

UNBOUNDED APOLLONIAN PACKINGS AND RESIDUAL POINTS

ABSTRACT. We begin discussing Apollonian circle packings by introducing the notion of a Descartes configuration of four mutually tangent circles. Apollonian circle packings and the configuration of four mutually tangent circles they contain can be described in at least three ways: geometrically, algebraically, and holographically. Building off the work of Lagarias et al., we primarily examine unbounded circle packings and non-trivial residual points in an algebraic sense. Using the generators of the Apollonian group and the principles behind the reduction algorithm, we construct sequences of infinite length to identify unique self-similar packings.

1. INTRODUCTION

We define a circle in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to be a circle in \mathbb{C} or a line in \mathbb{C} together with a point at ∞ . Each circle comes with a choice of interior between the two connected components of its complement in $\widehat{\mathbb{C}}$. Given a triplet of mutually tangent circles C_1, C_2 , and C_3 which lie in $\widehat{\mathbb{C}}$, there are only two possible circles, C_4 and C'_4 , that can lie tangent to the triplet with disjoint interiors. Choosing one of these possible circles, we form an *ordered Descartes configuration* $\mathcal{D} = (C_1, C_2, C_3, C_4)$, which is a set of four mutually tangent circles lying in $\widehat{\mathbb{C}}$. The four circles are mutually tangential when any two intersect tangentially and there are distinct points of tangency for each pair. A Descartes configuration may have a circle that contains the other three, in which case, we say that these circles have internal contact. We say the circles of a given Descartes configuration \mathcal{D} belongs to generation 0, and we make new generations of circles by replacing C_i with C'_i . Since the other three circles are mutually tangential and can be extended to another Descartes configuration via C_i , we say that the circles C'_1, C'_2, C'_3 , and C'_4 make up generation 1. By replacing circles in the first n generations with the other circles C' forming a Descartes configuration, we obtain circles in the $(n + 1)$ st generation. The collection of circles from all generations 0, 1, ... is an *Apollonian circle packing* \mathcal{P} . For any three mutually tangent circles in \mathcal{P} , the circles tangent to all three will also lie in \mathcal{P} .

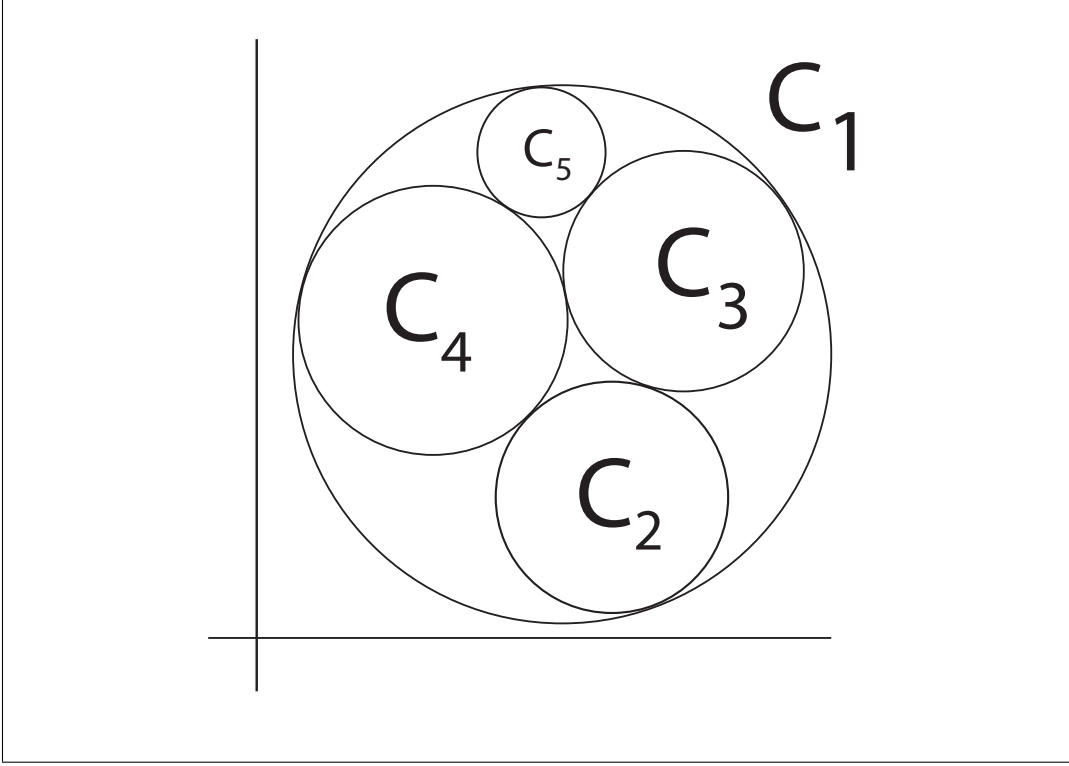


Figure 1: This is a Descartes configuration lying in $\widehat{\mathbb{C}}$, where C_1 is a bounding circle.

