fact that the function $f(x)$ as $x \to 0$ gets so steep so quickly, the δ's disappear as you choose ϵ.

Another example is

$$f(x) = x^2$$

Proposition 24. *If a function f is a continuous function defined on a closed and bounded set F, then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, $x \in F$ (where $f : F \to \mathbb{R}$), then there exists $\delta_x > 0$ such that if $|x-y| < \delta_x$ and $y \in F$ then $|f(x) - f(x)| < \frac{1}{2}\epsilon$. Let $I_x = (x - \frac{1}{2}\delta_x, x + \frac{1}{2}\delta_x)$, $x \in I_x$, is an open cover of F as $x \in F$. By Heine-Borel, there exists a finite subcover $\{I_{x_1}, ..., I_{x_n}\}$. Let $\delta = min\frac{1}{2}\delta_{x_1}, ...\frac{1}{2}\delta_{x_n}\} > 0 \in \mathbb{R}$. We let $y, z \in F$, and assume that $|y - z| < \delta$. There exists some I_{x_i} containing y and I_{x_j} containing z, which implies that

$$|z - x_i| \leq |z - y| + |y - x_i| < \delta + \frac{1}{2}\delta_{x_i} < \delta_{x_i}$$

This implies that $|f(z) - f(x_i)| < \frac{1}{2}\epsilon$. It is also true that $|f(y) - f(x_i)| < \frac{1}{2}\epsilon$. We now have that

$$|f(z) - f(y)| \leq |f(z) - f(x_i)| + |f(x_i) - f(y)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

so we are done. □

Definition. A sequence $\{f_n\}$ of functions defined on a set E converges point-wise on E to a function f if for each $x \in E$ we have that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Definition. A sequence of functions $\{f_n\}$ on E uniformly converges on E to a function f if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for every $x \in E$ we have that

$$|f_n(X) - f(x)| < \epsilon$$

for all $n \geq N$.
Example. Suppose that

$$f_n(X) = x^n$$

for $0 \leq x \leq 1$. In this case, f_n converges point-wise to $f(x) = 0$ for all $0 \leq x < 1$ and 1 for $x = 1$. However, f_n does not converge uniformly to f. This idea follows from what happens to the distance at the point $(1, 0)$: the functions f_n do not get closer to f nicely.

1.14 Borel Sets

Remark. A countable union of closed sets is not necessarily closed.
Definition. The collection $\mathcal{B}$ of Borel Sets is the smallest σ-algebra which contains all of the open sets. It is certain that $\mathcal{B}$ is generated by all closed sets. It is also true that $\mathcal{B}$ is generated by all open intervals.

A set is F_σ if it is a countable union of closed sets. Notice that every closed set is F_σ. Also, notice that every countable set F_σ. An example of a F_σ set that is not countable is the following set: $\mathbb{N}$, which follows directly from our last example. Alternatively,

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

is another example.
Definition. A set is G_δ if it is a countable intersection of open sets.

An open interval is F_σ, for one could do the following:

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

1.15 Lebesgue Measure

Taking the interval $I = [a, b]$, the length ℓ of $l(I)$ of I is

$$\ell(I) = b - a.$$

The length is an example of a set function (it associates a real number to each set in a collection subsets of a set). ℓ is a set function on the collection of intervals in $\mathbb{R}$. The length of an open set is defined as follows: the sum of the lengths of the open intervals of which it is composed.

Given a collection $\mathcal{M}$ of sets, with m a set function on $\mathcal{M}$ such that for every $E \in \mathcal{M}$, we would like to define m such that $m(E) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and such that m satisfies our "wish-list" of properties:

1. $m(E)$ defined for all $E \subset \mathbb{R}$.
2. For every interval I, we would like to have that $m(I) = \ell(I)$.
3. If $\{E_n\}$ is a disjoint sequence then $m(\cup_n E_n) = \sum_N m(E_n)$
4. $m(E + y) = m(E)$ for all E and for each $y \in \mathbb{R}$. In other words, we want E to be invariant under translation. With intervals, we can notice that

   $$[a, b] + y = [a + y, b + y]$$

   which preserves distance trivially.

Unfortunately, it is not possible to satisfy all four properties for any set function m. What we will end up doing is weakening (1): $m(E)$ will be defined on as many sets as possible, and we will keep properties 2-4.

We will require that $\mathcal{M}$ is a σ-algebra. Any such $\mathcal{M}$ has to contain a Borrel σ-algebra, by property number 2 (since the Borel σ-algebra is the smallest σ-algebra which contains all of the open sets in some collection).

1.15.1 Outer Measure

For any subset $A \subseteq R$, we will consider a countable collection $\{I_n\}$ of open intervals which covers A; in other words

$$A \subset \bigcup_n I_n$$

Definition. Lebesgue outer measure of A is as follows:

$$m^*(A) = \inf_{A \subset \cup_n I_n} \sum_n \ell(I_n)$$

From the definition, it is immediate that $m^*(\emptyset) = 0$. Also, if $A \subset B$, then $m^*(A) \leq m^*(B)$. It also follows that $m^*\{x\} = 0$. One way to see this would be to let

$$I_n = \left(x - \frac{\epsilon}{2^n + 1}, x + \frac{\epsilon}{2^n + 1}\right)$$

in which case,

$$\sum_{n=1}^{\infty} \ell(I_n) = \epsilon \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) = \epsilon$$

and since our choice of ϵ was arbitrary, we let $\epsilon \to 0$.
Proposition 25. *The outer measure of an interval is its length.*

Proof. Notice that $\emptyset$ is considered an open interval. We have $[a, b]$, and that for all $\epsilon > 0$, that $(a - \epsilon, b + \epsilon) \supset [a, b]$. This tells us that the outer measure

$$m^*\left([a, b]\right) \leq b - a + 2\epsilon$$

Where we define

$$I_n = (a - \epsilon, b + \epsilon), \qquad I_n = \emptyset, \quad n \geq 1$$

since this holds for all $\epsilon > 0$,it follows that $m^*([a, b]) \leq b - a$. It remains to be shown that $m^*([a, b]) \geq b - a$. Let $\{I_n\}$ be an arbitrary collection of open intervals that covers $[a, b]$. By the Heine-Borel Theorem, $\{I_n\}$ has a finite subcover

Since $a \in [a, b]$, there exists an interval (a_1, b_1) containing a which is in that finite subcover. If $b_1 < b$, let (a_2, b_2) be an interval in the finite sub-cover which contains b_1. Then

$$a_2 < b_1 < b_2$$

We can continue this process, were we would then say that

$$a_3 < b_2 < b_3$$

and we can continue this process until b is in an interval, which must happen (this process stops because the collection is finite, which we have from the Heine-Borel theorem). We have now that

$$a_i < b_{i-1} < b_i$$

So,

$$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{i=1}^{k} \ell((a_i, b_i)) = b_k - a_k + b_{k-} - a_{k-1} + ... + b_1 - a_1 = b_k - (a_k - b_{k-1}) - ... - (a_2 - b_1) - a_1$$

Notice that each term $a_k - b_{k-1}$ is negtive, so we can remove them:

$$\geq b_k - a_1 \geq b - a \Rightarrow m^*[a, b] \geq b - a$$

from which we conclude that

$$m^*[a, b] = b - a = \ell([a, b])$$

If I is any finite interval, then given $\epsilon > 0$ there exists a closed interval $J \subset I$ such that

$$\ell(J) > \ell(I) - \epsilon$$

Since

$$\ell(I) - \epsilon < \ell(J) = m^*(J) \leq m^*(I) \leq m^*(\overline{I}) = \ell(\overline{I}) = \ell(I)$$

This implies that as $\epsilon \to 0$, then

$$\ell(I) = m^*(I)$$

□

For homework, prove this when I is an interval: if I is any infinite interval, then given some $\epsilon > 0$ there exists a closed interval $J \subset I$ such that

$$\ell(J) > \ell(I) - \epsilon$$

1.16 Review Questions

Notice that

$$[a, b)$$

Is a Borel set, since it is a F_σ set. This can be shown by looking at the union:

$$\bigcup_{n=1}^{\infty} [a, b - \frac{1}{n}]$$

of closed sets.

Notice that

$$f_n(x) = x^n \quad 0 \leq x \leq \frac{1}{2}$$

uniformly converges to $f(x) = 0$. This takes some work to show: given $\epsilon > 0$, we need to find N such that if $n \geq N$ then

$$|x^n - 0| < \epsilon$$

First notice that

$$|x^n - 0| = |x^n| = x^n \leq \frac{1}{2^n} < \epsilon$$

for every x in the interval $x \in [0, \frac{1}{2}]$. We want to find n from the following:

$$\frac{1}{2^n} = \epsilon \Rightarrow n \log \frac{1}{2} = \log \epsilon$$

So,

$$N = [\frac{\log \epsilon}{\log \frac{1}{2}}] + 1$$

1.17 Outer Measure

If $A \subseteq \mathbb{R}$ then

$$m^*(A) = \inf_{A \subset \bigcup_n I_n} \sum \ell(I_n)$$

This definition has some consequences: for example, if $A \subset B$, then $m^*(A) \leq m^*(B)$. This follows directly from the fact that A has more covers than B does.

Proposition 26. *If I is an interval, then $m^*(I)$ is the length of I, $\ell(I)$.*

Proof. First, look at a closed interval $[a, b]$. We have to show that $m^*[a, b] \leq b - a$ and that $m^*[a, b] \geq b - a$. The first one follows quickly, choosing $I_1 = [a - \epsilon, b + \epsilon]$ and for all others to be $I_n = \emptyset$, $n = 2, 3, ...$ In this case, the sum of the lengths of I_n is $b - a + 2\epsilon$. Notice that

$$m^*[a, b] \leq \sum \ell(I_n) = b - a + 2\epsilon \Rightarrow m^*[a, b] \leq b - a$$

Since we let ϵ to go zero. For the second case, we have to work a bit harder - but we can us the Heine-Borel theorem. Suppose that we have a cover $\{I_n\}$. There must exists $I_1, ..I_k$, a finite sub-cover for $[a, b]$. Based on how this interval should look, it has to look like (a_1, b_1), containing a. If this set doesn't contain b, we look at point b_1. Since we have a cover, we have to add another set that contains b_1. This process continues k times, since we have a finite cover. We can then show that (thanks to the overlaps of these sets) that the sums of this alternative cover is greater than or equal to $b - a$.

If I is finite, in this case there exists a closed set J such that $J \subset I$, and that $\ell(J) \geq \ell(I) - \epsilon$ (there is a trick that you can do to make this work) given some $\epsilon > 0$. So,

$$m^*(J) = \ell(J)$$

since J is a closed interval. Since $J \subset I$, this implies that

$$m^*(J) \leq m^*(I)$$

But we also note that $I \subset \overline{I}$. Then,

$$m^*(I) \leq m^*(\overline{I}) = \ell(\overline{I}) = \ell(I)$$

Thus what we have is

$$\ell(I) - \epsilon \leq m^*(J) = \ell(J) \leq m^*(I) \leq m^*(\bar{I}) \leq \ell(\bar{I})$$

Which ultimately tells us that $m^*(I) = \ell(I)$.

We now approach the case in which I is infinite. Suppose we had something like (a, ∞) or $(-\infty, b)$. Well suppose had some infinite interval: in any case, given some $M > 0$ it is possible to find some closed set $J \subset I$ such that

$$\ell(J) = m^*(J) \geq M$$

This then implies that $m^*(I) \geq m^*(J) > M$ for all $M > 0$. This then implies that $m^*(I) = \infty$, which is the length of I, $\ell(I)$. $\square$

Proposition 27. *Let $\{A_n\}$ be a countable collection of sets of real numbers. Then*

$$m^*(\bigcup A_n) \leq \sum m^*(A_n)$$

In other words, m^* is 'countably sub-additive'.

Proof. For A_n, there exists a cover $\{I_n, i\}_{i=1}$ by open intervals such that

$$m^*(A_n) \geq \sum_i \ell(I_n, i) - \frac{\epsilon}{2^n}$$

So

$$\sum m^* A_n \geq \sum_n \left(\sum_i \ell(I_n, i) - \frac{\epsilon}{2^n} \right) = \left(\sum_n \sum_i \ell(I_n, i) \right) - \epsilon \geq m^*(\bigcup A_n) - \epsilon$$

Since

$$\sum_n \frac{\epsilon}{2^n} = \epsilon$$

Because $\{I_n, i\}_{n=1,i=1}^{\infty,\infty}$ is a cover of the union of all A_n. Notice that $\mathbb{N} \times \mathbb{N}$ is countable, therefore our set $\{I_n, i\}_{n=1,i=1}^{\infty,\infty}$ is countable. Let $\epsilon \to 0$, in which case you get the conclusion we wanted.

Just a s a remark, we need to consider the case when one $m^*(A_n) = \infty$ separately. However, it is a trivial case, we don't have anything to check. $\square$

Corollary 28. *If A is countable, then $m^*(A) = 0$. This follows from seeing that A is the countable union of singletons, and based on above, we have the fact that $m^*(A) = 0$.*
Corollary 29. *The set $[0, 1]$ is not countable. If it were, then $m^*[0, 1]$ would be 0- however, we know based on the fact that it is a closed interval that $m^*[0, 1] = 1$.*
Proposition 30. *Given any set A and any $\epsilon > 0$, there exists an open set O such that $A \subset O$ and $m^*(O) \leq m^*(A) + \epsilon$.*

There is a set G which is G_δ such that $A \subset G$ and

$$m^*(A) = m^*(G)$$

Proof. For homework! $\square$

1.18 Measurable Sets and Lebesgue Measure

Outer measure is countable sub-additive, as we showed, but not countably additive (in general). We restrict ourselves to a smaller family of subsets of $\mathbb{R}$ in order to get a measure.
Definition. A set E is said to be **measurable** if for each set A we have that

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

Remark. It is always true that $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C)$.

This follows directly from the fact that our measure is countably sub-additive. This tells us that E is measurable if and only if

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C)$$

for all $A \subset \mathbb{R}$.
Fact. If E is measurable, then E^C is measurable. This follows from the fact that $(E^C)^C = E$, and that addition is commutative.
Fact. The $\emptyset, \mathbb{R}$ are both measurable. This follows simply from plugging in these sets for E, and recognizing that at least one of the terms in our sum of m^* is zero.
Lemma 31. *If $m^*(E) = 0$, then E is measurable.*

Proof. We take $A \subset \mathbb{R}$. In this case,

$$A \cap E \subset E = \qquad A \cap E^C \subset A$$

So,

$$m^*(A \cap E) + m^*(A \cap E^C) \leq m^*(E) + m^*(A) = m^*(A)$$

Using the fact that $m^*(E) = 0$. This implies that E is measurable. □

Lemma 32. *If E_1 and E_2 are measurable, then $E_1 \cup E_2$ is measurable.*

Proof. Take $A \subset \mathbb{R}$, some arbitrary set. Since both E_2 is measurable, we have that

$$m^*(A \cap E_1^C) = m^*(A \cap E_1^C \cap E_2) + m^*(A \cap E_1^C \cap E_2^C)$$

Just as a remark: notice that

$$A \cap E_1^C \cap E_2^C = A \cap (E_1 \cup E_2)^C$$

Now looking at $A \cap (E_1 \cup E_2)$, we have that

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^C)$$

So,

$$m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^C)$$

(by the additivity of countable sets). Looking at the whole thing,

$$m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C) \leq m^*(A \cap E_1) + m^*(A \cap E_1^C \cap E_2) + m^*(A \cap E_1^C \cap E_2^C)$$

Now, what we really have is

$$= m^*(A \cap E_1) + m^*(A \cap E_1^C)$$

because E_2 is measurable. Now using the fact that E_1 is measurable,

$$= m^*(A)$$

This then implies that $E_1 \cup E_2$ is measurable. □

Corollary 33. *The family $\mathcal{M}$ of all measurable sets is an algebra.*

Lemma 34. *Let $A \subset \mathbb{R}$ be any set and let $E_1, ..., E_n$ be a finite sequence of disjoint measurable sets. Then,*

$$m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof. The poof follows by induction on n. It is true for $n = 1$, and we assume that it is correct for $n - 1$, we need to show that it is true for n. Well we have

$$A \cap [\bigcup_{i=1}^{n} E_i] \cap E_n = A \cap E_n$$

Because E_i's are all disjoint. Another thing that is true is as follows:

$$A \cap [\bigcup_{i=1}^{n} E_i] \cap E_n^C = A \cap [\bigcup_{i=1}^{n-1} E_i]$$

By assumption E_n is measurable, which implies that

$$m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = m^*(A \cap E_n) + m^*(A \cap [\bigcup_{i=1}^{n-1} E_i])$$

Using our inductive hypothesis,

$$= \sum_{i=1}^{n} m^*(A \cap E_1)$$

□

Recall from the last class that we had the following Lemma:

Lemma 35. *Given E_1, E_2 measurable, $E_1 \cup E_2$ is measurable.*

As a corollary to this, we have

Corollary 36. *The collection $\mathcal{M}$ of all such sets is an algebra.*

We also had the following:

Lemma 37. *If A is any set, $E_1, ..., E_n$ are measurable and disjoint sets, then*

$$m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

We now have the following:

Theorem 38. *The collection $\mathcal{M}$ of measurable sets is a $\sigma-$algebra.*