Each Descartes configuration $\mathcal{D} = (C_1, C_2, C_3, C_4)$ has a corresponding *Descartes quadruple* $\mathbf{q} = (a, b, c, d)$ describing the curvature of each C_i in \mathcal{D} . While \mathcal{D} describes four tangent circles in the plane, \mathbf{q} describes the curvature, i.e. inverse radius, of each circle. Descartes' theorem states that each such quadruple is contained in the kernel of the following quadratic form

$$Q_{\mathcal{D}}(w, x, y, z) := 2(w^2 + x^2 + y^2 + z^2) - (w + x + y + z)^2$$

Definition. A *projective variety* $\mathbb{P}\mathcal{D}$ is the set of non-zero solutions to a system of polynomial equations up to scaling defined thusly:

$$\mathbb{P}\mathcal{D} = \{(a, b, c, d) \in \mathbb{R}^4 : 2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2\} / \sim ,$$

where

$$(a, b, c, d) \sim (x, y, z, w) \leftrightarrow \exists \lambda \in \mathbb{R}^\times \text{ s.t. } \lambda a = x, \lambda b = y, \lambda c = z, \lambda d = w$$

Lagarias and others have shown that $\text{Möb}(2)$ can be identified with an index 2 subgroup of $\text{Aut}(Q_{\mathcal{D}})$.

Definition. We define the *general Möbius group* as $\text{Möb}(2) = \text{Möb}(2)_+ \cup \text{Möb}(2)_-$, where

$$\text{Möb}(2)_+ := \left\{ (z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0) \right\},$$

is the set of orientation-preserving Möbius transformations and where

$$\text{Möb}(2)_- := \left\{ (z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0) \right\}.$$

is the set of orientation-reversing Möbius transformations.

$\text{Möb}(2)$ acts on Descartes configurations, allowing for reflections (including complex conjugation). Since Möbius transformations take circles to circles and preserve angles, they can take an ordered Descartes configuration to any other Descartes configuration. (For more information on the general Möbius group, refer to Lagarias trilogy I, section 7). We observe that the Descartes configurations in each Apollonian packing form an orbit of a single Descartes configuration under a certain finitely generated discrete subgroup of $\text{Aut}(Q_D)$, which we call the *Apollonian group*.

Definition. The *Apollonian group* \mathcal{A} is the subgroup of $GL_2(\mathbb{Z})$ generated by the following four integer 4×4 matrices

$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

These generators form a Coxeter group with only the following relations: $S_1^2 = S_2^2 = S_3^2 = S_4^2 = I$. The *length* of an Apollonian group member is the number of generators that appear in a word such that $S_{i_j} \neq S_{i_{j+1}}$. For instance, we say $S_2 S_3 S_4 S_2 S_1$ has length 5 while $S_2 S_3 S_3 S_4 S_1$ has length 3 because $S_3 S_3 = I$.

If we view real Descartes quadruples $\mathbf{v} = (a, b, c, d)^T$ as column vectors, then the Apollonian group \mathcal{A} acts by matrix multiplication, sending \mathbf{v} to $M\mathbf{v}$, for any $M \in \mathcal{A}$. The geometric interpretation of the action of \mathcal{A} on a Descartes configuration \mathcal{D} has each S_i corresponding to inversion of the circle passing through the three tangent points that do not include the i th circle of the configuration. Since this inversion fixes the three circles and moves the fourth circle to the unique C'_i that is tangent to the other three, we call it the *reflection operator* (cite lagarias). In this sense, an Apollonian packing is just the orbit of a Descartes configuration under \mathcal{A} .

We say an Apollonian circle packing is *integral* if all the circles have an integer curvature. For each integral Apollonian circle packing and a given Descartes quadruple associated with

it, there is a reduction procedure that yields a sequence of Descartes quadruples manifested in the packing and halting at a unique *root quadruple*. We will cover the definition of the the reduction algorithm and the root quadruple in section (2). Geometrically, the Apollonian packing generated by actions of \mathcal{A} on the root quadruple $(-1, 2, 2, 3)$ can be represented by figure (1).