Proof. Since $\mathcal{M}$ is an algebra, it is enough to show that a countable union of elements of $\mathcal{M}$ is again an element of $\mathcal{M}$. For example suppose that $\{A_n\}_{n=1}^{\infty}$ is in $\mathcal{M}$. We want to show that their union is in $\mathcal{M}$, that is:

$$\bigcup_{n=1}^{\infty} A_n = E$$

where E is measurable. We need to show that

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

for any set A such that $A \subseteq \mathbb{R}$. Since E is the union of our collection of A's, and since all such $A \in \mathcal{M}$ (and $\mathcal{M}$ is an algebra) then

$$E = \bigcup_{n=1}^{\infty} E_n$$

Such that each F_n is pairwise disjoint, and in $\mathcal{M}$. This can be constructed in the following way:

$$E_1 = A_1, \quad E_2 = A_2 - E_1, \quad E_3 = A_3 - (E_1 \cup E_2), ...$$

Now, let

$$F_n = \bigcup_{i=1}^{n} E_i \in \mathcal{M}$$

We also know that F_n^C contains E^C. From this we can write the following:

$$m^*(A) = m^*(F_n \cap A) + m^*(F_n^C \cap A)$$

By the definition of a measurable set. notice that this line,

$$m^*(A) = m^*(F_n \cap A) + m^*(F_n^C \cap A) \geq m^*(F_n \cap A) + m^*(A \cap E^C)$$

But notice that

$$m^*(F_n \cap A) + m^*(A \cap E^C) = m^*([\bigcup_{i=1}^{n} E_n] \cap A) + m^*(A \cap E^C)$$

Using our lemma above, we have that

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^C)$$

which is true for all $n \in \mathbb{N}$, that

$$m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^C)$$

What we do is let $n \to \infty$, from which we get

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^C)$$

Now, by countable sub-additivity, we have that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C)$$

This implies that E is measurable. $\square$

Lemma 39. *The interval (a, ∞) is measurable.*

Proof. Let $A \subseteq \mathbb{R}$. Let $A_1 = A \cap (a, \infty)$ and let $A_2 = A \cap (a, \infty)^C = A \cap (-\infty, a]$. We need to show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2)$$

Of $m^*(A) = \infty$, then we are done. If $m^*(A) \leq \infty$, then given $\epsilon > 0$ there exists some $\{I_n\}$ cover of A by open intervals such that

$$m^*(A) \geq \left(\sum_{n=1}^{\infty} \ell(I_n) - \right) \epsilon$$

Let $I_n' = I_n \cap (a, \infty)$ and let $I_n'' = I_n \cap (-\infty, a]$. If we have that

$$A_1 \subset \bigcup_{n=1}^{\infty} I_n'$$

then

$$m^*(A_1) \leq m^*(\bigcup_{n=1}^{\infty} I_n') \leq \sum_{n=1}^{\infty} m^*(I_n)$$

Similarly,

$$A_2 \subset \bigcup_{n=1}^{\infty} I_n''$$

tells us that

$$m^*(A_2) \leq \sum_{n=1}^{\infty} m^*(I_n'')$$

Thus,

$$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} [m^*(I_n') + m^*(I_n'')]$$

Notice that

$$m^*(I_n') = \ell(I_n'), \qquad m^*(I_n'') = \ell(I_n'')$$

Since both I_n' and I_n'' are intervals. Thus,

$$\sum_{n=1}^{\infty} [m^*(I_n') + m^*(I_n'')] \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \epsilon$$

since our choice of ϵ was arbitrary, we let $\epsilon \to 0$, in which case

$$m * (A_1) + m^*(A_2) \leq m^*(A)$$

$\square$

The theorem that follows from this lemma is as follows:
Theorem 40. *Every Borel set is measurable. In particular, each open and each closed set is measurable.*

Proof. Recall that $\mathcal{M}$ is a σ-algebra. Since (a, ∞) are measurable, then $(-\infty, a]$ are measurable. From this, it follows that sets of the form $(-\infty, b)$ are measurable as well. This follows from seeing that

$$(-\infty, b] = (-\infty, b) \cup \{b\}$$

and since both $(-\infty, b]$ and $\{b\}$ are measurable, thus $\{b\}^C$ is measurable, and since

$$(-\infty, b) = (-\infty, b] \cap \{b\}^c$$

our set is measurable.

More directly, notice that

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n})$$

and each of these sets is measurable. From this it would follows that $(-\infty, b)$ is measurable.

We see that as long as $a < b$, that

$$(-\infty, b) \cap (a, \infty) = (a, b)$$

It then follows that $(a, b) \in \mathcal{M}$, and that $\mathcal{M}$, a σ-algebra, contains all open intervals, and implies that $\mathcal{M}$ contains all Borel sets. □

Definition. If E is a measurable set, then the **Lebesgue measure** $m(E)$ is the outer measure $m^*(E)$.

Proposition 41. *Let $\{E_i\}$ be a sequence of measurable sets. Then,*

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

If the E_i's are pairwise disjoint, then

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

This is the property of **countable additivity**.

Proof. The first part of our proposition follows from countable sub-additivity of the outer measure.

Suppose we have the sequence $\{E_i\}_{i=1}^{\infty}$. Then

$$m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

$A = \mathbb{R}$ tells us that

$$m^*(\bigcup_{i=1}^{\infty}) = \sum_{i=1}^{n} m^*(E_i)$$

So

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$$

Thus

$$m^*(\bigcup_{i=1}^{\infty} E_i) \geq m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i)$$

From this we conclude that

$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{n} m(E_i)$$

Letting $n \to \infty$, we have that

$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$$

Which follows from seeing that "this is an infinite sequence (increasing) that is bounded above, so it must have a limit". □

Proposition 42. *Let $\{E_n\}_{n=1}^{\infty}$ be a infinite decreasing sequence of measurable sets. Let $m(E_i) < \infty$. Then,*

$$m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$$

We have that $\{E_n\}$ is decreasing if $E_n \supseteq E_{n+1}$ for all n.

Proof. We call

$$E = \bigcap_{i=1}^{\infty} E_i$$

Let

$$F_i = E_i - E_{i+1}$$

We have the following:

$$E_1 - E = \bigcup_{i=1}^{\infty} F_i$$

which are pairwise disjoint. Thus, according to countable additivity,

$$m(E_1 - E) = \sum_{i=1}^{\infty} m(F_i) = \sum_{i=1}^{\infty} m(E_i - E_{i+1})$$

Notice that

$$E_i = E_{i+1} \cup (E_i - E_{i+1}), \quad E_{i+1} \subset E_1$$

And that these two are disjoint. We get

$$m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$$

Since we are under the assumption that $m(E_i)$ is finite for all i, we have that

$$m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1})$$

So,

$$\sum_{i=1}^{\infty} m(E_i - E_{i+1} = \sum_{n=1}^{\infty}(m(E_i) - m(E_{i+1})) = \lim_{n\to\infty} \sum_{i=1}^{n}[m(E_i) - m(E_{i+1})] = \lim_{n\to\infty}[m(E_1) - m(E_{n+1})]$$

$$= m(E_1) - \lim_{n\to\infty} m(E_{n+1})$$

So,

$$m(E_1 - E) = m(E_1) - \lim_{n\to\infty} m(E_n)$$

Thus

$$m(E_1) - m(E_1 - E) = m(E) = \lim_{n\to\infty} m(E_n)$$

□

1.19 Exam

The exam will be on Thursday, the 29^{th}.

1.20 Measurable Sets (continued)

A set E is measurable if for all $A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

We said that if E is measurable, then the Lebesgue measure of E is given as

$$m(e) = m^*(E)$$

We said that if we have $\{E_n\}$ where $E_n \supseteq E_{n+1}$ where each E_n is measurable, then

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

This was all shown in our last class. We have the following now:
Proposition 43. *Let E be any set. The following are equivalent:*

1. *E is measurable*
2. *Given $\epsilon > 0$, there exists an open set $O \supseteq E$ such that*

$$m^*(O - E) < \epsilon$$

3. *Given $\epsilon > 0$ there exists a closed set $F \subset E$ such that*

$$m^*(E - F) < \epsilon$$

4. *There exists G in G_δ with $E \subset G$ and*

$$m^*(G - E) = 0$$

5. *There exists F in F_σ with $F \subset E$ and*

$$m^*(E - F) = 0$$

If $m^*(E) < \infty$, the above are equivalent to number (6):

6. *Given $\epsilon > 0$, there exists a finite union U of open intervals such that*

$$m^*(U \Delta E) < \epsilon \qquad ^*$$

Proof. Assume first that $m^*(E) < \epsilon$. We first want to show that (1) $\Rightarrow$ (2).

We know that $m^*(E)$ is finite, given $\epsilon > 0$ we can conclude that there exists a cover $\{I_n\}$ such that

$$\sum_{n=1}^{\infty} \ell(I_n) > m^*(E) + \epsilon$$

Since by definition,

$$m^*(E) = \inf_{E \subset \bigcup_{n=1}^{\infty} I_n} \sum \ell(I_n)$$

Let

$$O = \bigcup_{n=1}^{\infty} I_n,$$

which is an open set. O is measurable because each I_n is measurable (open sets are measurable). We also know that $E \subset O$. Then,

$$m(O) \leq m^*(E) + \epsilon$$

Which is true because $\{I_n\}$ is a cover of O, so

$$m(O) = m^*(O) \leq \sum \ell(I_n) \leq m^*(E) + \epsilon$$

Since E is measurable, we have that

$$m(O) = m^*(O) = m^*(O \cap E) + m^*(O \cap E^C)$$

But this is the same as saying

$$= m(E) + m(O - E)$$

Since $E \subset O$. From above, we have that

$$m(E) + m(O - E) \leq m(E) + \epsilon$$

Canceling out since $m(E)$ is finite, we have that

$$m^*(O - E) < \epsilon$$

$\square$

Now we want to show that (2) $\Rightarrow$ (4).

*This is the symmetric difference of sets

We have that there exists O_i open, such that $E \subset O_i$ and

$$m^*(O_i - E) < \frac{1}{i}, \quad \forall i = 1, 2, 3....$$

We now take $\bigcap_{i=1}^{\infty} O_i = G$, and we notice that G is G_δ. We know that $G \supset E$, because $O_i \supset E$ for all i. So we need to look at

$$m^*(G - E)$$

and need to show that this outer measure is 0. We know that $G \subset O_i$, which implies that

$$G - E \subset O_i - E, \qquad \forall i$$

Which tells us that

$$m^*(G - E) \leq m^*(O_i - E) < \frac{1}{i} \to 0, i \to \infty$$

So by letting $i \to \infty$,

$$m^*(G - E) = 0$$

□

We now want to show that (4) ⇒ (1). We have that $G \supset E$, and that $m^*(G - E) = 0$. This in itself implies that $G - E$ is measurable, and we know that G is measurable because it is a Borel set. We can write the set E as follows:

$$E = G - (G - E)$$

Since both G and $G - E$ are measurable, their difference is measurable: E is measurable. □

Now we want to show that (1) ⇒ (2). We now assume that $m^*(E) = \infty$. For all covers $\{I_n\}$ of E by open intervals,

$$\sum_{n=1}^{\infty} \ell(I_n) = \infty$$

We have a set $E_n = [-n, n]$. E_n has finite outer measure $m^*(E_n) < \infty$, since E_n is a subset of $[-n, n]$ which has finite order $(m^*([-n, n]) = 2n)$.

We know that given $\epsilon > 0$, there exists O_n, open, such that $O_n \supset E_n$, and $m^*(O_n - E_n) < \epsilon$. Let $O = \bigcup_{n=1}^{\infty} O_n$. Then $O \supset O_n \supset E_n$ for all n. This implies that $O \supset E = \bigcup_{n=1}^{\infty} E_n$. We now have to look at $O - E$. We claim that

$$O - E \subseteq \bigcup_{n=1}^{\infty} [O_n - E_n]$$

By countable sub-additivity,

$$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n)$$

We change our trick above to saying that instead, $m^*(O_n - E_n) < \frac{\epsilon}{2^n}$: so we have

$$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

□

We now want to prove that (2) $\Rightarrow$ (4). For all $i > 0$, there exists $O_i \supseteq E$, O_i open such that

$$m^*(O_i - E) < \frac{1}{i}$$

Then, we define $G = \bigcap_{i=1}^{\infty} O_i$, which is a G_δ set. It is true that $G \supset E$, and we have that

$$m^*(G - E) \leq m^*(O_i - E) < \frac{1}{i}, \quad \forall i$$

So let $i \to \infty$, in which case $\frac{1}{i} \to 0$, and we have that $m^*(G - E) = 0$. $\square$

As before, the proof that (4) $\Rightarrow$ (1) follows in the same manner. Thus we have

$$(1) \iff (2) \iff (4)$$

For an arbitrary E. Now let us see that (1) $\Rightarrow$ (3): We have that E is measurable, this implies that E^C is measurable. We use that (1) $\Rightarrow$ (2) for E^C (in other words, E^C satisfies (2)). Given $\epsilon > 0$, there exists $O \supseteq E$, where O is open such that

$$m^*(O - E^C) < \epsilon$$

So, $O \supseteq E^C$. Thus, $O^C \supset E$. We define $F = O^C$. Thus, $O - E^C \supseteq E - O^C = E - F$. What we're really saying is that $O \cap E = E \cap O$. Thus, (1) $\Rightarrow$ (3). $\square$

We now claim that (3) $\Rightarrow$ (5). For all i, let $F_i \subseteq E$, F_i closed such that

$$m^*(E - F_i) \leq \frac{1}{i}$$

Then, we define

$$F = \bigcup_{i=1}^{\infty} F_i$$

Which is a F_σ set (it is a countable union of closed sets). We have that

$$E - F \subset E - F_i \leq \frac{1}{i} \quad \forall i$$

Thus, we let $i \to \infty$, so

$$m^*(E - F) = 0$$

$\square$

We want to show now that (5) $\Rightarrow$ (1).

We have that $E - F$ is measurable, because $m^*(E - F) = 0$. F is a Borel set because it is a countable union of closed sets (i.e., f_σ-sets), and is thus measurable. We then notice that

$$E = (E - F) \cup F$$

And therefore, E is measurable. In total at this point, we have that

$$(1) \iff (2) \iff (3) \iff (4) \iff (5)$$

For (6) recall first that

$$U \Delta E = (U - E) \cup (E - U) = (U \cup E) - (U \cap E)$$

We will discuss (6) next class. $\square$

1.21 Measurable Functions

As a nice reminder, see if you can figure out why

$$[0,1] - \mathbb{Q}$$

Is measurable, and of measure 0.

We would like to prove property (iii) for $[0,1] - \mathbb{Q}$ without referring to our proposition from last section.

Proposition 44. *Let f be an extended real-valued function whose domain is a measurable set, which we can think of it as*

$$f : D \to \overline{\mathbb{R}} \cup \{+\infty, -\infty\}$$

(An extended real valued function is a function that can handle $\pm\infty$.) Then the following are equivalent:

1. *For each $\alpha \in \mathbb{R}$, the set*
$$\{x \in D : f(x) > \alpha\}$$
Is measurable.
2. *For each $\alpha \in \mathbb{R}$ the set*
$$\{x : f(x) \geq \alpha\}$$
Is measurable.
3. *For each $\alpha \in \mathbb{R}$ the set*
$$\{x : f(x) < \alpha\} = f^{-1}([-\infty, \alpha))$$
Is measurable
4. *For each $\alpha \in \mathbb{R}$ the set*
$$\{x : f(x) \leq \alpha\} = f^{-1}([-\infty, \alpha])$$
Is measurable. In addition, these statements imply statement (5).
5. *For all extended real numbers α,*
$$\{x : f(x) = \alpha\}$$
Is measurable.

Proof.
$(1 \Rightarrow 4)$ Let $\alpha \in \mathbb{R}$, we see that

$$\{x : f(x) \leq \alpha\} = D - \{x : f(x) > \alpha\}$$

Since our sets $\{x : f(x) > \alpha\}$ and D are measurable by assumption, $\{x : f(x) \leq \alpha\}$ is measurable.

$(4 \Rightarrow 1)$ This is the reverse of our previous argument

$(2 \Rightarrow 3), (3 \Rightarrow 2)$ These proofs follows exactly as the proof for $(1 \Rightarrow 4)$.

(1 ⇒ 2) Notice that

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$$

and equality follows since $x \in (\bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\})$ if $f(x) > \alpha - \frac{1}{n}$ for all n. So we let $n \to \infty$, and we have the set $\{x : f(x) \geq \alpha\}$. Since we have the countable intersection of (assumed) measurable sets, we conclude our proof.

(2) ⇒ (1)

Notice that

$$\{x : f(x) > \alpha\} \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + \frac{1}{n}\}$$

This proof follows as the one above.

$(1 \iff 2 \iff 3 \iff 4 \iff 5)$

In case one, suppose that $\alpha \in \mathbb{R}$.

$$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$$

In case two, we have that $\alpha = \pm\infty$. Then,

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$$

and

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

which is measurable by (3). □

Definition. An extended real valued function f is (Lebesgue) measurable if its domain is a measurable set and one of the four conditions hold (from the proposition above). In other words, for all $\alpha \in \mathbb{R}$

$$\{x : f(X) > \alpha\}$$

Is measurable.
Remark. A continuous function with a measurable domain is a measurable function.

Proof. For the sake of example, suppose $f : \mathbb{R} \to \mathbb{R}$. Then

$$\{x : f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

Is open, because f is continuous. Therefore, this set is measurable, and thus f is measurable.

If $f : D \to \mathbb{R}$, where D is a measurable set Recall that $IU \subseteq D$ is open in D if there exists V open in $\mathbb{R}$ such that $U = D \cap V$.