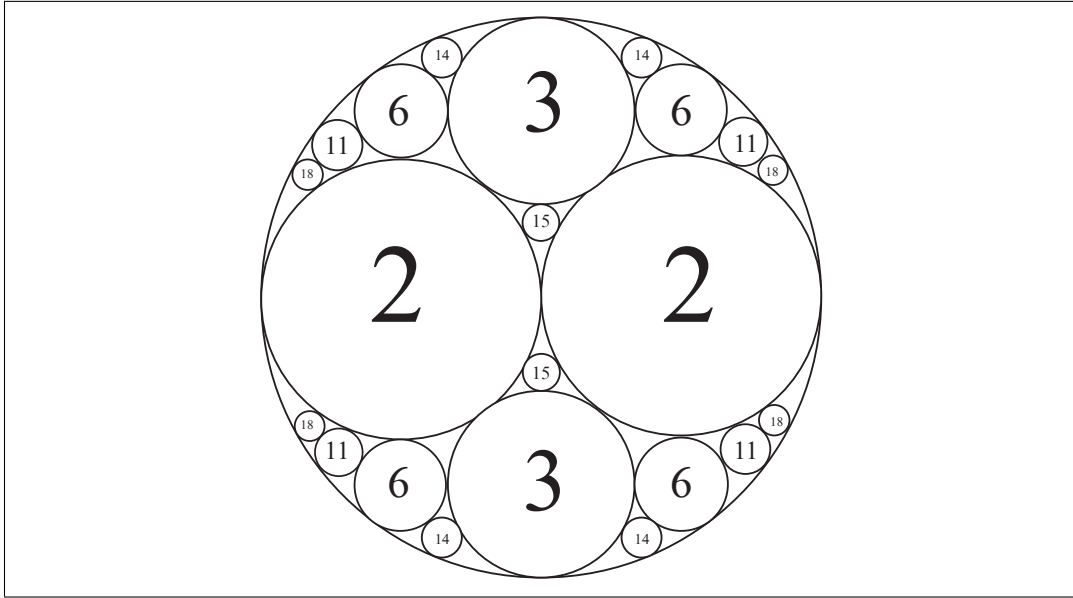


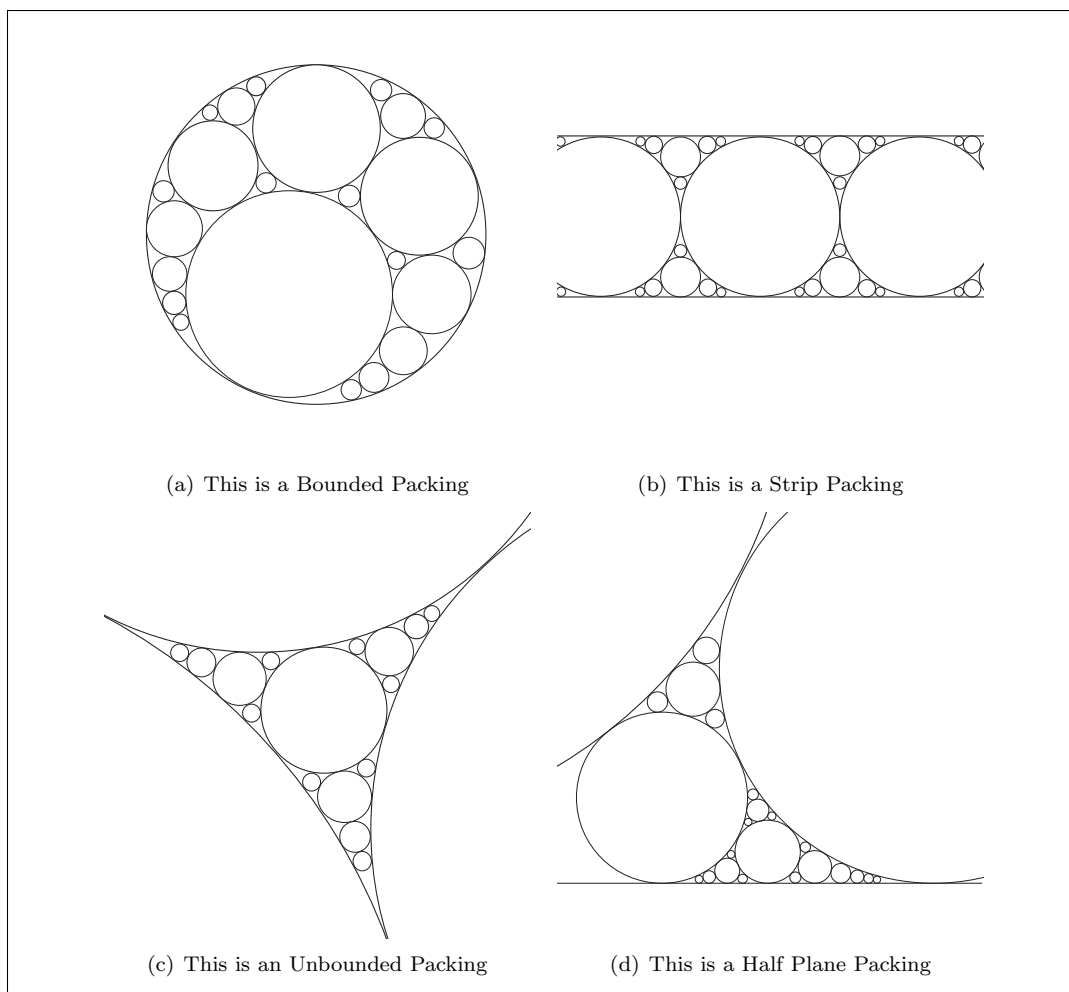
Figure 2: This is the geometric representation of an Apollonian packing with a root quadruple of $(-1, 2, 2, 3)$

Of particular interest to us will be the *residual points* of \mathcal{P} .

Definition. The **residual set** $\Lambda(\mathcal{P}; \mathcal{D})$ of an Apollonian packing \mathcal{P} with the Descartes configuration \mathcal{D} is the complement of the interiors of all circles of \mathcal{P} . A point $z_0 \in \Lambda(\mathcal{P}; \mathcal{D})$ is a *non-trivial residual point* if and only if z_0 is not in a circle of \mathcal{P} .

If we treat \mathcal{P} as a collection of open disks in the plane, $\Lambda(\mathcal{P}; \mathcal{D})$ is the plane with all of these disks removed. The main focus of this paper is to describe how to find the non-trivial residual points that make up $\Lambda(\mathcal{P}; \mathcal{D})$. We know that there are non-trivial residual points as previous research (cite Lagarias papers) has shown that \mathcal{P} has a Hausdorff dimension between 1 and 2.