This means that $f^{-1}((\alpha\infty))$ is open in D, so there exists V_α open in $\mathbb{R}$ such that

$$f^{-1}((\alpha, \infty)) = D \cap V_\alpha$$

Since D is open by assumption and V_α is open (and thus measurable), their intersection is open. □

Definition. The function $\varphi : [a, b] \to \mathbb{R}$ is a **step function** if there exists a subdivision

$$x_0 = a < x_1 < x_2 < \dots < x_n < b$$

and $c_i \in \mathbb{R}$ for $i = 1, \dots, n$ such that

$$\varphi(x) = c_i$$

For all $x_{i-1} < x < x_i$, for $i = 1, 2, \dots, n$.
Claim. Step functions are measurable; if $\varphi : [a, b] \to \mathbb{R}$ is a step function, then it is measurable.

Proof. We want to look at

$$\varphi^{-1}((\alpha, \infty))$$

Take $\alpha \in \mathbb{R}$. We want to show that this set is measurable. Notice that this set is a finite union of intervals and singletons- so it is a Borel set. □

Remark. It f is a measurable function, and E a measurable subset of the domain of f, then $f|_E$ is measurable [†].

Proof. Taking $\alpha \in \mathbb{R}$, We want to show that

$$(f|_E)^{-1}((\alpha, \infty))$$

Is measurable. Notice that we can write this set as :

$$\{x \in E : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cap E$$

These two are measurable, since f is a measurable function, and E is measurable by assumption. Therefore, their intersection is measurable.

□

Proposition 45. *Let c be a constant, and let f and G be two measurable real-valued functions. Then,*

$$f + c, \quad f + g, \quad cf, \quad f - g, \quad f \cdot g$$

Are measurable

Proof. Suppose we had $f(x) + c$. We want to examine the set

$$\{x : f(x) + c > \alpha\}$$

Let $\alpha \in \mathbb{R}$, we want to show that this set is measurable. Well, we have that

$$\{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\}$$

Which is measurable, since f is measurable. Thus, $f(x) + c$ is measurable.

Now suppose we had $f(x) + g(x)$. Fixing some $\alpha \in \mathbb{R}$, we look at

$$\{x : f(x) + g(x) > \alpha\}$$

[†] this is 'f restricted to E'

Notice that we can write $f(x) + g(x) > \alpha$ as $f(x) > \alpha - g(x)$. Using the fact that between any two real numbers there is a rational number, we know that there exists $r \in \mathbb{Q}$ such that

$$f(x) > r > \alpha - g(x)$$

Since the rational numbers are countable, we can write our set as:

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x : f(x) > r\} \cap \{g(x) > \alpha - r\})$$

The two sets are measurable since f and g are measurable functions, and since a countable union of measurable sets is measurable, we are done.

Supposing now that we have $cf(x)$. Our first case is that $c > 0$, then

$$\{x : f(X) > \frac{\alpha}{c}\}$$

If $c < 0$, then we have that

$$\{x : f(x) < \frac{\alpha}{c}\}$$

And if $c = 0$ we have

$$\{x : 0 > \alpha\}$$

Notice that this set is either empty, or the entire domain- either way, it is measurable. □

1.22 Homework

Page 70 question 18

1.23 Continued

Proposition 46. *If f, g are measurable and $c \in \mathbb{R}$, then*

$$f + c, f + g, cf, g - f$$

are all measurable.

Proof. We have that

$$g - f = g + (-1)g$$

How can we show that fg is measurable? Well, we want to show that f^2 is measurable, so for $\alpha \in \mathbb{R}$ we have that

$$\{x : [f(x)]^2 > \alpha\}$$

If $\alpha > 0$, thne we have the inequality $f(x) > \sqrt{\alpha}$, which is equivalent to saying

$$f(x) < -\sqrt{\alpha}$$

So we have
$$\{x : [f(x)]^2 > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \bigcup \{i : f(x) < -\sqrt{alpha}\}$$
And since f is measurable, this union is measurable. Alternatively if $\alpha = 0$, we have

$$\{x : [f(x)]^2 = 0\} = \{x : f(x) = 0\}$$

which is measurable, and if $\alpha < 0$ then we have

$$\{x : [f(x)]^2 < \alpha\} = \text{Domain}$$

Similarly,
$$\frac{1}{2}(f + g)^2 - f^2 - g^2 = fg \text{ which is measurable}$$
This shows that all these algebraic combinations are really measurable.

□

Theorem 47. *Let $\{f_n\}$ be a sequence of measurable functions with the same domains. Then the functions*
$$\inf_n \{f_1, ..., f_n\}, \qquad \sup_n \{f_1, ..., f_n\}$$
are measurable. From this we also have that

$$\overline{\lim_{n\to\infty}} f_n, \qquad \underline{\lim_{n\to\infty}} f_n$$

are also measurable.

Proof. Suppose that
$$h(x) = \inf\{f_1(x), f_2(x), .., f_n(x)\}$$
If we then look at the set
$$\{x : h(x) > \alpha\}$$
for $\alpha \in \mathbb{R}$, we need to show that this set is measurable. Notice that we can do the following:

$$\{x : h(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

Each of these sets is measurable, and thus their intersection is measurable. Similarly, suppose

$$g(x) = \inf_n f_n(x) = \bigcup_{n=1}^{\infty} (\cap_{i=1}^{\infty} \underbrace{\{x : f_i(x) < \alpha + \frac{1}{n}\}}_{A_n})$$

From this, we have that if $x \in A_n$, then

$$\inf_i f_i(x) \geq \alpha + \frac{1}{n}$$

Now, we examine why the following set is measurable:

$$\{x : \overline{\lim_{n\to\infty}} f_n(x) > \alpha\}$$

This follows from seeing that

$$\overline{\lim_{n\to\infty}} f_n(x) = \inf_n \sup_{k\geq n} f_k(x)$$

If we call $\sup_{k\geq n} f_k(x) = g_n(x)$, we have that each $g_n(x)$ is measurable by what we showed above. Taking the infimum over these measurable functions, we have something measurable.

□

Definition. A property holds almost everywhere ($a.e$) if the set of points were it fails to hold is a set of measure 0.

For example, $f = g$ a.e. if f and g have the same domain and the measure of $\{x : f(x) \neq g(x)\}$ is 0.

We say that f_n converges to g almost everywhere if there exists a set E with $m(E) = 0$ such that

$$f_n(x) \to g(x) \text{ as } n \to \infty$$

For $x \notin E$.

Proposition 48. *If f is a measurable function and $f = g$ almost everywhere, then g is a measurable function.*

(for example: the characteristic function χ (look up the definition) has the following property:

$$\chi_{\mathbb{R}} = \chi_{\mathbb{R}-\mathbb{Q}} \textit{almost everywhere}$$

because $m(\mathbb{Q}) = 0$.)

Proof. Let $E = \{x : f(x) \neq g(x)\}$. Then $m(E) = 0$, by definition. Then for $\alpha \in \mathbb{R}$,

$$\{x : g(x) > \alpha\} = [\{\{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}] - \{x \in E : g(x) \leq \alpha\}$$

Each one of these sets is measurable. □

Proposition 49. *Let $f : [a, b] \to \mathbb{R}$ be measurable, and assume that $m\{x : f(x) = \pm\infty\} = 0$. Then given $\epsilon > 0$, a step function g and a continuous function h such that*

$$|f - g| < \epsilon \quad \textit{and} \quad |f - h| < \epsilon$$

except on a set of measure $< \epsilon$. If, in addition, $m \leq f(x) \leq< M$, then we can choose g and h such that

$$m \leq g \leq M, \quad m \leq h \leq M$$

Proof. We have a function $f : [a, b] \to \mathbb{R}$ and a set $E = \{x : f(x) = \pm\infty\}$, where $m(E) = 0$. So, let's just worry about

$$f : [a, b] - E \to \mathbb{R}$$

From this, we can assume that f is finite. Since E is measure 0, it doesn't affect anything.

Claim. There exists $M > 0$ such that $|f(x)| \leq M$ except on a set F of measure $< \frac{\epsilon}{3}$.

Proof Of Claim. We let $C_n = \{x : n-1 \leq |f(x)| < n\}$. Then C_n's are disjoint, and $\bigcup_{n=1}^{\infty} C_n$ is the domain of f. We then have

$$m(\bigcup_{n=1}^{\infty} C_n) = \sum_{n=1}^{\infty} m(C_n)$$

Then, there exists $N : \sum_{n=N+1}^{\infty} m(C_n) < \frac{\epsilon}{3}$. So, we take

$$f = \bigcup_{n=1}^{N} C_n, \qquad M = N; \text{ for } x \in F, |f(x)| < N$$

So

$$D - F = \bigcup_{n=N+1}^{\infty} C_n$$

□

Definition.

$$\varphi : [a,b] \to \mathbb{R}$$

Is a simple function if it is measurable, and assumes only a finite number of values.

Example. $\chi_{\mathbb{Q}}$ is simple, but not a step function. If $\{c_1, ..., c_n\}$ are the values of a simple function φ, then

$$\varphi(X) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

Where $A_i = \{x : P\varphi(x) = c_i\} = f^{-1}(\{c_i\})$.

Claim. (assume f is bounded) There is a simple function φ such that $|f - \varphi| < \epsilon$ outside of a set of measure $< \frac{\epsilon}{3}$.

Proof of Second Claim. We know that

$$|f(x)| \leq M \forall x$$

We know that there exists a$n \in \mathbb{N}$ such that

$$\frac{M+1|}{n} < \epsilon$$

We then look at the set

$$A_i = \{x : f(x) \in (i \cdot \frac{M+1}{n}, (i+1)\frac{M+1}{n})\}$$

We can then write

$$\varphi(x) = \sum_{i=-n}^{n} i \frac{M+1}{n} \chi_{A_i}(x)$$

Which is a simple function, and we can say that

$$|f(x) - \varphi(x)| < (i+1)\frac{M+1}{n} - (i)\frac{M+1}{n} = \frac{M+1}{n} < \epsilon$$

Going from a simple to a step function, we then say that there exist finitely many intervals $I_j, j = 1, 2, 3, ...k_i$ suc that

$$m(A_i \Delta \bigcup I_j) > \frac{\epsilon}{6n}$$

Then we have

$$\psi(x) = \sum_{i=-n}^{n} i\frac{M+1}{n} \sum_{j=1}^{k_i} \chi_{I_j}$$

which is a step function. We have that

$$\psi(x) = \varphi(x) \text{ outside of } \bigcup_{i=-n}^{n} (A_i \Delta \bigcup_{j=1}^{k_i} I_j)$$

And we have that

$$m(\bigcup_{i=-n}^{n} (A_i \Delta \bigcup_{j}^{k_i} I_j)) \leq \sum_{i=-n}^{n} m((A_i \Delta \bigcup_{j}^{k_i} I_j) \leq 2n \cdot \frac{\epsilon}{6n} = \frac{\epsilon}{3}$$

□

□

1.24 Littlewood's Three Principles

1. Every measurable set is "nearly" the union of finitely many intervals.
2. Every measurable function is "nearly" continuous
3. Every convergent sequence of measurable functions is "nearly" uniformly convergent.

Proposition 50. *Let E be a measurable set with $m(E) < \infty$ and $\{f_n\}$ a sequence is measurable function defined on E. Let $f : E \to \mathbb{R}$ such that for all $x \in E$, $f_n(x) \to f(x)$ as $n \to \infty$. Then, given $\epsilon > 0$ and $\delta > 0$, there exists $A \subset E$ (measurable) with $m(A) < \delta$, and there exists $N \in \mathbb{N}$ such that for all $x \notin A$ and $\forall n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon$$

Remark. Notice that $f_n \rightrightarrows f$ on $E - A$.

Proof. Define a set $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$. We then let $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$ for some $n \geq N$. Notice that $E_{n+1} \subset E_n$. Notice that $m(E_n) < \infty$ for all n, since $m(E) < \epsilon$. Then,

$$\bigcup_{N=1}^{\infty} E_N = \emptyset$$

because $\lim_{n\to\infty} f_n(x) = f(x)$ for all $\epsilon > 0$, there exists $N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$, for all $n \geq N(x, \epsilon)$.

Recall: For $\{A_n\}$ measurable, $m(A_1) < \infty$, $A_{n+1} \subset A_n$ for all n, then

$$m(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$$

Applying this fact to what we have here,

$$0 = m(\bigcap_{N=1}^{\infty} E_n) = \lim_{N\to\infty} m(E_n)$$

This implies that there exists N_0 such that $m(E_{N_0}) < \delta$, and $A := E_{N_0}$. $\square$

1.25 Homework

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1.26 Continued

Definition. If there exists $B \subset E$ with $m(B) = 0$ such that $f_n \to f$ point-wise on $E - B$, then $f_n \to f$ almost everywhere.

Proposition 51. *Let E be measurable with $m(E) < \infty$ and $\{f_n\}$ a sequence of measurable functions that converge to f almost everywhere on E. Then given $\epsilon > 0$ and $\delta > 0$, there exists $A \subset E$ with $m(A) < \delta$ and there exists $N \in \mathbb{N}$ such that for all $x \in A$ and for all $n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon$$

Chapter 2

The Lebesgue Integral

2.1 The Riemann Integral

We assume $f : [a, b] \to \mathbb{R}$, a bounded function. There exists some $M > 0, |f(x)| < M$ for all $x \in [a, b]$.

Subdivision: $a = C_0 < C_1 < C_2 < ... < C_n = b$, and

$$S = \sum_{i=1}^{n} (C_i - C_{i-1}) M_i$$

And

$$s = \sum_{i=1}^{n} (C_i - C_{i-1}) m_i$$

Where

$$M_i = \sup_{C_{i-1} < x \leq C_i} f(x), \qquad m_i = \inf_{C_{i-1} < x \leq C_i} f(x$$

The upper Riemann integral

$$R \overline{\int_a^b} f(x) dx = \inf S$$

and the lower Riemann integral:

$$R \underline{\int_a^b} f(x) dx = \sup s$$

Definition. When the upper and the loewr Riemann integral of f are equal, then f is **Riemann Integrable**.

2.2 Step Functions

A step function has the property that $\phi(x) = c_i$, $\mathcal{C}_{i=1} < x < C_i$. Then

$$\int_a^b \psi(x) dx = \sum_{i=1}^{n} c_i (C_i - C_{i-1})$$

We can set

$$R\overline{\int_a^b} f(x)dx = \inf_{\psi(x)\geq f(x), \text{ is a step function}} \int_a^b \psi(x)dx$$

We say that

$$S = \int_a^b \psi(x)dx$$

Similarly

$$R\underline{\int_a^b} f(x)dx = inf_{\varphi(x)\leq f(x) \text{ is a step function}} \int_a^b \varphi(x)dx$$

2.3 Lebesgue Integration

The Lebesgue Integral of a bounded function over a set of finite measure

Given $\varphi(x)$, a simple function that assumes finitely many values is measurable. We can write it as follows:

$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$

Which is a representation that is not unique. If $\varphi(x)$ assumes non-zero values $\{c_1, ..., c_k\}$, let $A_i = \varphi^{-1}(\{c_i\}) = \{x : \varphi(x) = c_i\}$. Then we have

$$\varphi(x) = \sum_{i=1}^{k} c_i \chi_{A_i}(x)$$

This is a unique representation.

If φ vanishes outside a set of finite measure,

$$\int \varphi(x)dx = \sum_{i=1}^{k} c_i m(A_i)$$

If E is any measurable set, then

$$\int_E \varphi(x)dx = \int \varphi(x)\chi_E(x)dx$$

Lemma 52. *Let*

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

With $E_i \cap E_j = \emptyset$ for $i \neq j$. Suppose each E_i is measurable with finite measure. Then

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i m(E_i)$$

Proof. The set $A_a = \{x : \varphi(x) = a\} = \bigcup_{a_i=a} E_i$. Then,

$$a \cdot m(A_a) = \sum_{a_i=a} a_i m(E_i)$$

Therefore

$$\int \varphi(x)dx = \sum a \cdot m(A_a) = \sum_{i=1}^{n} a_i m(E_i)$$

□

2.3.1 Additional homework

Show that $\chi_A(x), A = [0,1] - \mathbb{Q}$ is not Riemann Integrable.

Whenever E is measurable,

$$\chi_E(x) = 1(x \in E), 0(x \notin E)$$

should be measurable.

2.4 The Lebesgue Integral of a Bounded Function over a set of Finite Measure

We had that if $\phi(x)$ is a simple function, then

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

Where $A_i = \varphi^{-1}\{a_i\}$, and $a_i \neq 0$, and take all $\{a_1, a_2, .., a_n\}$ non-zero values of φ, we have that $A_i \subset \mathbb{R}$, and we have that $\varphi^{-1}\{a_i\}$ is measurable, since φ is measurable. This is called our 'canonical representation'.