Any Apollonian circle packing \mathcal{P} comes in four basic shapes: **bounded**, **half – plane**, **strip**, and **full – plane**, as illustrated in figures 3(a), 3(b), 3(c), 3(d). We are interested in finding full-plane unbounded packings, and plan on demonstrating their connection with residual points. While *a priori* the two do not relate, we seek to show that the residual



points of a fixed packing are in a bijective correspondence with unbounded packings with a marked configuration up to similarity.

Definition. Similarity conditions between packings can be represented through a collection of transformations, specifically translation, rotation, dilation, and reflection. More formally the following sets represent orientation-preserving similarities and orientation-reversing similarities of $\widehat{\mathbb{C}}$:

$$\mathcal{S}_+ = \{az + b : a, b \in \mathbb{C}, a \neq 0\} \quad \mathcal{S}_- = \{a\bar{z} + b : a, b \in \mathbb{C}, a \neq 0\}$$

and we take $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$

Definition. Two packings $\mathcal{P}_1, \mathcal{P}_2$ are **similar** with respective marked Descartes configurations \mathcal{D}_1 and \mathcal{D}_2 if there exists some transformation T such that $T(\mathcal{P}_1) = \mathcal{P}_2$ (implying

$T(\mathcal{D}_1) = \mathcal{D}_2$) where T belongs to \mathcal{S} . A packing \mathcal{P} is **self-similar** if there exists a transformation T' such that:

$$T'(\mathcal{P}) = \mathcal{P}$$

Where T' is again in either \mathcal{S}_+ or \mathcal{S}_- .

Definition. \mathcal{U} is the set of unbounded packings in $\widehat{\mathbb{C}}$ with a marked Descartes configuration up to similarity transformation. That is

$\mathcal{U} = \{(\mathcal{P}', Q') \mid \mathcal{P}' \text{ is an unbounded circle packing and } Q' \text{ is a Descartes configuration of } \mathcal{P}'\} / \sim$
where \sim denotes similarity.

Another useful tool will be infinite sequences of the generators which we call *Apollonian sequences*. We

Definition. The set of *Apollonian sequences* \mathcal{A}_∞ is defined as follows:

$$\mathcal{A}_\infty = \{\dots S_{i_3} S_{i_2} S_{i_1} \mid i_1, i_2, \dots \in \{1, 2, 3, 4\}, S_{i_{j+1}} \neq S_{i_j}\}.$$

Definition. $\overline{\mathcal{A}}$ is the set of *Apollonian sequences* that eventually repeat. That is

$$\overline{\mathcal{A}} = \{\dots S_{i_3} S_{i_2} S_{i_1} \in \mathcal{A}_\infty \mid \exists n \text{ such that } \forall j \gg 0, i_j = i_{j+n}\}.$$

It is clear that $\overline{\mathcal{A}} \subset \mathcal{A}_\infty$. Given a Descartes configuration \mathcal{D} , we associate circles contained in \mathcal{P} to some $a \in \mathcal{A} \setminus \{1\}$. A sequence of circles associated with the sequence of generators $\dots S_{i_3} S_{i_2} S_{i_1}$ is just the sequence of circles associated to S_{i_1} , $S_{i_2} S_{i_1}$, $S_{i_3} S_{i_2} S_{i_1}$ acting on the given configuration \mathcal{D}

Definition. Let $(\mathcal{P}, \mathcal{D})$ be a circle packing with marked configuration. Let $a_\infty = \dots S_{i_3} S_{i_2} S_{i_1}$ be an element of \mathcal{A}_∞ . We may associate to this sequence of generators, a sequence of circles C_n which corresponds to the n th circle added to the packing via applying S_{i_1} then S_{i_2} , etc. So $C_n = S_{i_n} \dots S_{i_1}(\mathcal{D}) \setminus S_{i_{n-1}} \dots S_{i_1}(\mathcal{D})$. In particular, C_n is associated to the n th generator in a_∞ .

Our primary tool in this paper will be the following sequence of \mathcal{A} -equivariant maps:

$$\overline{\mathcal{A}} \hookrightarrow \mathcal{A}_\infty \xrightarrow{F_1} \Lambda(\mathcal{P}; \mathcal{D}) \xrightarrow{F_2} \mathcal{U} \xrightarrow{F_3} \mathbb{P}\mathcal{D} \xrightarrow{F_4} \mathcal{S}^2$$

With these tools we have arrived at a proof that each Apollonian group member of length ≥ 2 has an eigenvector which is a Descartes quadruple of a self-similar unbounded packing and that all unbounded self-similar packings appear in this fashion. We also have shown that using F_2 in the diagram above, one can obtain non-trivial residual points from full-plane unbounded packings.

2. THE REDUCTION ALGORITHM AND ARRIVING AT RESIDUAL POINTS OF AN APOLLONIAN PACKING

Through a unique sequence of actions of the generators of \mathcal{A} on a Descartes quadruple \mathbf{q} , we can successively reduce the curvatures of a quadruple \mathbf{q} in some packing \mathcal{P} . This process can be found for a given quadruple by the reduction algorithm:

Definition. Given a real Descartes quadruple (a, b, c, d) with $a + b + c + d > 0$,

- (1) Test to see in order $1 \leq i \leq 4$ whether some S_i decreases the sum $(a + b + c + d)$. If so, apply that generator to the quadruple \mathbf{q} to produce a new quadruple, and continue the algorithm.
- (2) If no S_i decreases the sum, halt.

The following facts about the Reduction Algorithm can be shown to be true:

- (1) The Reduction Algorithm is unique; there exists only one configuration that decreases the curvature at each step given some quadruple \mathbf{q} .
- (2) If the algorithm terminates, it must terminate at a unique root quadruple. If a root quadruple exists, the algorithm must reach it.
- (3) The algorithm does not halt if and only if the packing is an unbounded packing that isn't the strip packing.

Definition. A **root quadruple** $\mathbf{r} = (a, b, c, d)$ is a quadruple belonging to some packing \mathcal{P} that has the following properties:

- (1) $a \leq 0 \leq b \leq c \leq d$
- (2) $a + b + c \geq d$

By considering the actions of an infinite chain of generators of \mathcal{A} on a quadruple in a fashion that reverses this algorithm, we arrive at the following theorem:

Theorem 1. *Given a bounded Apollonian circle packing P and an element $\mathcal{S} = (\dots S_{i_3} S_{i_2} S_{i_1})$ in \mathcal{A}_∞ , the following are true:*

- (1) *Given two points z_n and z'_n on the circles associated with the element $S_{i_n} S_{i_{n-1}} \dots S_{i_1}$ on the packing P with root quadruple \mathbf{r} , the following limits exist*

$$\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} z'_n$$

And converge to a residual point in the packing \mathcal{P} .

- (2) *The number of S_i 's appearing finitely many times in \mathcal{S} is equal to the number of circles of \mathcal{P} through $\lim_{n \rightarrow \infty} z_n$.*

Proof.

(1) By looking at the generators $\{S_1, S_2, S_3, S_4\}$ of the Apollonian group \mathcal{A} , we can see that given an infinite chain of such generators and having them act on a Descartes root quadruple $\mathbf{r}=(a, b, c, d)$, we are performing the reduction algorithm backwards. This follows inductively from the infinite actions of the generators of \mathcal{A} acting on a root quadruple associated with a bounded packing. In other words, as the generators in the chain \mathcal{S} act on \mathbf{q} successively, each generator increases the curvature value of some element in \mathbf{q} . To see that this infinite sequence converges to a specific point, we look to [1], which considers a slowest-case situation in which the authors conclude that the minimal growth of a reduced word of length $2n$ on a quadruple \mathbf{v} is:

$$|W_{2n}\mathbf{v}|_\infty = n(n+1)(a+b) - nc + (n-1)d$$

which clearly grows quadratically with n . We see that even the slowest growing case converges. Since the points are on the boundary of the circle and based on the definition of the packing, we can conclude that this sequence of points z_n converges to some residual point. This follows from the fact that

$$|z_n - z_m| < 2 \sum (\text{radii from the } n^{\text{th}} \text{ circle to the } m^{\text{th}} \text{ circle})$$

which goes to zero, since $\{z_n\}$ is a Cauchy sequence and converges since $\widehat{\mathbb{C}}$ is complete.

(2) By applying the generators that appear only a finite number of times to a quadruple \mathbf{q} , we arrive at some new configuration on which we would now like to apply the generators that appear an infinite number of times. By choosing z_n to be the points of tangency associated with the circle whose curvature will no longer be changed by the sequence of generators, we see that the limit of the sequence of points on the circles associated with the infinite sequence will converge to that point of tangency, which lies on the boundary of the circles whose curvatures are not altered by the infinite sequence of generators (since the number of generators that appear infinitely many times must be at least two, it follows that this point will remain fixed). Therefore, the number of generators that appear a finite number of times designates the number of lines that pass through the residual point associated with the infinite sequence of generators. \square

As we can see, this then allows us to construct a map

$$F_1 : \mathcal{A}_\infty \rightarrow \Lambda(\mathcal{P}; \mathcal{D}) \quad a_\infty \in \mathcal{A}_\infty \mapsto a_\infty Q \in \Lambda(\mathcal{P}; \mathcal{D})$$

By associating the residual point in $\Lambda(\mathcal{P}; \mathcal{D})$ with the residual point that the element \mathcal{S} arrives at when acting on the element \mathcal{D} in the packing \mathcal{P} .

Corollary 2. *Theorem 2 holds for a packing \mathcal{P} that is not necessarily bounded with a Descartes configuration \mathbf{q} not necessarily a root quadruple.*

Proof. Recall that any two Apollonian packings are equivalent under some Möbius transformation. By taking an unbounded packing to a bounded packing via some transform, the sequence of generators that approached a residual point starting from some root quadruple \mathbf{r} in a Descartes configuration \mathcal{D} must still approach a residual point in the new unbounded packing with Descartes configuration \mathcal{D}' with associated quadruple \mathbf{q} . From this, our corollary follows. \square

2.1. Injectivity and Surjectivity of F_1 . A natural question that follows from the construction of the map F_1 is whether or not this map is injective. What this boils down to is whether or not there exists a unique path of circles to a specific residual point.