We could have written the following:

$$\varphi(x) = \sum_{i=1}^{k} c_i \chi_{E_i}(x)$$

but unless you know something especially helpful about E, this isn't a great idea. We had the following last time:

Lemma 53. *We said let $\varphi = \sum c_i \chi_{E_i}$ with $E_i \neq E)j$ when $i \ngeq j$, then*

$$\int \varphi = \sum c_i m(E_i)$$

We had the following definition:

Definition.

$$\int_a^b \varphi(x)dx = \sum_{i=1}^{n} a_i m(A_i)$$

Proposition 54. *Let φ and ψ be simple functions which vanish outside a set of finite measure. Then,*

$$\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$$

and if $\varphi \geq \psi$ almost everywhere, then

$$\int \varphi \geq \int \psi$$

Proof. Take $\{A_i\}$, $\{B_j\}$ sets occurring in canonical decompositions for φ and ψ. Let A_0 and B_0 be sets where φ and ψ are 0. Let E_k be the pairwise intersections of families $\{A_n\} \cup \{A_0\}$ and $\{B_j\} \cup \{B_0\}$. There are finitely many E_k's, and each is measurable. Suppose now that we want to write φ as follows:

$$\varphi = \sum c_k \chi_{E_k}$$

And we do the same with ψ:

$$\psi = \sum d_k X_{E_k}$$

In this case,

$$a\varphi + b\psi = \sum_k (ac_k + bd_k)\chi_{E_k}$$

so,

$$\int (a\varphi + b\psi) = \sum_k (ac_k + bd_k)m(E_k) = a\sum C_k m(E_k) + b\sum d_k m(E_k) = a\int \varphi + b\int \psi$$

We also have that $\varphi \geq \psi \Rightarrow c_k \geq d_k \forall k$, then

$$\int \varphi = \sum e_k m(E_k) \geq \sum d_k m(E_k) = \int \psi$$

□

Remark. It follows that if $\varphi - \sum a_i \chi_{A_i}$, then

$$\int \varphi = \sum a_i m(A_i)$$

without the assumption that $A_i \cap A_j = \emptyset$.
Definition. Suppose we have $f : E \to \mathbb{R}$ where f is bounded and measurable, we would like to say that if

$$\inf_{\psi \geq f} \int_E \psi \stackrel{?}{=} \sup_{\varphi \leq f} \int_e \varphi$$

then f is Lebesgue integrable, where φ and ψ are integrable functions.
Proposition 55. *Let f be a bounded function on a measurable set E with $m(E) < \infty$. Then,*

$$\inf_{f \leq \psi} \int_e \psi(x)dx = \sup_{\varphi \leq f} \int_E \varphi(x)dx$$

for all simple functions φ, ψ if and only if f is measurable on E.

Proof. Suppose that $|f| \leq M$, and f is measurable. Let $E_k = \{x \in E : \frac{kM}{n} \geq f(x) > \frac{k-1}{n}M\}$, $-n \leq k \leq n$. These sets are measurable, disjoint, and their union is E. We then have

$$\sum_{k=-n}^{n} m(E_k) = m(E)$$

and we define $\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_k}(x)$ which is the upper bound, and we have

$$\varphi_n(X) = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\chi_{E_k}(x)$$

as a lower bound, where these functions are both simple. Then we look at

$$\inf_{f\leq\psi}\int_e \psi(x)dx \leq \int_e \psi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} k\cdot m(E_k)$$

and

$$\sup_{\varphi\leq f}\int_E \varphi(x)dx \geq \int_E \varphi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n}(k-1)m(E_k)$$

Looking then at

$$\begin{aligned} 0 &\leq \inf_{f\leq\psi}\int_E \varphi(x)dx - \sup_{\varphi\leq f}\int_E \varphi(x)dx \\ &\leq \int_E \psi_n(x)dx - \int_E \varphi(x)dx = \frac{M}{n}\sum_{k=-n}^{n} m(E_k) \\ &= \frac{M}{n}m(E) \to 0 \text{ as } n\to\infty \end{aligned}$$

Now going in the other direction, we assume that $\inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{\varphi\leq f}\int_E \varphi(x)dx$. Then for all $n\in\mathbb{N}$, there exist simple functions π_n, φ_n such that

$$\varphi_n(x) \leq f(x) \leq \psi_n(x) \quad \forall x\in E$$

and

$$\int \psi(x)dx - \int \varphi_n(x) < \frac{1}{n}$$

We then define

$$\psi_*(x) := \inf_n \psi_n(x), \qquad \varphi_*(x) = \sup_n \varphi_n(x)$$

which are both measurable functions. We have that

$$\varphi_*(x) \leq f(x) \leq \psi_*(x)$$

And we want to look at the set

$$\Delta = \{x : \varphi_*(x) < \psi_*(x)\}$$

We represent it as

$$\Delta = \bigcup_{\mu\in\mathbb{N}} \underbrace{\{x : \varphi_*(x) < \psi_*(x) - \frac{1}{\mu}\}}_{\Delta_\mu}$$

Now, we have that

$$\int_E \psi_n - \int_E \varphi_n < \frac{1}{n}$$

This is really the same as

$$\int_E (\psi_n - \varphi_n)$$

Which is a simple, non-negative function–whose integration is therefore non-negative.

Remark. We have $\varphi \geq 0$, which is a simple function. Then,

$$\varphi(x) = \sum c_i \chi_{E_i}, c_i > 0$$

Then,

$$\int_E \varphi(x)dx = \sum c_i m(E_i) \geq 0$$

So, we can do the following:

$$\int_{\Delta_\mu \cap E} \varphi(x)dx = \sum c_i m(E_i \cap \Delta_\mu) \leq \sum c_i m(E_i) = \int_E \varphi(x)dx$$

By this remark, we have that

$$\int_{\Delta_\mu} (\psi_n - \varphi_n) \leq \int_E (\psi_n - \varphi_n) \leq \int_{\Delta_\mu} (\psi_n - \varphi_n) < \frac{1}{n}$$

So we have that

$$\frac{1}{\mu} m(\Delta_\mu) \leq \int_{\Delta_\mu} (\psi_n - \varphi_n) < \frac{1}{n}$$

Which tells us that

$$m(\Delta_\mu) < \frac{1}{n}$$

Letting $n \to \infty$, we get that $m(\Delta_\mu) \leq 0$, which implies that $m(\Delta_\mu) = 0$. From this, countable sub-additivity, we have that $m(\Delta) = 0$. Therefore $f = \psi_*$ on $E - \Delta$, and since Δ has measure zero, we have that f is measurable. □

Definition. If f is a bounded and measurable function defined on a measurable set E with $m(E) < \infty$, then f is **Lebesgue integrable** and

$$\int_E f(x)dx = \inf_{f \leq \psi} \int_E \psi(x)dx$$

Where the infimum is over all simple functions $\psi \geq f$.

Proposition 56. *Let f be a bounded function on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is measurable and*

$$R \int_A^b f(x)dx = L \int_a^b f(x)dx$$

(R and L here represent 'Riemann' and 'Lebesgue').

Proof. Since every step function is a simple function, we have that

$$R\underline{\int_a^b} f(x)dx \leq \sup_{\varphi \leq f} \int_a^b \varphi(x)dx \leq \inf_{f \leq \psi} \int_a^b \psi(x)dx \leq R\overline{\int_a^b} f(x)dx$$

Where φ, ψ are simple. Equalities hold everywhere, because f is Riemann integrable. This implies that f is Lebesgue integrable. □

Proposition 57. *If f and g are bounded, measurable functions defined on E with $m(E) < \infty$, then*

1. $\int(af + bg) = a\int f + b\int g$
2. *If $f = g$ almost everywhere, then $\int f = \int g$*
3. *If $f \leq g$ almost everywhere, then $\int f \leq \int g$. In particular, this implies that $|\int_E f| \leq \int_E |f|$.*
4. *If $A \leq f(x) \leq B$, for all $x \in E$, then*
$$A \cdot m(E) \leq \int_E f \leq B \cdot m(E)$$
5. *If A and B are disjoint, measurable subsets of E, then*
$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof. 1. We will first try to show that $\int af = a\int f$. We know that if $a > 0$, then
$$\int af = \inf_{f\leq\psi} \int_E a\psi = a \cdot \inf_f \int_E \psi = a\int_E f$$
Alternatively, if $a < 0$, then
$$\int_E af = \inf_{\varphi\leq f\equiv a\varphi\leq af} \int a\varphi = a \sup_{\varphi\leq f} \int_E \varphi = a\int_E f$$
Where ψ, φ are simple. Now looking at $f + g$, we choose simple functions ψ_1, ψ_2 such that $f \leq \psi_1, g \leq \psi_2$, from which it follows that $f + g \leq \psi_1 + \psi_2$. In this case,
$$\int_E (f + g) \leq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2$$
This implies that
$$\int_E (f + g) \leq \int_E f + \int_E g$$
By taking the infimums for all $\psi_1 \geq f, \psi_2 \geq g$ in the previous equation. Now going from below, we have that $\varphi_1 \leq f, \varphi_2 \leq g$, which are simple functions. In this case, $\varphi_1+\varphi_2 \leq f+g$, and so
$$\int_E \varphi_1 + \int_E \varphi_2 = \int_E (\varphi_1 + \varphi_2) \leq \int_E (f + g)$$
Taking the supremums over all φ_1, φ_2, we have that
$$\int_E f + \int_E g \leq \int_E (f + g)$$
Which proves (1).

2. We need to show that if $f = 0$ almost everywhere, then $\int_E f = 0$. Recall that $f = g$ almost everywhere is equivalent to saying that $f - g = 0$ almost everywhere. We showed in (1) that integration can be taken over sums, so we have that $\psi \geq f$, ψ simple, implies that
$$\psi \geq 0 \text{ almost everywhere.}$$

In this case,
$$\int \psi \geq 0 \Rightarrow \int_E f \geq 0$$
We have that $\varphi \leq f$, and φ is simple. Since $\varphi \leq 0$ almost everywhere,
$$\int_E \varphi \leq 0, \quad \int \varphi = \sum c_i m(E_i) \leq 0$$
Taking the supremum over all ϕ, we end up showing that
$$\int_E f \leq 0$$
Implying that $f = 0$, and we are done with (2).

3. $f \leq g$ almost everywhere means that $f - g \leq 0$ almost everywhere. Let φ be a simple function such that $\varphi \leq f - g$. This implies that $\varphi \leq 0$ almost everywhere. We then know that
$$\int_E \varphi \leq 0$$
This follows from the fact that φ is a simple function, i.e.,
$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$
which we call our 'canonical representation/decomposition'. Recall that $\{c_1, c_2, ..., c_n\}$ are a set of values of φ, except 0. In this case,
$$E_i = \varphi^{-1}(\{c_i\})$$
Notice that $E_i \cap E_j = \emptyset$ for $i \neq j$, as a result of this definition. In this case,
$$\int_E \varphi = \sum_{i=1}^{n} c_i m(E_i)$$
If any c_i's are positive, since $\varphi \leq 0$ almost everywhere, this would imply that $m(E_i) = 0$. Therefore,
$$\int_E \varphi = \sum_{i=1}^{n} c_i m(E_i) \leq 0$$
we have that
$$0 \geq \sup_{\varphi \leq f - g} = \int (f - g)$$
where φ is simple. By (1), $\int f - \int g \leq 0, \Rightarrow \int f \leq \int g$.

4. By (3),
$$\int_E f \leq \int_E B = B \cdot m(E)$$
The other inequality follows in the same way.

5. The characteristic function $\chi_{A \cup B} = \chi_A + \chi_B$, because A and B are disjoint. We have that
$$\int_{A \cup B} f = \int_{\mathbb{R}} \chi_{A \cup B} f = \int (\chi_A + \chi_B) f = \int (\chi_A f + \chi_B f) = \int \chi_A f + \int \chi_B f = \int_A f + \int_B f$$

□

Proposition 58. The Bounded Convergence Theorem: *Let $\{f_n\}$ be a sequence of measurable functions defined on E, with $m(E) < \infty$. Assume that there is $M > 0$ such that $|f_n(x)| < M$ for all $x \in E$ and n. If $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then*

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof. Take $\epsilon > 0$. Then

$$|\int_E f_n - \int_E f|$$

($f(x) = \lim_{n\to\infty} f_n(x) = \overline{\lim_{n\to\infty}} f_n(x)$, so $f(x)$ is measurable- this is something we proved a while ago). For a fixed x, we have that $-M < f_n(x) < M$, so $-M < \lim_{n\to\infty} f_n(x) < M$, so $|f(x)| \leq M$- which is true for every x.

Proposition 59. *Recall: (this is a piece of an old proposition) Given $\delta > 0, \epsilon > 0$, then there exists $N \in \mathbb{N}$, and $A \subseteq E$ such that $m(A) < \delta$ and*

$$|f_n(x) - f(x)| < \epsilon$$

For $x \notin A$.

Using this proposition in our proof,

$$|\int_E f_n - \int_E f| = |\int_E (f_n - f)| = |\int_{E-A} (f_n - f) + \int_A (f_n - f)|$$
$$\leq {}^*|\int_{E_A} (f_n - f)| + |\int_A (f_n - f)| \leq \int_{E-A} |f_n - f| + \int_A |f_n - f|$$

Which also follows from the fact that

$$|\int f| \leq \int |f|$$

From this it follows that

$$|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq 2M$$

So, we have that

$$\int_{E-A} |f_n - f| + \int_A |f_n - f| \leq \epsilon \cdot m(E_A) + 2Mm(A) \leq \epsilon m(E) + 2Mm(A)$$

To make some progress, we take our proposition and adapt it so that

$$\delta = \frac{\epsilon}{4M}, \quad |f_n(x) - f(X)| < \frac{\epsilon}{2m(E)}$$

Let $N = N(\frac{\epsilon}{2m(E)} \cdot \frac{\epsilon}{4M}$ be from the proposition. Then $n \geq N$, for all $A \subset E$, and we have in our chain of equations that

$$\epsilon \cdot m(E_A) + 2Mm(A) \leq \frac{\epsilon}{2m(E)} m(E) + 2Mm(A) \leq \frac{\epsilon}{2} + 2 < \frac{\epsilon}{4M} = \epsilon$$

□

Proposition 60. *A bounded function f on $m[a, b]$ is* **Riemann Integrable** *if and only if the set of points at which f is discontinuous is of Lebesgue measure 0.*

This proposition tells us immediately that $\chi_{[0,1]-\mathbb{Q}}$ is not Riemann integrable. This follows from the fact that this function is never continuous, and the measure of $[0, 1] - \mathbb{Q}$ is 1.

2.5 The Integral of a Non-negative Function

We assume that $f \geq 0$, and is a measurable function defined on a measurable set E.
Definition.

$$\int_E f = \sup_{h \leq f} \int_E h$$

Where h is a bounded, measurable function such that $m(\{x : h(x) \neq 0\}) < \infty$. This set is called 'the support' of h.
Proposition 61. *If f and g are non-negative measurable functions, then*

1. $$\int_E cf = c \int_E f$$

2. $$\int_E (f+g) = \int_E f + \int_E g$$

3. *If $f \leq g$ almost everywhere, then*
$$\int_E f \leq \int_E g$$

Proof. 1. If you have $h \leq f$, $c > 0$, $ch \leq cf$. Then

$$\int_E cf := \sup_{h \leq f} \int_E ch = \sup_{h \leq f} c \int_E h = c \sup_{h \leq f} \int_E h = c \int_E f$$

2. If $h \leq f$ and $k \leq g$, then $h + k \leq f + g$. So, we have that

$$\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g) \tag{2.5.1}$$
$$\sup_{h \leq f} \int_E h + \int_E K \leq \int_E (f+g) \tag{2.5.2}$$
$$\sup_{h \leq k} \int_E h + \sup_{k \leq g} \int_E k \leq \int_E (f+g) \tag{2.5.3}$$
$$\int_E f + \int_E g \int_E (f+g) \tag{2.5.4}$$

3. If $f \leq g$ almost everywhere, then $0 \leq f - g$ almost everywhere. We again use our definitions: if we have $h \leq f$, then $h \leq g$. In this case,

$$\int_E f = \sup_{h \leq f} \int_E h \leq \sup_{h' \leq g} \int_E h' = \int_E g$$

□

Let $l \leq f + g$, $m(\{x : l(x) \neq 0\}) < \infty$, and l is bounded. We form $h(x) = min(f(x), l(x))$, and $k(x) = l(x) - h(x)$. The minimum is non-negative. We have that $h(x) \leq f(x), k(x) \leq g(x)$, and

$$\int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g$$

Taking the supremum over all l, we have

$$\int_E (f+g) = \sup_{l \leq f+g} \int_E l \leq \int_E f + \int_E g$$

The test material is up to this section (page 85 in the book).

2.6 Review for Midterm

2.6.1 Solution to (1)

We have that $f : E \to \mathbb{R}$, such that $m(E) < \infty$, and f is non-negative and bounded. We need to prove that $\inf_E f(x)dx = 0$ implies that $f = 0$ almost everywhere.

Proof by contradiction: we assume that there exists some set $A \subset E$ with $m(A) > 0$ such that $f(x) > 0$ for all $x \in A$. We have that

$$\int_E f(x)dx = \geq \int_A f(x)dx$$

which follows from the fact that $f(x) \geq 0$, so

$$\int_E f(x)dx = \int_{E_A} f(x)dx + \int_A f(x)dx$$

where $\int_{E-A} f(x)dx \geq 0$, implying that

$$\int_{E-A} f(x)dx \geq 0 + \int_A f(x)dx \geq \int_A f(x)dx$$

This follows from the definition:

$$\int_{E-A} f(x)dx = \inf_{\psi \geq f} \int_{E_A} \psi \geq 0$$

(f is a simple, bounded function). This follows by recalling that

$$\psi = \sum c_i \chi_{E_i}$$

and since $\psi \geq 0$, we have that each $c_i \geq 0$ so the integral of ψ is

$$\int \psi = \sum c_i m(E_i) \geq 0$$

We have that f is bounded, by definition this means that there exists $M > 0$ such that $|f| \leq M$. We have the set $A_n \subset A$ as follows:

$$A_n = \{x \in A : \frac{M}{n+1} < f(x) \leq \frac{M}{n}\}$$

We have that $A_n \cap A_m$ if $n \neq m$. We also know that $\cup A_n = A$. We have that

$$m(A) = \sum_{n=1}^{n} m(A_n)$$

but before we do this, we have to show that each A_n is measurable- which follows from the fact that

$$A_n = f^{-1}((\frac{M}{n+1}, \frac{M}{n}]) = f^{-1}((\frac{M}{n+1}, \infty]) \cap f^{-1}([-\infty, \frac{M}{n}])$$

These two sets are measurable, therefore their intersection is measurable. We have that

$$0 \leq m(A) = \sum_{n=1}^{\infty} m(A_n)$$

Of course for all n, $m(A_n) \geq 0$. This implies that there exists n_0 such that $m(A_{m_0}) > 0$. Which would mean that

$$\int_E f(x)dx \geq \int_{A_{n_0}} f(x)dx$$

Notice that $x \in A_0 \Rightarrow \frac{M}{n_0+1} < f(x)$. So,

$$\int_E f(x)dx \geq \int_{A_{n_0}} f(x)dx \geq \frac{M}{n_0+1} m(A_{n_0}) > 0$$

Which is a contradiction, and we are done.

2.6.2 Solution to (2)

We have that $f \geq 0$, and we have an increasing sequence φ_n such that $\varphi_n \geq 0$ and are simple. We want to show that

$$f(x) = \lim_{n \to \infty} \varphi_n(x) \quad \forall x$$

We can assume that $f : \mathbb{R} \to \mathbb{R}$. Given n, we look at $[\frac{1}{n}, n]$, more specifically $[\frac{k}{n}, \frac{k+1}{n}]$, $k = 1, ..., n^2$. We define $A_k = f^{-1}([\frac{k}{n}, \frac{k+1}{n}]) \cap [-n, n]$ (the intersection is to keep the set of finite measure). Then w define

$$\varphi_n(x) = \sum_{k=1}^{n^2} \frac{k}{n} \chi_{A_k}(x)$$

Notice that $\varphi_n(x) \leq f(x)$, since we choose the lower value. We have that

$$|f(x) = \varphi_n(x)| \leq \frac{1}{n} \quad \text{on} \quad \bigcup_{k=1}^{n^2} A_k$$

since $\varphi_n(x)$ is increasing, we are essentially finished. This doesn't work with step functions: for example, we can't do this over $\chi_{\mathbb{Q}}$, for if we wanted to approximate it over an interval, the step function can only take on value on every interval, and $\chi_{\mathbb{Q}}$ takes on two values at every interval.

2.6.3 Solution to (3)

We have that

$$\lim_{n \to \infty} \int_0^1 \frac{log(x+n)}{n} e^{-x} cos(x) dx$$

Recall that we had something called the bounded-convergence theorem, which tells us that if we have f_n measurable and bounded on a closed interval $[a, b]$,

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e.$$

Then f is measurable, and bounded, and

$$\int_{[a,b]} f(x)dx = \lim_{n\to\infty} \int_{[a,b]} f(x)dx$$

Notice that

$$\left| \frac{log(x+n)}{n} e^{-x} cos(x) \right| \leq 1 \quad \forall x, \forall n$$

since

$$\sup_x \left| \frac{log(x+n)}{n} \right| = \frac{log(1+n)}{n} \to 0 \quad \text{ as } n \to \infty$$

This implies that there exists some $M > 0$ such that

$$\frac{log(x+n)}{n} \leq \frac{log(1+n)}{n} \leq M$$

We also have that f_n is continuous, so we have that this set is measurable. Since

$$\lim_{n\to\infty} \frac{log(x+n)}{n} e^{-x} cos(x) = 0$$

So, the integral is 0.

2.6.4 Solution to (1)

We have $f : [0, 1] \to \mathbb{R}$, and the definition:

$$f(x) = 0, x \neq \frac{1}{2^n}, 0 \qquad \text{if } x = \frac{1}{2^n} \tag{2.6.1}$$

We want to show that f is Riemann integral. One way to do this is to show that the lower integral is equal to the upper integral, i.e.,

$$\inf \sum_{i=1}^{k} M_i(\xi_i - \xi_{i-1})$$

Where $M_i = \sup_{\xi_{i-1} \leq x < \xi_i} f(x)$.

We take the division $\xi_i = [\frac{i}{2^k}, \frac{i+1}{2^k})$.

We can eventually show that

$$\inf \sum_{i=1}^{k} M_i(\xi_i - \xi_{i-1}) = 0$$

This function is also Lebesgue integrable, simply because it is Riemann Integrable.

2.6.5 Solution to the other (2)

We have $f_n : [a, b] \to \mathbb{R}$, a sequence of bounded functions that converges uniformly to f on $[a, b]$. We want to show if each $f)n$ is Riemann integrable over $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

We have that f_n converges to f uniformly, and that $\sup_{x\in[a,b]} f_n(x) < \infty$. Given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for every $x \in [a, b]$ and for every $n \geq \mathbb{N}$. From here, we conclude that

$$|f(x)| < |f_n(x)| + \epsilon$$

Taking the supremum,

$$\sup_{x\in[a,b]} |f(x)| \leq \sup_{x\in[a,b]} |f_n(x)| + \epsilon$$

So f is a bounded function, and it is even uniformly bounded, for

$$|f_n(x)| < |f(x)| + \epsilon$$

For all $n \geq N$, for all x. (notice that Lebesgue measurable + bounded implies Riemann integrable). We apply the bounded convergence theorem, which will tell us that our integrals at the beginning of the question are equal.

We sort of screwed up and forgot to show that f is Riemann integrable. We have the following: we take $\varphi \leq f_n$, where φ is a step function. We have that

$$\int_a^b f_n(x)dx = \sup_{\varphi\leq f_n} \int_a^b \varphi(x)dx$$

Since we have that

$$|f_n(x) - f(x)| < \epsilon$$

for some $n \geq N(\epsilon)$ and for all $x \in [a, b]$, then this is equivalent to saying that

$$f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$$

Then

$$\varphi(x) - \epsilon \leq f_n(x) - \epsilon < f(x)$$

In this case,

$$\int_a^b [\varphi(x) - \epsilon]dx \leq \int_a^b [f_n(x) - \epsilon]dx = \int_a^b f_n(x)dx - \epsilon(b - a)$$

Given another step function, we have that

$$\int_a^b f_n(x)dx = \sup_{\varphi\leq f_n} \int_a^b \varphi(x)dx = \inf_{\psi\geq f_n} \int_a^b \psi(x)dx$$

So

$$\int_a^b [\varphi(x) - \epsilon]dx \leq \int_a^b [f_n(x) - \epsilon]dx = \int_a^b f_n(x)dx - \epsilon(b - a)$$

$$< \int_A^b f_n(x)dx + \epsilon(b-a) = \int_a^b (f_n(x) + \epsilon_d x\ leq \int_a^b (\psi(x)+\epsilon)dx = \int_a^b \psi(x)dx + \epsilon(b-a)$$

What this tells us is that

$$\sup_{\varphi_1 \leq f} \int_a^b \varphi_1(x)dx \leq \inf_{\psi_1 \geq f} \int_a^b \psi_1(x)dx$$

which is always the case. Since we have that

$$\varphi(x) - \epsilon$$

is one of those 'φ_1's', we know that

$$\int_a^b (\varphi(x)-\epsilon) \leq \sup_{\varphi_1 \leq f} \int_a^b \varphi_1(x)dx \leq \inf_{\psi_1 \geq f} \int_a^b \psi_1(x)dx \leq \int_a^b (\psi(x)+\epsilon)dx$$

Screw above. He redid the proof. See below:

Since f_n is Riemann integrable, there exist φ_n, ψ_n step functions such that

$$\varphi_n \leq f_n, \quad f_n \leq \psi_n$$

and

$$\int_a^b \psi_n(x)dx - \int_a^b \varphi_n(x)dx \leq \frac{1}{n}$$

Let $\epsilon_n = \sup_{x\in[a,b]} |f(x) - f_n(x)$. Then we have that

$$\varphi_n(x) - \epsilon_n \leq f_n(x) - \epsilon_n \leq f(x) \leq f_n(x) + \epsilon_n \leq \psi_n(x) + \epsilon_n$$

So,

$$\int (\varphi_n(x)-\epsilon_n)dx \leq \sup_{\varphi^* \leq f} \int_a^b \varphi^*(x)dx \leq \inf_{f \leq \psi^*} \int_a^b \psi^*(x)dx \leq \int_a^b (\psi_n(x)+\epsilon_n)dx$$

Looking at

$$\int_a^b [(\psi_n(x)+\epsilon_n) - (\varphi_n(x)-\epsilon_n)]dx = \int_a^b \psi_n(x)dx = \int_a^b \varphi_n(x)dx + 2\epsilon_n < \frac{1}{n} + \epsilon_n(b-a) \to 0, n \to \infty$$

Since $|f(x) - f_n(x)| \to 0$ as $n \to \infty$.

2.7 Post Exam, Integral of a Nonnegative Function

We have that $f \geq 0$, and is measurable on a set E- this will be a long standing assumption for this section. As a remark, we allow $m(E)$ to possibly be infinite. We have that

$$\int_E f = \sup_{h \geq f} \int_E h$$

Where h is bounded and measurable, and $m(supp(h)) < \infty$. Recall that

$$supp(h) = \{x \in E : h(x) \neq 0\}$$

Proposition 62. *We have that if f, g are nonnegative and bounded, then*

1.
$$\int_E cf = c\int_E f, c > 0$$

2.
$$\int_E (f+g) = \int_E f + \int_E g$$

3. *If $f \leq g$ almost everywhere, then*
$$\int_E f \leq \int_E g$$

This proposition was demonstrated last time.

Theorem 63. Fatou's Lemma*: If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on E, as $n \to \infty$, then*

$$\int_E f \leq \varliminf_{n\to\infty} \int_E f_n$$

Proof. We can assume that $f_n(x) \to f(x)$ on the whole set E. Since h is bounded and measurable on E, $h(x) \leq f(x) \forall x \in E$. In this case $supp(h) = E'$, and $m(E') < \infty$. We then define the new function

$$h_n(x) = min\{h(x), f_n(x)\}$$

So $supp(h_n) \subset E'$. We then have the $h_n(x)$ is bounded by the bound for h (another way to say this is that 'h_n is uniformly bounded), measurable (since it's the minimum of two measurable functions). We now apply the Bounded convergence Theorem, which we would like to apply to show that $h_n(x) \to h(x)$ for all $x \in E$. This last statement follows from taking $x \in E$, knowing that $h(x) \leq f(x)$, we have the following:

1. If $h(x) < f(x)$, then there exists some $n(x) : h(x) < f_{n(x)}(x)$, for all $n \geq n(x)$.
2. $h_n(x) = f(x)$.

So, by the bounded convergence theorem,

$$\int_E h = \lim_{n\to\infty} \int_E h_n \leq \int_E$$

Since the following are true

$$h_n(x) \leq h(x) \leq f(x)$$
$$h_n(x) \leq f_n(x)$$
$$\int_E h_n \leq \int_E f_n$$
$$\int_E h_n \leq \int_E f_n$$

Taking the supremum over all $h \leq f$, we have that

$$\int_e f \leq \varliminf_{n\to\infty} \int_E f_n$$

□

Theorem 64. The Monotone Convergence Theorem *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere. Then,*

$$\int f = \lim_{n\to\infty} \int f_n$$

Proof. By Fatou's lemma, we have that

$$\int f \leq \varliminf_{n\to\infty} \int f_n$$

So, we just need to show the opposite inequality. We have that $f_n(x) \leq f(x)$, almost everywhere, since our sequence is increasing. This means that

$$\int f_n \leq \int f$$

Taking the limit superior on both sides,

$$\varlimsup_{n\to\infty} \int f_n \leq \int f \leq \varliminf_{n\to\infty} \int f_n \leq \varlimsup_{n\to\infty} \int f_n$$

Thus, our proof is complete. ☐

Corollary 65. *Let u_n be nonnegative measurable functions, and let $f = \sum_{n=1}^{\infty} u_n$. Then,*

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

Proposition 66. *Let $f \geq 0$ be a measurable function defined on E, and let $\{E_i\}$ be a sequence of disjoint and measurable sets such that $E - \bigcup_i E_i$. Then,*

$$\int_E f = \sum_i \int_{E_i} f$$

Proof. We define

$$\int_E f = \int_{\mathbb{R}} f\chi_E$$

And since we know that $\chi_E = \sum_i \chi_{E_i}$, we just apply our last corollary to get that

$$f\chi_E = \sum_i f\chi_{E_i}$$

Then we have that

$$\sum_i \int f \cdot \chi_{E_i} = \sum_i \int_{E_i} f$$

☐

Proposition 67. *Let f and g be two nonnegative measurable functions. If f is integrable over E (which means that $\int_E f < \infty$), and $f(x) \geq g(x)$ almost everywhere on E¡ then g is integrable and*

$$\int_E (f - g) = \int_E f - \int_E g$$

Proof. We know that
$$\int_E (f+g) = \int_E f + \int_E g$$
We write $f = (f-g)+g$. Notice that $f-g \geq 0$, and $g \geq 0$ by assumption. In this case,
$$\int_E f = \int_E (f-g) + \int_E g$$
Both of these integrals on the right hand side are less than infinity, which implies that g is integrable. This line above is equivalent to
$$\int_E f - \int_E g = \int_E (f+g)$$
□

Proposition 68. *Let $f \geq 0$ be measurable, and integrable over a set E. Then given $\epsilon > 0$ there exists $\delta > 0$ such that $\forall A^\dagger \subset E$ if $m(A) < \delta$, then*
$$\int_A f < \epsilon$$

Proof. If f is bounded by some M, then
$$\int_A f \leq \int_A M = Mm(A)$$
(prove that $\int_A 1 = \int_A m(A)$

Alternatively, let f not be bounded. Then
$$f_n(x) = f(x) \text{ if } f(x) \leq n \quad n \text{ if } f(x) > n$$
f_n bounded, and increasing, since $f_n(x) \to f(x)$ as $n \to \infty$, we use the monotone convergence theorem to show that
$$\int_E f_n \to \int_E f, n \to \infty$$
Given $\epsilon > 0$, let $N \in \mathbb{N}$ such that
$$\int_E f_n > \int_E f - \frac{\epsilon}{2}$$
We know that
$$|f_n| \leq N$$
Applying what we know about bounded functions, if we chose $\delta = \frac{\epsilon}{2N}$, then if $m(A) < \delta$, then we have that
$$\int_A f_N < \frac{\epsilon}{2} < N\delta = N \cdot \frac{\epsilon}{2N} = \frac{\epsilon}{2}$$
Taking this, we have that
$$\int_A f < \int_A f_n + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
For all $m(A) < \delta$. □

†measurable

Homework

On page 89, do questions 3,7, and 8.

2.8 The General Lebesgue Integral

Now, we will mean that f is a measurable function.
Definition. $f^+(x) = max\{f(x), 0\}$, and $f^-(x) = max\{-f(x), 0\}$.
Fact.