As is geometrically obvious from figure 3, it is clear that there exist distinct paths of circles that converge to the same residual point. More formally, this case can be expressed as actions on the quadruple $\mathbf{v} = (1, 2, 3, 4)$ in the following way:

$$\lim_{n \rightarrow \infty} (S_3 S_4)^n \mathbf{v} = \lim_{n \rightarrow \infty} (S_4 S_3)^n \mathbf{v}$$

This rules out any hope for injectivity on elements of \mathcal{A}_∞ that have only two generators appearing infinitely many times. From this we have the following proposition:

Proposition 3. *The map $F_1 : \mathcal{A}_\infty \rightarrow \Lambda(\mathcal{P}; \mathcal{D})$ is 2-1 on elements in \mathcal{A}_∞ where only two generators appear infinitely many times.*

Proof. Circles are in bijective correspondence with lunes they are inscribed in when introduced to the packing generationally. Tangency points are contained in two lunes after the first generation, so there are exactly two choices for a circle to approach the residual point. The next circle must be tangent to the first one and so must be contained the lunes created by adding the first circle. There is exactly one of those lunes that still contains our residual point. Inductively, the remaining circles approaching the residual point are uniquely determined. \square

However, another natural question to ask is whether or not the map F_1 is surjective. Based on a similar argument to Theorem 1, we have the following proposition:

Proposition 4. *The map $F_1 : \mathcal{A}_\infty \rightarrow \Lambda(\mathcal{P}; \mathcal{D})$ is surjective.*

Proof. Consider some generation n in a packing. Based on the definition of a residual point, it is clear that residual points are in the lunes constructed by the circles of the packing in the first n generations. Then consider the next generation $(n+1)$; it is clear that the sum of the three lunes created by inscribing a circle into that lune is contained in the original lune. Once again, based on the definition of a residual point, we know that whatever points are

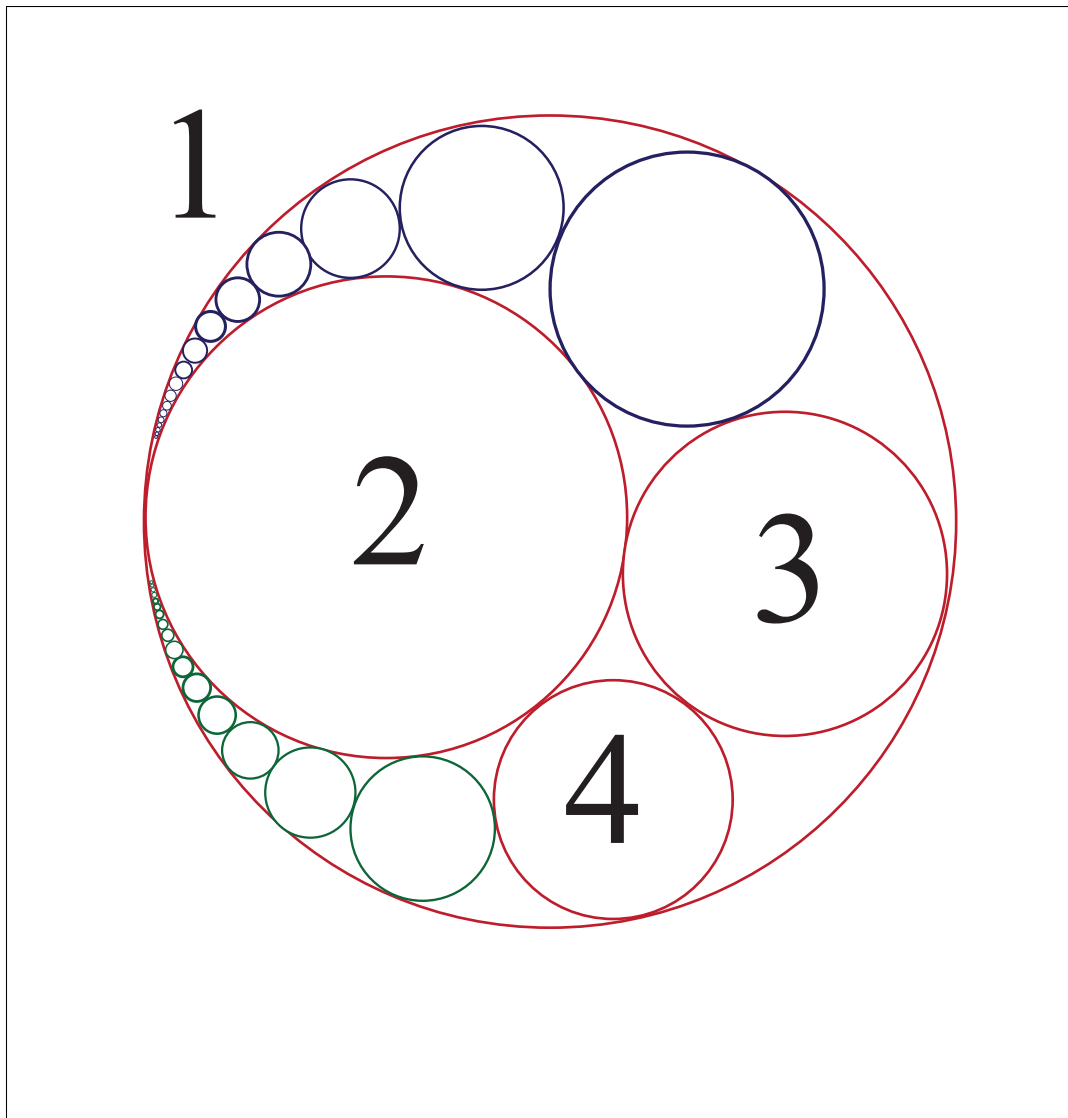


Figure 3: This is a packing generated by the root quadruple represented by circles (1,2,3,4)