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-$$

Definition. A measurable function f is **integrable** over a measurable set E if f^+ and f^- are both integrable over E, and in this case,

$$\int_E f = \int_E f^+ - \int_E f^-$$

Proposition 69. *Let f and g be integrable over E, and $c \in \mathbb{R}$. Then,*

1. *cf is integrable over E, and*
$$\int_E cf = c \int_E f$$

2. *The function $f + g$ is integrable over E, and*
$$\int_E (f+g) = \int_E f + \int_E g$$

3. *If $f \leq g$ almost everywhere, then*
$$\int_E f \leq \int_E g$$

4. *If A and B are disjoint measurable subsets in E, then*
$$\int_{A \cup B} = \int_A f + \int_B f$$

Proof. We have that $\int_E f^+, \int_E f^- < \infty$. We now want to look at cf, where $c \in \mathbb{R}$. Like before, we have that

$$cf = (cf)^+ - (cf)^-$$

If $c \geq 0$, then

$$(cf)^+ = cf^+, (cf)^- = cf^-$$

In this case,

$$\int_(cf)^+ = \int cf^+ = c \int f^+ < \infty$$

Similarly, we can do the same for f^-. Thus,

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^- = c(\int_E f^+ - \int_E f^0) = c \int_E f$$

Alternatively, if $c < 0$, then $(cf)^+ = cf^-$, and $(cf)^- = -cf^+$. Thus,

$$\int -cf^- = -c\int_E f^- < \infty$$

So both pieces are integrable. Then,

$$\int_E cf = \int_e (cf)^+ - \int_E (cf)^- = \int_E -cf^- - \int_E cf^+ = c\int_E f^+ - c\int_E f^- = c\int_E f$$

We would like both $(f+g)^+, (f+g)^-$ to be integrable. It follows that

$$f + g = (f^+ + g^+) - (f^- + g^-)$$

We have to prove that if $f_1 \geq 0, f_2 \geq -$, and $f = f_1 - f_2$, we would need to show that

$$\int f = \int f_1 - \int f_2$$

Which are integrals of non-negative functions. We have by assumption, that

$$f = f^+ - f^- = f_1 - f_2$$

This implies that

$$f^+ + f_2 = f_1 + f^-$$

We have that

$$\int (f^+ + f_2) = \int (f_1 + f^-) \Rightarrow \int f^+ \int f_2 = \int f_1 + \int f^- \Rightarrow \int f^= - \int f^- = \int f_1 - \int f_2$$

Where the last line follows from integrability. By the claim, if we know that $f+g$ is integrable, it follows that

$$\int (f+g) = \int (f^+ + g^+) - \int (f^- + g^-) \tag{2.8.1}$$
$$= \int f^+ + \int g^+ - (\int f^- + \int g^-) = \int f^+ - \int f^- + \int g^+ - \int g^- \tag{2.8.2}$$
$$= \int f + \int g \tag{2.8.3}$$

2. We have $f \leq g$; Then $\int_E (g - f) \geq 0$. Using property 2 and 1 with $c = -1$, we have that

$$\int_E (g - f) \geq 0 = \int_E g + \int_E (-f) = \int_E g = \int_E f$$

3. We have

$$\int_{A \cup B} f = \int f\chi_{A\cup B} = \int f(\chi_A + \chi_B) = \int f\chi_A + \int f\chi_B = \int_A f + \int_B f$$

□

2.9 The Lebesgue Convergence Theorem

Theorem 70. The Lebesgue Convergence Theorem: *Let g be an integrable (non-negative) function over a measurable set E, and let $\{f_n\}$ be a sequence of measurable functions over E such that $|f_n| \leq g$ almost everywhere on E, and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

For almost all $x \in E$, then

$$\int_E f = \lim_{n\to\infty} \int_E f$$

Proof. We know that $|f_n| \leq g$ almost everywhere. Then $f_n \leq |f_n| \leq g$. We can then write

$$g - f_n \geq 0$$

This is a non-negative function, on which we can apply Fatou's lemma. This gives us that

$$\varliminf_{n\to\infty} \int_E (g - f_n) \geq \int_E (g - f)$$

Thus

$$\varliminf_{n\to\infty} (\int_E g - \int_E f_n) \geq \int_E g - \int_E f$$

moving the limit inferior inside, we have that

$$\int_E g - \varlimsup_{n\to\infty} \int_E f_n \geq \int_E g - \int_E f \Rightarrow \varlimsup_{n\to\infty} \int_E f_n \leq \int_E f$$

We have that $|f_n| \leq g$, then $-f_n \leq |f_n| \leq g$. Thus, $g + f_n \geq 0$. By Fatou's lemma,

$$\varliminf_{n\to\infty} \int_E (g + f_n) \geq \int_E (g + f)$$

Going with the integral inside, we have that

$$\varliminf_{n\to\infty} (\int_E g + \int_E f_n) \geq \int_E g + \int_E g$$

$$\int_E g + \varliminf_{n\to\infty} \int f_n \geq \int_E g + \int_E f \Rightarrow \varliminf_{n\to\infty} \int_E f_n \geq \int_E f$$

Putting these two things together, we have that

$$\varlimsup_{n\to\infty} \int_E f_n \leq \int f \leq \varliminf_{n\to\infty} \int_E f_n \leq \varlimsup_{n\to\infty} \int_E f_n$$

Thus, the whole chain has to be equal. This implies that there exists $\lim_{n\to\infty} \int f_n$, and

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

$\square$

2.10 Back from the break, Quick Review

2.10.1 The General Lebesgue Integral

We had the following definition: given a function $f : E \to \mathbb{R}$ which is measurable, it's Lebesgue integral is as follows: we define

$$f^+(x) = max\{f(x), 0\} \qquad f^-(x) = max\{-f(x), 0\}$$

In this case, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. Noting that both f^+, f^- are nonnegative functions, we can define their Lebesgue integrals. We then say that f is Lebesgue integrable if

$$\int f^+, \int f^- < \infty$$

and in this case

$$\int_E f = \int_E f^+ - \int_E f^-$$

Example. Notice that $|f| \geq 0$, but if f is integrable this means that $\int f^+, \int f^- < \infty$ so

$$\int |f| = \int f^+ + \int f^- < \infty$$

So, $|f|$ is integrable. It is interesting to notice that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \text{ but} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} < \infty$$

2.10.2 The Lebesgue Convergence Theorem

Given $g : E \to \mathbb{R}$ integrable, we take $f_n : E \to \mathbb{R}$ is integrable and assume that $|f_n| \leq g$ almost everywhere, and we have that $f_n(x) \to f(x)$ almost everywhere on E as $n \to \infty$. Then

$$\int_E f(x) = \lim_{n\to\infty} \int_E f_n$$

2.11 Differentiation of Monotone functions

We denote by $\mathcal{I}$ a collection of intervals.

Definition. We have that $\mathcal{I}$ covers a set $E \subset \mathbb{R}$ in the sense of Vitali if for all $\epsilon > 0$, and for each $x \in E$ there exists $I \in \mathcal{I}$ such that $x \in I$ and $\ell(I) < \epsilon$. Here, I can be an open, half open, or closed interval.

Theorem 71. *Let $m^*(E) < \infty$, and $\mathcal{I}$ a collection of intervals that cover E in the sense of Vitali. Then for every $\epsilon > 0$, there exists a finite disjoint collection $\{I_1, I_2, ..., I_n\}$ of intervals in $\mathcal{I}$ such that*

$$m^*(E - \cup_{n=1}^{N} I_n) < \epsilon$$

Proof. We can assume that intervals in $\mathcal{I}$ are closed. Let O be an open set such that $O \supset E$ and $m(O) < \infty$. We then drop the elements of $\mathcal{I}$ which are not contained in O. This new collection, called $\mathcal{I}$ again, is also a covering of E in the sense of of Vitali.

Our goal is to chose a disjoint sequence $\{I_n\}$ in $\mathcal{I}$ as follows:

1. I_1 is any interval in $\mathcal{I}$. Suppose that $I_1, ..., I_n$ are already chosen. Let $k_m =$ sup of the lengths of the intervals of $\mathcal{I}$ that to not meet any of $I_1, ..., I_n$. Notice that $0 < k_n \leq m(O)$, because each interval in I is contained in O.

Unless $I \subset \bigcup_{i=1}^{n} I_i$, then there exists $I_{n+1} \in \mathcal{I}$ with $\ell(I_{n+1}) > \frac{1}{2}k_n$, and I_{n+1} is disjoint from all $I_1, .., I_n$. Notice that $\bigcup_{n=1}^{\infty} I_n \subset O$. SInce $\{I_n\}$ is a disjoint sequence, we have that

$$\sum_{n=1}^{\infty} \ell(I_n) = m(\bigcup_{n=1}^{\infty} I_n) \leq m(O) < \infty$$

Hence there exists N such that

$$\sum_{n=N+1}^{\infty} \ell(I_n) < \frac{\epsilon}{5}$$

Let $R = E - \bigcup_{n=1}^{N} I_n$. Let $x \in R$. Since $\bigcup_{n=1}^{N} I_n$ is a closed set not containing x, there exists $I \in \mathcal{I}$ with $x \in I$ and I does not intersect $I_1, .., I_n$. If $I \cap I_i = \emptyset$, for all $i \leq n$, then we can conclude that $\ell(I) \leq k_n < 2\ell(I_{n+1})$.

We also have that $\lim_{n\to\infty} \ell(I_n) = 0$, which follows from the fact that the sums of the lengths are finite.

This implies that I has to meet at least one I_n. Let n be the smallest such that $I \cap I_n \neq \emptyset$. Then $n > N$ and

$$\ell(I) \leq k_{n-1} < 2\ell(I_n)$$

We have that

$$dist(midpoint(I_n), x) \leq \ell(I) + \frac{1}{2}\ell(I_n) < \frac{5}{2}\ell(I_n)$$

We make J_n the interval whose midpoint is the midpoint of I_n, and $\ell(J_n) = 5\ell(I_n)$. Then, $R \subset \bigcup_{n+1}^{\infty} J_n$. Based on how we chose our sequence of I_n, we have that

$$\sum_{n=N+1}^{\infty} \ell(J_n) = 5 \sum_{n=N+1}^{\infty} \ell(I_n) < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

□

Definition. $\mathcal{I}$ is a cover of a set E in the sense of Vitali if for every ϵ and $x \in E$¡ there exists $I \in \mathcal{I}$ such that

$$\ell(I) < \epsilon, x \in I$$

Lemma 72. *Due to Vitali, we have that if $m^*(E) < \infty$ and $\mathcal{I}$ (a cover in the sense of Vitali for E), then there exist $I_1, ..., I_n$ in $\mathcal{I}$ such that*

$$m^*(E - \bigcup_{i=1}^{n} I_i) < \epsilon$$

Definition.

$$D^+f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h}$$

$$D^-f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x)-f(x+h)}{h}$$

$$D_+f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h}$$

$$D_-f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x)-f(x+h)}{h}$$

Fact.

$$D^+f(x) \geq D_+f(x), \qquad D^-f(x) \geq D_-f(x)$$

Definition. If $D^+f(x) = Df(x) = D^-f(x) = D_-f(x) \neq \pm\infty$, then f is differentiable and

$$f'(x) = D^+f(x)$$

Similarly, if $D^+f(x) = D_+f(x)$ then f is differentiable from the right, denoted $f'(x+) = D^+f(x)$-often called the right-hand derivative.

If $D^-f(x) = D_-f(x)$, then f is differentiable from the left, and $f'(x-) = D^-f(x)$, and is called the left-hand derivative.

Proposition 73. *If f is continuous on $[a,b]$ and one of its derivatives is everywhere non-negative on (a,b) then f is non decreasing on $[a,b]$.*

Theorem 74. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then, f is differentiable almost everywhere on (a,b). The derivative f' is measurable and*

$$\int_a^b f'(x)dx \leq f(b) - f(a)$$

Proof. Let $E = \{x : D^+f(X) > D_-f(x)\}$. We want to show that the measure of this set is zero. We let $E = \bigcup_{u,v\in\mathbb{Q}} E_{u,v}$ where $E_{u,v} = \{x : D^+f(x) > u > v > D_-f(x)\}$. It is enough to show that $m^*(E_{u,v}) = 0$ for all $u, v \in \mathbb{Q}$. Let $s = m^*(E_{u,v}) < \infty$ because $E_{u,v} \subset [a,b]$. Take $\epsilon > 0$. Then there exists an open set O such that $O \supset E_{u,v}$ and $m(O) < s + \epsilon$. Then for all $x \in E_{u,v}$ there exists an interval $[x-h, x] \subset O$ such that

$$f(x) - f(x-h) < v \cdot h$$

The family of all such $[x-h, x]$ is a Vitali cover of $E_{u,v}$. By the previous lemma, there exists finitely many intervals $\{I_1, ..., I_n\}$ whose interiors cover a subset A of $E_{u,v}$ with

$$m^*(A) > S - \epsilon$$

$$A \subset \bigcup_{n=1}^{N} \text{interiors } (I_n)$$

So, we look at the following:

$$I_n = [x_n - h_n, x_n], \quad \sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \cdot \sum_{n=1}^{N} h_n < vm(O) < v(s+\epsilon)$$

Let $y \in A$. Then $\exists k > 0$ such that

$$(y, y+k) \subseteq I_n \text{ for some } n = 1, ..., N$$

We have that $D^+ fx) = \overline{\lim}_{h \to 0^+} \frac{f(x+h)-f(x)}{h} > u$, and

$$f(y+k) - f(y) > u \cdot k$$

Consider $[y, y+k]$ for $y \in A$ with the above choice of k; this is a Vitali cover of A.

Proof. By Vitali's lemma, there exists a finite collection $\{J_1, .., J_M\}$ such that their union contains a subset of A of outer measure greater than $S - 2\epsilon$. So, $J_i = (y_i, y_i + k_u)$. So,

$$\sum_{i=1}^{M} [f(y_i + ki) - f(y_i)] > u \cdot \sum_{i=1}^{n} k_i > u(s - 2\epsilon)$$

Note that $J_i \subset I_n$ (for every i, there is an n such that this is true). We take the sum over all i with $J_i \subset I_n$ and we get (fix n):

$$\sum_{i: J_i \subset I_n} [f(y_i + k_i) - f(y_i)] \leq f(x_n) - f(x_n + k_n)$$

Because f is increasing. This implies

$$u(s - 2\epsilon) < \sum_{i=1}^{M} [f(y_i k_i) - f(y_i)] \leq \sum_{n=1}^{N} [f(x_n) - f(x_n + k_n)]$$

Which implies that

$$u(s - 2\epsilon) < v(s + \epsilon)$$

Since this is true for all ϵ, let $\epsilon \to 0$. In this case, we show that $u < v$, which is a contradiction. This means that $s = 0$, so the measure of our set is indeed zero. In other words, $m^*(E) = 0$, which implies that f is differentiable almost everywhere.

2.12 Homework

On page 101, Question number 3.

2.13 Continued

We have that

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

defined almost everywhere, and f is differentiable when $g(x) \neq \pm\infty$. Let

$$g_n(x) = \frac{1}{\frac{1}{n}} [f(x + \frac{1}{n}) - f(x)]$$

where $f(x) := f(b)$ for $x \geq b$. Then $g)n(x) \to g(x)$ almost everywhere, and this implies that g is measurable.

We know that f is increasing, this implies that $g)n(x) \geq 0$. To this we can apply Fatou's lemma, which tells us that

$$\int_a^b f \leq \underline{\lim} \int_a^b g_n = \underline{\lim} n \int_a^b [\frac{1}{\frac{1}{n}} f(x + \frac{1}{n}) - f(x)]dx$$

$$= \underline{\lim} n [\int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx]$$

$$= \underline{\lim} [f(b) - n \int_a^{a+\frac{1}{n}} f(x)dx] \leq f(b) - f(a)$$

From this, our proof is finished. □

□

2.14 Functions of Bounded Variations

Suppose we take $f : [a, b] \to \mathbb{R}$, and we take a subdivision $a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k = b$ of $[a, b]$.
Definition. If $r \in \mathbb{R}$ is a real number, we define

$$r^+ = \begin{cases} r \text{ if } r \geq 0 \\ 0 \text{ if } r < 0 \end{cases}$$

similarly

$$r^- = \begin{cases} -r \text{ if } r \leq 0 \\ 0 \text{ if } r > 0 \end{cases}$$

In this case, $r = r^+ - r^-$, and $|r| = r^+ + r^-$. Letting

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

and

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

And we define

$$t := p + n = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

Fact.

$$f(b) - f(a) = p - n = \sum_{i=1}^{k} \{[f(x_i) - f(x_{i-1})]^+ - [f(x_i) - f(x_{i-1})]^-\} = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]$$

$$= f(x_k) - f(x_0) = f(a) - f(b)$$

Definition. $P = sup(p), N = sup(n), T = sup(t)$, and all of these supremums are taken over all subdivisions of $[a, b]$.

Notice that $P, N \leq T$. This is because $p, n \leq t$, since both p and n are positive and t is their sum. It turns out to also be true that

$$P, N \leq T \leq P + N$$

which is true because $sup(A+B) \leq sup(A) + sup(B)$. P, N, and T are positive negative and total variations of f , respectively.

Definition. If $T < \infty$, then f is of bounded variation on $[a, b]$, sometimes denoted $f \in BV$.

Lemma 75. *If f is of bounded variation on $[a, b]$, then*

$$T_a^b = P_a^b + N_a^b$$

and

$$f(b) - f(a) = P_a^b - N_a^b$$

Proof. We always have that $T_a^b \leq P_a^b + N_a^b$. Recall that for any subdivision of $[a, b]$, $p = n + f(b) - f(a)$. Taking the supremum over all subdivisions of $[a, b)$, we get

$$P = N + f(b) - f(a)$$

Notice that

$$P = sup(p) = sup[n + f(b) - f(a)] = sup(n) + [f(b) - f(a)] = N + f(b) - f(a)$$

Doing some algebra, $t = p + n$ so using our fact that $f(b) - f(A) = p - n$ we say that

$$t = p + n = 2p - [f(b) - f(a)]$$

Taking the supremum,

$$T = 2P - [f(b) - f(a)] = 2P - (P - N) = P + N$$

Which gives us that indeed,

$$T_a^b = P_a^b + N_a^b$$

The second part of the lemma follows directly from the second line above. □

Theorem 76. *A function f on $[a, b]$ is of bounded variation if and only if f is the difference of two monotone real valued functions on $[a, b]$.*

Proof. Assume first that f of bounded variation. Define $g(x) = P_a^x$ (the positive variation of the function f on the interval $[a, x]$). Also define $h(x) = N_a^x$ (the negative variation of the function f on the interval $[a, x]$. We have that

$$f(x) - f(a) = P_a^x - N_a^x = g(x) - h(x)$$

From here, we can say that

$$f(x) = g(x) - [h(x) - f(a)]$$

It is true that $g(x)$ is an increasing function, since $g(x) = P_a^x$. This is increasing simply because as x increases, we have a larger interval, and P_a^x is the supremum over all the interval's subdivisions.

Similarly, if $x_2 > x_1$, then $N_a^{x_1} \leq N_a^{x_2}$. It also follows that $h(x) - f(a)$ is increasing, for a monotonic function minus a constant is still monotonic.

Now proving the other direction, assume that $f(x) = g(x) - h(x)$ where $g(x)$ and $h(x)$ are monotonic. Looking at

$$\sum_{i=1}^{k}[f(x_i) - f(x_{i-1})] = \sum_{i=1}^{k}[g(x_i) - g(x_{i-1})] - \sum_{i=1}^{k}[h(x_i) - h(x_{i-1})]$$

This means that

$$\sum_{i=1}^{k}|g(x_i) - g(x_{i-1})| = |g(b) - g(a)|$$

and the same is true of $h(x)$. Noticing that

$$|f(x_i) - f(x_{i-1})| = |[g(x_i) - g(x_{i-1})] - [h(x_i) - h(x_{i-1})]| \leq |g(x_i) - g(x_{i-1})| + |h(x_i) - h(x_{i-1})|$$

which tells us that

$$t \leq \sum_{i=1}^{k}|g(x_i) - g(x_{i-1})| \sum_{i=1}^{k}|h(x_i) - h(x_{i-1})| \leq |g(x_i) - g(x_{i-1})| + |h(x_i) - h(x_{i-1})|$$

which holds for any subdivision, and thus $f \in BV$. ☐

Corollary 77. *If f is of bounded variation on $[a, b]$ then $f'(x)$ exists for almost all $x \in (a, b)$.*

2.15 Differentiation of an Integral

We assume that f is integrable on $[a, b]$ and define

$$F(x) = \int_a^x f(t)dt$$

Lemma 78. *If f is integrable on $[a, b]$, then the function*

$$F(x) = \int_A^x f(t)dt$$

is a continuous function of bounded variation on $[a, b]$.

Proof. $F(x)$ is continuous- Assume that $f \geq 0$. Taking $x, y \in [a, b]$, we want

$$|F(x) - F(y)|$$

If $x < y$, then

$$|F(x) - F(y)| = F(y) - F(x) = \int_a^y f(t)dt - \int_A^x f(t)dt = \int_a^x f(t)dt + \int_a^y f(t)dt - \int_a^x f(t)dt$$

$$= \int_a^y f(t)dt < \epsilon$$

for $|x - y| < \delta$.

Since f is integrable over $[a,b]$, then by the proposition from before, we have that $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\int_c^d f < \epsilon \text{whenever} |d-c| < \delta$$

We take a subdivision $a = x_0 < x_1 < ... < x_k = b$ of $[a,b]$, and look at

$$\sum_{i=1}^k |F(x_i) - F(x_{i-1})| = \sum_{i=1}^k |\int_{x_i-1}^{x_i} f(t)dt| \leq \sum_{i=1}^k \int_{x_i-1}^{x_i} |f(t)|dt = \int_a^b |f(t)|dt < \infty$$

because f is integrable, therefore $f \in BV$. ☐

Lemma 79. *If f is integrable over $[a,b]$ and*

$$\int_a^x f(t)dt = 0$$

for all $x \in [a,b]$, then $f(x) = 0$ almost everywhere on $[a,b]$.

2.16 Homework

Question 8 on page 104

2.17 The Differentiation of an Integral

Take f to be integrabl on $[a,b]$. We have that $F(x) = \int)a^x f(t)dt$, and we're trying to show that $F(x)$ is a differentiable function. We had the following lemma:

Lemma 80. *If f is integrable on the closed interval $[a,b]$, then the function*

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of bounded variation on $[a,b]$.

Proof. In showing continuity, let $x < y$, then

$$|F(x) - F(y)| = \int_a^y f(t)dt - \int_a^x f(t)dt = |\int_x^y f(t)dt| \leq \int_x^y |f(t)|dt < \epsilon$$

For $|x - y| > \delta$.

For bounded variation, we look at the subdivision $a = x_0 < x_1 < .. < x_k = b$, and examine

$$\int_{i=1}^k |F(x_i) - F(x_{i-1})| = \sum_{i=1}^k |\int_{x_{i-1}}^{x_i} f(t)dt| \leq \sum_{i=1}^k \int_{x_{i-1}}^{x_i} |f(t)|dt = \int_a 6b|f(t)|dt < \infty$$

Recall that

$$T = sup(t) \leq \int_a^b |f(t)|dt$$

which implies that f is of bounded variation. ☐

Lemma 81. *If f is integrable on $[a, b]$ and*

$$\int_a^x f(t)dt = 0$$

for all $x \in [a, b]$ then $f(t) = 0$ almost everywhere in $[a, b]$.

Proof. Suppose that $f(x) > 0$ on E with $m(E) > 0$. Then there exists a closed set $F \subset E$ with $m(F) > 0$. We define an open set $O = [a, b] - F$. We have that

$$0 = \int_a^b f(t)dt = \int_O f(t)dt + \int_F f(t)dt$$

From the homework, we know that

$$\int_O f(t)dt = -\int_F f(t)dt =\neq 0$$

O being open implies that it is a countable union of open disjoint intervals $\{(a_n, b_n)\}$. So,

$$\int_O f(t)dt = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(t)dt \Rightarrow \int_{a_n}^{b_n} f(t)dt \neq 0$$

This means that

$$\int_{a_n}^{b_n} f(t)dt = \int_a^{b_n} f(t)dt - \int_a^{a_n} f(t)dt$$

Which implies that either

$$\int_a^{b_n} \neq 0 \quad \text{or} \quad \int_a^{a_n} f(t)dt = 0$$

which is a contradiction, and thus f is 0 almost everywhere. □

Lemma 82. *If f is bounded and measurable on $[a, b]$ and*

$$F(x) = \int_a^x f(t)dt + F(a)$$

then $F'(x) = f(x)$ for almost all $x \in [a, b]$.

Proof. Since F is of bounded variation on $[a, b]$, we know that $F(x)$ has a derivative almost everywhere. Let $K > 0$ be such that $|f(x)| \leq K$ for all $x \in [a, b]$. We then let

$$f_n(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}}$$

So, we get that

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t)dt \leq nK \cdot (x + \frac{1}{n} - x) = K$$

And we know that f_n are each uniformly bounded by K. Notice that $f_n(x) \to F'(x)$ almost everywhere. We use the bounded converge theorem on f_n Let $c \in [a, b]$. Then

$$\int_a^c F'(x) = \lim_{n \to \infty} \int_a^c f_n(t)dt$$

$$= \lim_{n\to\infty} n \int_a^c [F(t+\frac{1}{n}) - F(t)]dt$$

Notice that

$$\int_a^c F(t+\frac{1}{n})dt = \int_{a+\frac{1}{n}}^{c+\frac{1}{n}} F(t)dt$$

Thus

$$\lim_{n\to\infty} n[\int_c^{c+\frac{1}{n}} F(t)dt - \int_a^{a+\frac{1}{n}} F(t)dt] = \lim_{n\to\infty}[n\int_c^{c+\frac{1}{n}} F(t)dt - n\int_a^{a+\frac{1}{n}} F(t)dt]$$

$F(x)$ is continuous by our previous lemma, and the above line is equal to

$$F(c) - F(a) = \int_a^c f(t)dt$$

This tells us that

$$\int_a^c [F'(t) - f(t)]dt 0 \forall c \in [a,b]$$

Which implies that $F'(t) - f(t)$ almost everywhere on $[a,b]$. □

Theorem 83. *Let f be an integrable function and suppose that $F(x) = F(a) + \int_a^x f(t)dt$. Then, $F'(x) = f(x)$ for almost all $x \in [a,b]$.*

Proof. Assume that $f > 0$.

$$f_n(x) = \begin{cases} f(x) \text{ if } f(x) \leq n \\ n \text{ if } f(x) > n \end{cases}$$

Note that $f(x) - f_n(x) \geq 0$ and

$$G_n(x) = \int_a^x [f(t) - f_n(t)]dt$$

is increasing in x (you have a positive function, the more you integrate it, the larger it gets). This implies that $G_n(x)$ has a derivative almost everywhere and $G_n'(x) \geq 0$, We have that $|f_n(x)| \geq n$ for all $x \in [a,b]$, so by the previous lemma

$$\frac{d}{dx}\int_a^x f_n(t)dt = f_n(x)$$

almost everywhere.

$$F(x) = F(a) + \int_a^x [f(t) - f_n(t)]dt + \int_a^x f_n(t)$$

which means that $F'(x)$ exists, and

$$F'(x) = G_n'(x) + f_n(x) \geq f_n(x)$$

almost everywhere, for all n. Let $n \to \infty$, in which case $F'(x) \geq f(x)$ almost everywhere. We take

$$\int) a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a)$$

Since $f \geq 0$ by assumption, $F(x)$ is monotonically increasing. By the theorem for increasing functions, we have that

$$\int_a^b F'(t)dt \leq F(b) - F(a)$$

This gives us that

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

This implies that

$$\int_a^b [F'(x) - f(x)]dx = 0 \Rightarrow F'(x) = f(x)$$

almost everywhere. □

Remark. An increasing function can have at most countably many points of discontinuity.

2.18 Absolute Continuity

Definition. $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if $\forall \epsilon > 0 \exists \delta > 0$ such that

$$\sum_{i=1}^{k} |f(x_i') - f(x_i)| < \epsilon$$

for every collection $\{(x_i, x_i')\}$ of non-overlapping intervals with sum

$$\sum_{i-1}^{k} |x_i' - x_i| < \delta$$

Lemma 84. *If f is absolutely continuous, on $[a, b]$, then it is of bounded variation on $[a, b]$.*

Proof. Since f is absolutely continuous, we set $\epsilon = 1$ and choose $\delta = \delta(1) > 0$ from the definition of absolute continuity. Start with a subdivision of $[a, b], a = x_0 < x_2 < ... < x_n = b$ and look at

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

Let $K = 1 + \frac{b-a}{\delta}$. Then,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq K \cdot 1$$

And this implies that f is of bounded variation. □

Corollary 85. *If f is absolutely continuous, then f has a derivative almost everywhere.*
Lemma 86. *If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ almost everywhere, then $f(x)$ is constant.*

Proof. Take $c \in [a, b]$; we need to show that $f(a) = f(c) \forall c$. Let $E := \{x :\in [a, c] | f'(x) = 0\} \subset [a, c]$ thus, $m(E) = c - a$. Since the derivative is 0 at each such pint, this means that

$$\lim_{h \to \infty} \frac{f(x + h) - f(x)}{h} = 0 \quad (*)$$

We fix $\eta, \epsilon > 0$. For all $x \in E$, there exists an arbitrary small interval $[x, x + h] \subset [a, c]$ such that

$$|f(x + h) - f(x) < \eta \cdot h$$

There exists some finite collection $\{[x_k, y_k]\}$ of non-overlapping intervals which satisfy property (*), and which cover E, except for a set of measure $\delta(\epsilon) > 0$, where δ is the constant corresponding to ϵ and f in the definition of the absolute continuity of f. We assume that $x_i < x_{i+1}$ for the convenience of notation. Then,

$$x_i < y_i < x_{i+1} < y_{i+1}$$

Then

$$\sum_{i=1}^{k} |f(y_i) - f(x_i)| \leq \sum_{i=1}^{k} \eta \cdot |y_i - x_i| < \eta(c-a)$$

Alternatively,

$$\sum_{i=0}^{k} |f(y_i) - f(x_{i+1})| < \epsilon$$

which follows from absolute continuity, and that

$$\sum_{i=0}^{k} |y_i - x_{i+1}| < \delta$$

From the triangle inequality, we have that

$$|f(c) - f(a)| \leq \sum_{i=1} 6k|f(x_i) - f(y_i)| + \sum_{i=0}^{k} |f(y_i) - f(x_{i+1})| < \eta(c-a) + \epsilon$$

And since both η and ϵ were chosen arbitrarily, we let them go to 0.This then implies that $f(c) = f(a)$. Also, c is arbitrary from $[a, b]$, thus f is constant.

□

Theorem 87. *A function f is absolutely continuous if and only if it is an indefinite integral.*

Proof. If F is an indefinite integral, then use $\epsilon - \delta$ for the integral of an integrable function.

In the other direction, we suppose that F is absolutely continuous on $[a, b]$. Then F is of bounded variation, which means that $F = F_1 - F_2$ where both F_1, F_2 are increasing. In this case, $F'(x)$ exists almost everywhere, and

$$|F'(x)| \leq F_1'(X) + F_2'(x)$$

So

$$\int_a^b |F'(x)|dx \leq \int_a^b [F_1'(x)F_2'(x)]dx \leq F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

Which gives us that $|F'(x)|$ is integrable, which tells us that $F'(x)$ is integrable. We let

$$G(x) = \int_a^x F'(x)dx$$

and that then $G(x)$ is absolutely continuous, which implies that $F(x) - G(x)$ is also absolutely continuous, which we will call f. We take the derivative of f:

$$f'(x) = F'(x) - F'(x) = 0 \quad \text{almost everywhere}$$

By the previous lemma, we have that $f(x)$ is constant, which gives us that $F(x) = G(x) = \int_a^x F'(t)dt + F(a)$. Therefore F is an indefinite integral (in fact, it is the indefinite integral of its derivative), and we are done. □

Corollary 88. *Every absolutely continuous function is the indefinite integral of its derivative.*

2.19 Convex Functions

Definition. φ"$(a,b) \to \mathbb{R}$ is **convex** if for every $x, y \in (a,b)$ and for every $\lambda, 0 \leq \lambda \leq 1$, we have that

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

Lemma 89. *If $\varphi : (a,b) \to \mathbb{R}$ is convex, and if $x, y, x', y' \in (a,b)$ with $x \leq x' < y$ and $x < y \leq y'$ then*

$$\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(y') - \varphi(x')}{y' - x'}$$

Proof. This proof is mainly computation. Let there be two points on the real line x_1, x_2, and $x \in (x_1, x_2)$. Then

$$x = \lambda x_1 + (1-\lambda)x_2$$

where $0 \leq \lambda \leq 1$. So $x = \lambda(x_1 - x_2) + x_2$ and $\lambda = \frac{x_2 - x_1}{x_2 - x_1}$. So,

$$1 - \lambda = \frac{x - x_1}{x_2 - x_1}$$

So we look at $\varphi(xx_1 +)(1-\lambda)x_2) \leq \lambda\varphi(x_1) +)1 - \lambda_\varphi(x_2)$. Then,

$$\varphi(\lambda x_1 + (1 - \lambda x_2) \leq \frac{x_2 - x}{x_2 - x_1}\varphi(x_1) + \frac{x - x_1}{x_2 - x_1}\varphi(x_2)$$

Dividing by $x_2 - x_1$, we have

$$(x_2 - x_1)\varphi(x) \leq (x_2 - x)\varphi(x_1) + (x - x_1)\varphi(x_2)$$

Subtracting $(x_2 - x_1)\varphi(x_2)$, we get

$$(x_2 - x_1)(\varphi(x) - \varphi(x_1)) \leq (x_1 - x)(\varphi(x_1)) + (x - x_1)\varphi(x_1)$$

so

$$(x_2 - x_1)[\varphi(x) - \varphi(x_1)] \leq (x - x_1)[\varphi(x_2) - \varphi(x_1)]$$
$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1}$$

and we are done for the first case. The second follows similarly. □

Proposition 90. *If φ is convex on (a,b) then φ is absolutely continuous on each closed sub-interval of (a,b). The right and the left-hand derivatives of φ exist at each point of (a,b) and are equal to each other except on a countable set. The left and the right-hand derivatives are monotonically increasing functions, and at each point the left-hand derivative is smaller or equal to the right-hand derivative.*

Proof. Take $[c,d] \subset (a,b)$. We know that

$$\frac{\varphi(c) - \varphi(a)}{c - a} \leq \frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(b) - \varphi(d)}{b - d} \forall x, y \in [c,d]$$

Thus
$$\frac{\varphi(c)-\varphi(a)}{c-a} \leq \varphi(y)-\varphi(x) \leq \frac{\varphi(b)-\varphi(d)}{b-d}(y-x)$$
Which tells us there exists some M such that
$$M := max\{|\frac{\varphi(c)-\varphi(a)}{c-a}|, |\frac{\varphi(b)-\varphi(d)}{b-d}|\} > 0$$
Which tells us that
$$|\varphi(y)-\varphi(x)| \leq M|y-x| \Rightarrow \varphi \text{ is absolutely continuous}$$
because
$$\sum |\varphi(x_i)-\varphi(x_i')| \leq \sum M|x_i - x_i'| = M\sum |x_i - x_i'| < M\delta = \epsilon$$
which fits in the with the definition of absolute continuity. We then have that for $x_0 \in (a,b)$, that
$$\frac{\varphi(x)-\varphi(x_0)}{x-x_0}$$
is increasing by the previous lemma. The limits as x approaches x_0 from the right and from the left exist and are finite. Thus φ has left and right-hand derivatives at each point of (a,b). From this it is then clear that the left-hand derivative is less than or equal to the right hand derivative at each point.

Take $x_0 < y_0, x < y_0, x_0 < y$. The following inequality still persists:
$$\frac{\varphi(x)-\varphi(x_0)}{x-x_0} \leq \frac{\varphi(y)-\varphi(y_0)}{y-y_0}$$
Either the derivative at x_0 is less than or equal to either derivative at y_0. Each derivative function is a monotone increasing function. Monotone functions are continuous except almost everywhere on a countable set. We then let φ'_+ be the right-hand derivative function, and let $c \in (a,b)$ be a point of continuity of φ'_+. We have that
$$\varphi'_+(c-h) \leq \varphi'_-(c) \leq \varphi'_+(c)$$
for $h > 0$. Then
$$\lim_{h\to 0} \varphi'_+(c \cdot h) = \varphi'_+(c)$$
because $\varphi'_- = \varphi'_+(c) \Rightarrow \varphi'(c)$ exist for each point of continuity of φ'_+. We are then done with our proof.