in the packing are contained in one of the three smaller lunes. From this we can construct an infinite sequence of lunes, which gets smaller as n grows. To show that this sequence of lunes must converge to a point, we briefly introduce the concept of a **dual** circle. The dual circle is the new circle constructed by taking the three tangent points of three mutually tangential circles as points on a circle. This is represented in figure 4. Since the diameter of any lune is less than the diameter of the dual circle encapsulating it, it suffices to show that the diameter of the dual circle goes to zero. This follows from the diameter of the dual circle being represented by the elements in the Apollonian packing you started with in the

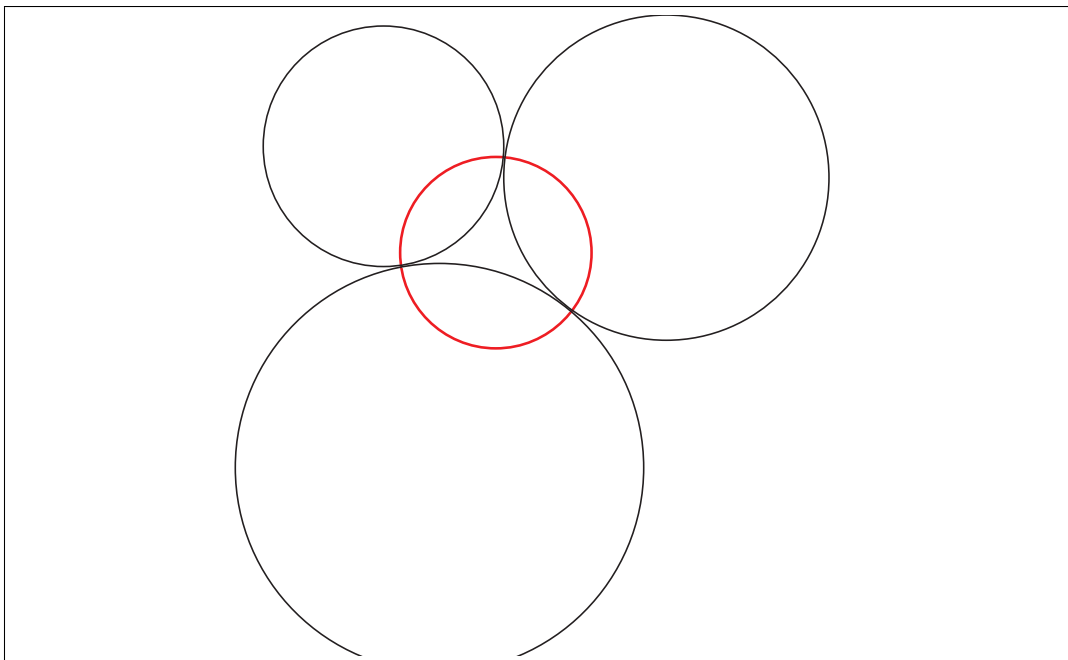


Figure 4: Here the Red Circle represents the Dual circle of the three black circles

following way:

$$d' = ab + bc + ac$$

Where d is the diameter of the dual circle. From this relation we can conclude that the diameter of the dual circle goes to zero. This then tells us that the lunes of our original packing must also converge to a point. \square

A harder question to answer is if the map F_1 is injective with respect to the elements in \mathcal{A}_∞ where at least three generators are repeated infinitely many times. We have the following proposition:

Proposition 5. *The Map $F_1 : \mathcal{A}_\infty \rightarrow \Lambda(\mathcal{P}; \mathcal{D})$ is injective on elements in \mathcal{A}_∞ that have at least three generators appearing infinitely many times.*

Proof. Unlike the case in which the residual point is at the intersection of two circles in the packing, the case where the point is on the boundary of exactly one or no circles cannot be contained in two lunes since the intersection of two lunes at most consists of one point, which is at the intersection of two circles. By definition, when looking for the lunes that encapsulate the point at which we would like to arrive, we have to disregard the intersection of two sides of the lune as a possible residual point. Because the circles used to approach the residual point are determined by the lunes containing that residual point, there is exactly one sequence of circles in the packing that approaches our point. Thus, one cannot construct

two distinct paths arriving at the same point from lunes without breaking our assumption that the point we arrive at is not at the intersection of two circles in the packing. \square