□

Proposition 91. *If φ is a continuous function on (a,b) and if one of the derivatives (say, $D^+\varphi$) of φ is nondecreasing then φ is convex.*

Proof. We take $a < y < x < b$ and form functions of the form
$$\psi(t) := \varphi(ty + (1-t)x) - t\varphi(y) - (1-t)\varphi(x)$$
We want to show that $\psi(t) \leq 0$. So we take $\psi(0) = \psi(1) = 0$. We use the chain rule to get:
$$D^+\psi = (D^+\varphi)\dot{(}x-y) - \varphi(x) + \varphi(y)$$

which gives us that

$$D^+\varphi \uparrow \Rightarrow D^+\psi \uparrow$$

so,

$$\psi : [0,1] \to \mathbb{R} \Rightarrow \text{ for a fixed x,y, } \psi \text{ is continuous on } [0,1]$$

Thus there exists a maximum of ψ on $[0,1]$. Let $\gamma \in [0,1]$ be the point where the maximum is achieved. At the maximum, we know that

$$D^+\psi(\gamma) < 0, D^+\psi \uparrow$$

$$\Rightarrow D^+\psi(t) \leq 0 \forall t \in [0,\gamma] \Rightarrow \psi \downarrow \text{ on } [0,\gamma]$$

Then, $\psi(0) = 0 \Rightarrow \psi(\gamma) \leq =$, and since $\psi(\gamma)$ is the maximum, this implies that $\psi(t) \leq 0 \forall t \in [0,1]$. Hence, we are finished. □

Corollary 92. *Assume that φ has the second derivative at every point in (a,b). Then, φ is convex if and only if $\varphi'' \geq 0 \ \ \forall x \in (a,b)$.*
Proposition 93. *(Jensen's Inequality) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and let $f : [0,1] \to \mathbb{R}$ be integrable. Then*

$$\int_0^1 \varphi(f(t))dt \geq \varphi\left(\int_0^1 f(t)dt\right)$$

Definition. $\varphi : (a,b) \to \mathbb{R}$ is convex, $x_0 \in (a,b)$ the **supporting line** of φ at x_0 is

$$y = m(x - x_0)\varphi(x_0)$$

Such that the graph of $\varphi(x)$ lies above this line. In other words,

$$\varphi(x) \geq m(x - x_0) + \varphi(x_0) \quad \forall x \in (a,b)$$

where

$$\varphi'_-(x_0) \leq m \leq \varphi'_+(x_0)$$

There is at least one support line (I think he meant to write, "There is at least one support line for a convex function".)

Proof. We define

$$\alpha = \int_0^1 f(t)dt$$

We take the supporting line at α, which is the line

$$\varphi(x) \geq y = m(x - \alpha) + \varphi(\alpha)$$

Then we know that

$$\varphi(f(t)) \geq m(f(t) - \alpha) + \varphi(\alpha)$$

Integrating,

$$\int_0^1 \varphi(f(t))dt \geq \varphi(\int_0^1 f(t)dt)$$

□

Chapter 3

End of Class, L^P-Spaces

Take $p > 0, \in \mathbb{R}$, and fixed. Then $L^p = L^p([0,1]) = \{f : [0,1] \to \mathbb{R} | \int_0^1 |f|^p dm < \infty\}$. As a result, L^1 consists of all Lebesgue's integrable functions.
Example. Notice that $f(x) = \frac{1}{x}$ for $0 < x \leq 1$ and $f(x) = 0$ when $x = 0$ is not in L^1, but *is* in L^2.
Remark. Taking $|f+g|^p \leq 2^p(|f|^p + |g|^p)$ because

$$|f+g|^p \leq (|f| + |g|)^p \leq [2 \cdot max\{|f|, |g|\}]^p = 2^p max\{|f|^p, |g|^p\} \leq 2^p[|f|^p + |g|^p]$$

Remark. Given $\alpha \in \mathbb{R}$, then $|\alpha f|^p = |\alpha|^p \cdot |f|^p$.
Fact. From these two remarks, if $f, g \in L^p, \alpha, \beta \in \mathbb{R}$, then

$$\alpha f + \beta g \in L^p$$

L^p is a linear space (or infinite dimensional vector space). Then, the $:^p$ **norm** is as follows:

$$||f||_p = \left(\int_0^1 |f|^p dm\right)^{\frac{1}{p}}$$

Fact. $||f||_p = 0$ if and only if $f = 0$ almost everywhere. Also, if $\alpha \in \mathbb{R}$, then

$$||\alpha f||_p = |\alpha| \cdot ||f||_p$$

Definition. A linear space X is a **normal linear space** if for every $f \in X$ there exists an assignment $||f|| \geq 0$ such that the following three properties hold:

1. $||\alpha f|| = |\alpha| \cdot ||f||$
2. $||f+g|| \leq ||f|| + ||g||$
3. $||f|| = 0 \iff f = 0$.

Definition. Two measurable functions are equivalent if and only if they are equal almost everywhere.
Example. $O(x) \equiv 0$ for all x,

$$f(x) = \begin{cases} n, x = \frac{1}{n} \\ 0, \text{ elsewhere} \end{cases}$$

interestingly, $O(x)$ and $f(x)$ are equal almost everywhere so they are equivalent.

Let $p = \infty$. Then, L^∞ is all measurable functions on $[0,1]$ which are bounded except on a subset of measure 0. We identify equivalent functions as we did before, but introduce the L^∞ norm:

$$||f||_\infty = \text{ essential } \sup |f|$$

Definition.

$$ess \ \sup \ f := \inf\{ \sup_{t\in[0,1]} g(t) : g(t) = f(t), \ a.e.\}$$

Alternatively,

$$ess \ \sup f := \int \{M \in \mathbb{R} : m(\{t \in [0,1] : f(t) > M\}) = 0\}$$

3.1 Minkowski's Inequality

If $f, g \in L^p$, with $1 \leq p \leq \infty$, then $f + g \in L^p$, and

$$||f + g||_p \leq ||f||_p + ||g||_p$$

If $1 < p < \infty$, then the inequality can be an equality if and only if there exists $\alpha, \beta \geq 0$ such that $\beta f = \alpha g$.

Proof. If $p = 1$, then

$$\int_0^1 |f + g| dm \leq \int_0^1 |f| dm + \int_0^1 |g| dm$$

We'll worry about the case in which $1 < p < \infty$. If $||g||_p = 0$ or $||g||_p = 0$, then we are really done. We will assume that $||g||_p = \alpha \neq 0$, $||g||_p = \beta \neq 0$. We define functions $f_0, g_0 : [0,1] \to \mathbb{R}$ by the equations

$$|f| = \alpha f_0, \qquad |g| = \beta g_0$$

from which we claim that $||f_0||_p = 1 = ||g_0||_p$. Define $\lambda = \frac{\alpha}{\alpha+\beta}$. So, $1 - \lambda = \frac{\beta}{\alpha+\beta}$. We know that $0 < \lambda < 1$, based on the way we defined α and β.

$$(1) \quad |f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p = (\alpha f_0(x) + \beta g_0(x))^p$$

$$= (\alpha + \beta)^p \left[\frac{\alpha}{\alpha + \beta} f_0(x) + \frac{\beta}{\beta + \alpha} g_0(x) \right] = (x + \beta)^p [\lambda f_0(x) + (1 - \lambda) g_0(x)]^p$$

We have that $F(x) = x^p$ is a convex function for $p \geq 1$, so we have that the line above is less than or equal to:

$$\leq (\alpha + \beta)^p [\lambda f_0(x)^p + (1 - \lambda) g_0(x)^p]$$

As a result,

$$\int_0^1 |f(x) + g(x)|^P dm \leq (\alpha + \beta)^p [\lambda \int_0^1 f_0^p dm + (1 - \lambda) \int_0^1 g_0^p du]$$

so

$$||f + g||_p \leq ||f||_p + ||g||_p$$

The inequality at (1) equality if $sign(f) = sign(g)$ almost everywhere, so you have that $f_0(x) = g_0(x)$, so $\beta|f| = alpha|g|$ almost everywhere, which gives you that $\beta f = \alpha g$. $\square$

3.2 Minkowski's Inequality for $0 < p < 1$

Let $f, g \geq 0$, $f, g \in L^p$, $0 < p < 1$. Then,

$$||f + g||_p \geq ||f||_p + ||g||_p$$

Lemma 94. *Let $1 \leq p < \infty$. Then for $a, b, t \geq 0$ we have that*

$$(a + tb)^p \geq a^p + ptba^{p-1}$$

Proof.

$$\varphi(t) = (a + tb)^p - a^p - ptba^{p-1}$$

So $\varphi(0) = 0$, if we show that φ is a decreasing function, then we're done.

$$\varphi'(t) = pb(a + tb)^{p-1} - pba^{p-1} = pb[(a + b)^{p-1} - a^{p-1}] \geq 0, t \geq 0$$

□

3.3 Hölder Inequality

If $0 \leq p \leq q \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and if $f \in L^p, g \in L^q$, then $f \cdot g \in L^1$ and

$$||fg||_1 = \int_0^1 |fg| dm \leq \left(\int_0^1 |f|^p dm \right)^{\frac{1}{p}} \cdot \left(\int_0^1 |g|^q dm \right)^{\frac{1}{q}}$$

which says that

$$\int |fg| \leq ||f||_p ||g||_q$$

The equality holds if and only if for some constants α and β not both equal to 0

Proof. If $p = 1$ and $q = \infty$, then

$$\int |fg| \leq \left(\int |f| \right) \cdot ||g||_\infty$$

because

$$|f(x)g(x)| \leq |f(x)| \cdot ess \ \sup |g| \quad \text{a.e.}$$

If $1 < p < \infty$, then this implies that $1 < q < \infty$. Assume $f \geq 0, g \geq 0$. Define $h(x) = g(x)^{q-1}$. This can be written as $g(x)^{\frac{q}{p}}$, because $\frac{1}{p} + \frac{1}{q} = 1$. Also note that from here, $g(x) = h(x)^{p-1} = h(x)^{\frac{p}{q}}$. Applying our lemma, we have that

$$ptf(x)g(x) = ptf(x)h(x)^{p-1} \leq [h(x) + tf(x)]^p - h(x)^p$$

From here we take the integration of this inequality,

$$pt\int fg \leq \int |f+tg|^p - \int h^p$$

which gives us that

$$pt\int fg \leq ||h+tf||_p^p - ||h||_p^p$$

from our last inequality,

$$\leq (||h||_p + ||t||f||_p)^p - |h||_p^p$$

taking the derivative at $t = 0$, we get that

$$p\int fg \leq p||f||_p \cdot ||h||_p^{p-1}$$

finally, we get that $||h||_p^{p-1} = (\int |h|^p)^{1/p})^{p-1}$, so we get that

$$\int fg \leq ||f||_p \cdot ||g||_q$$

□

3.4 Convergence and Completeness

Definition. $\{f_n\}$ in a normal space X converges to $f \in X$ if $\forall \epsilon > 0$ there exists $N : \forall n \geq N$,

$$||f - f_n|| < \epsilon$$

and

$$\lim_{n\to\infty} f_n = f, f_n \to f$$

Let $X = L^p$, $1 < p < \infty$, f_n converges to f if $||f - f_n||_p \to 0$ as $n \to \infty$ is the convergence in the mean of order p.

If $X = L^\infty$, then f_n converges to f if

$$||f_n - f||_\infty \to 0, n \to \infty$$

nearly uniformly convergence.

A sequence of functions converging at each point of the domain (pointwise convergence).
Definition. If $\{f_n\}$ is a sequence in a normed space X, then it is Cauchy if $\forall \epsilon > 0$ there exists $N : \forall n, m > N$,

$$||f_n - f_m|| < \epsilon$$

Definition. A normed space is complete if every Cauchy sequence converges. (Complete normed space are called **Banach** space).
Definition. $\{f_n\}$ is summable to a sum s if

$$||s - \sum_{i=1}^{n} f_i|| \to 0, n \to \infty$$

Definition. $\{f_n\}$ is absolutely summable if $\sum_{i=1}^{\infty} ||f_n|| < \infty$.
Proposition 95. *A normal space X is complete if and only if every absolutely summable series is summable.*

Proof. X complete, $\{f_n\}$ is also summable, so

$$\sum_{n=1}^{\infty} ||f_n|| = M < \infty$$

$$\Rightarrow \epsilon > 0, \exists N : \sum_{n=N}^{\infty} ||f_n|| < \epsilon$$

Let $s_n = \sum_{i=1}^{n} f_i$. Let $m, n \geq M$,

$$||s_n - s_m|| = ||\sum_{i=m}^{n} f_i|| \leq \sum_{i=1}^{n} ||f_i|| \leq \sum_{i=N}^{\infty} ||f_n|| M\epsilon$$

which means that $\{s_n\}$ is Cauchy. X is complete implies that $s_n \to s$. Then, $s = \sum_{i=1}^{\infty} f_i$.

In the other direction, take $\{f_n\}$ to be a Cauchy sequence in X. For all $k \in \mathbb{N}$, $\exists n_k$ such that $||f_n - f_m|| < 2^{-k}$ for every $n, m \geq n_k$. Assume $n_{k+1} > n_k$. So, $\{f_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}_{n=1}^{\infty}$. Let $g_1 = f_{n_1}$. So $g_k = f_{n_k} - f_n k - 1$, for $k > 1$. This tells us that

$$\sum_{i=1}^{k} g_i = f_{n_k}$$

So $||g_k|| - ||f_{n_k} - f_{n_{k-1}}|| < 2^{-k+1}$ for $k \geq 1$, so

$$\sum_{k=1}^{\infty} ||g_k|| \leq ||g_1|| + \sum_{k=2}^{\infty} 2^{-k+1} = ||g_k|| + 1$$

which implies that $\{g_k\}$ is absolutely summable. This implies that there exists an element $f \in X$ such that

$$\sum_{i=1}^{k} g_i \to f$$

so $f_{n_k} \to f$ as $k \to \infty$. □

Proof. We show $f = \lim_{n\to\infty} f_n$, $\epsilon > 0$, there exists $N : \forall n, m > M$

$$||f_n - f_m|| < \frac{\epsilon}{2}$$

$f_{n_k} \to f$, $k \to \infty$. There exists K such that for all $k \geq K$, ,

$$||f_{n_k} - f|| < \frac{\epsilon}{2}$$

so $\{f_n\}$ is a Cauchy sequence in X.

Choose k large enough such that $k > K$, and $n_k > N$. Then for each $n \geq N$,

$$||f_n - f|| \leq ||f_n - f_{n_k}|| + ||f_{n_k} - f|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Theorem 96. The Riesz-Fischer Theorem: *The L^p-spaces are complete, $1 \leq p \leq \infty$.*

Proof. If $p = \infty$, this is clear. Else, $\{f_n\}$ is a sequence in L^p such that $\sum_{n=1}^{\infty} ||f_n|| = M < \infty$. We define

$$g(x) = \sum_{k=1}^{n} |f_k(x)|$$

Notice,

$$||g_n||_p = \left(\int_0^1 (|f_k(x)|)^p \, dx\right)^{\frac{1}{p}} \leq \sum_{k=1}^{n} ||f_k||_p$$

(the last part follows from Minkowski's inequality). We have that

$$\int (g_n)^p \leq < M^p$$

For all $x \in [0,1]$, $\{g_n(x)\}_{n=1}^{\infty} \uparrow$. Let $g(x) = \lim_{n\to\infty} g_n(x)$ (the limit in the sense of extended real numbers). $g(x)$ is measurable. By Fatou's lemma, we have that

$$\int (g)^p = \int \underline{\lim}(g_n)^p \leq \underline{\lim} \int g_n^p \leq M^p$$

This gives us that $g \in L^p$. This means that g is finite almost everywhere. For every x where $g(x)$ is finite,

$$\sum_{n=1}^{\infty} f_n(x)$$

is an absolutely convergent series of real numbers. Then,

$$\sum_{n=1}^{\infty} f_n(x)$$

converges for each x where $g(x)$ is finite- which is almost everywhere- and let

$$s(x) = \sum_{k=1}^{\infty} f_k(x)$$

when it exists and $s(x) = 0$ where $g(x) = \infty$.

$$s_n = \sum_{k=1}^{\infty} f_k \to s \text{ a.e.}$$

implies that s is measurable.

$$|s_n(x)| \leq g_n(x) \leq g(x) \forall x$$

By the Lebesgue Dominated Convergence theorem applied to,

$$|s_j|^p \leq g^p$$

we have that

$$\int g^p = \lim_{n\to\infty} \int |s_n|^p = \int \lim_{n\to\infty} |s_n|^p = \int |s|^p$$

this implies that $s \in L^p$. So

$$|s - s_n|^p \leq (|s| + |s_n|)^p \leq (2g)^p = 2^p g^p$$

by the Lebesgueq dominated convergence theorem, we have

$$\lim_{n\to\infty} |s - s_n|^p = \int \lim_{n\to\infty} |s - s_n|^p = \int 0 = 0$$

□

3.5 Approximations in L^p

Proposition 97. *Given $f \in L^p$, where $1 \leq p < \infty$ and $\epsilon > 0$, there is a step function φ and a continuous function ψ such that*

$$||f - \varphi||_p < \infty, \qquad ||f - \psi||_p < \epsilon$$

3.6 Bounded Linear Functionals on the L^p Spaces

If X is some normed space, and $F : X \to \mathbb{R}$, such that $F(\alpha f + \beta g) = \alpha F(t) + \beta F(g)$ for all $f, g \in X$ and every $\alpha, \beta \in \mathbb{R}$, then F is said to be **linear-functional.**

F is bounded if there exists M such that $|F(f)| \leq M \cdot ||f||$. If this is true for every $f \in X$, then F is a bounded linear functional, and

$$||F|| = \sum_{f \in X - \{o\}} \frac{|F(f)|}{||f||}$$