3. ARRIVING AT UNBOUNDED PACKINGS FROM RESIDUAL POINTS

For this section, fix an Apollonian packing \mathcal{P} and one of its Descartes configurations \mathcal{D} . Let $\Lambda(\mathcal{P}; \mathcal{D})$ denote the set of residual points of \mathcal{P} (specifying \mathcal{D} will help us define an action of \mathcal{A} on these residual points). To each residual point z_0 in $\Lambda(\mathcal{P}; \mathcal{D})$, we may associate a Möbius transformation $\phi_{z_0} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ defined as follows

$$\phi_{z_0}(z) = \frac{1}{z - z_0} \quad \text{with} \quad \phi_{z_0}(z_0) = \infty$$

For convenience, we define \mathcal{U} as follows:

$$\mathcal{U} = \{(\mathcal{P}', \mathcal{D}') \mid \mathcal{P}' \text{ is an unbounded packing, } \mathcal{D}' \text{ is a Descartes configuration of } \mathcal{P}'\} / \sim$$

where \sim denotes similarity. Given a residual point z_0 , we may now define the map $F_2 : \Lambda(\mathcal{P}, \mathcal{D}) \rightarrow \mathcal{U}$ as follows:

$$z_0 \mapsto (\phi_{z_0}(\mathcal{P}), \phi_{z_0}(\mathcal{D}))$$

Thus, F_2 maps a residual point z_0 to the unbounded packing obtained by moving z_0 to ∞ . This map is a bijection tying the study of residual points and the study of unbounded Apollonian packings together.

Proposition 6. *The map $F_2 : \Lambda(\mathcal{P}; \mathcal{D}) \rightarrow \mathcal{U}$ is bijective.*

Proof. Given a residual point z_0 , denote $F_2(z_0)$ by $(\mathcal{P}_{z_0}, \mathcal{D}_{z_0})$. We define a map $G_2 : \mathcal{U} \rightarrow \Lambda(\mathcal{P}, \mathcal{D})$ having the property that $G_2(F_2(z_0)) = z_0$ and $F_2(G_2(\mathcal{P}', \mathcal{D}')) = (\mathcal{P}', \mathcal{D}')$ for all z_0 in $\Lambda(\mathcal{P}; \mathcal{D})$ and all $(\mathcal{P}', \mathcal{D}')$ in \mathcal{U} .

Let G_2 be the map that associates an unbounded packing $(\mathcal{P}'; \mathcal{D}')$ to a residual point in $\Lambda(\mathcal{P}; \mathcal{D})$ as follows:

$$(\mathcal{P}', \mathcal{D}') \mapsto \phi_{\mathcal{D}'}(\infty)$$

where $\phi_{\mathcal{D}'}$ is a generalized Möbius transformation with the property that $\phi_{\mathcal{D}'}(\mathcal{D}') = \mathcal{D}$; here \mathcal{D} is the fixed Descartes configuration of our fixed bounded packing. Such a transformation exists by the fact that there exists a bijection $H_{\mathcal{D}} : \text{Möb}(2) \rightarrow \{\text{Set of all Descartes configurations}\}$ defined by

$$\phi \mapsto \phi(\mathcal{D}) = \mathcal{D}'$$

Intuitively speaking, this means that given any fixed Descartes configuration \mathcal{D} , one can obtain any other Descartes configuration \mathcal{D}' via an element of $\text{Möb}(2)$.

We first show that this map G_2 is well-defined, up to the relation of similarity in \mathcal{U} . Let $(\mathcal{P}', \mathcal{D}'), (\mathcal{P}'', \mathcal{D}'')$ be any two unbounded packings such that there exists a similarity transformation ψ with $\psi(\mathcal{P}') = \mathcal{P}''$ so that the following is true:

$$G_2(\mathcal{P}') = G_2'(\mathcal{P}'') = z_0,$$

Setting the following two generalized Möbius transformations:

$$\phi_{\mathcal{D}'} : (\mathcal{P}', \mathcal{D}') \rightarrow (\mathcal{P}, \mathcal{D}) \text{ and } \phi_{\mathcal{D}''} : (\mathcal{P}'', \mathcal{D}'') \rightarrow (\mathcal{P}, \mathcal{D})$$

we obtain the following identity:

$$\phi_{\mathcal{D}''}(\infty) = \phi_{\mathcal{D}''} \circ \psi(\infty) = \phi_{\mathcal{D}'}(\infty).$$

This follows from ψ being a similarity transformation. Thus, G_2 is well-defined.

We now show that G_2 is indeed the inverse of F_2 . We first show that $G_2(F_2(z_0)) = z_0$ for all z_0 in $\Lambda(\mathcal{P}; \mathcal{D})$. Given a $(\mathcal{P}_{z_0}, \mathcal{D}_{z_0})$ st. $F_2(z_0) = (\mathcal{P}_{z_0}, \mathcal{D}_{z_0})$, we may define a generalized Möbius transformation $\phi_{\mathcal{D}_{z_0}} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with the property that $\phi_{\mathcal{D}_{z_0}}(\mathcal{D}'') = (\mathcal{D})$. It follows that $\phi_{\mathcal{D}_{z_0}} \circ \phi_{z_0}$ fixes the Descartes configuration \mathcal{D} and is therefore the identity map. Indeed, this shows that $G_2(F_2(z_0)) = z_0$

Finally, we show that $F_2(G_2(\mathcal{P}', \mathcal{D}')) = (\mathcal{P}', \mathcal{D}')$ for all unbounded packings in \mathcal{U} . We show that any $(\mathcal{P}', \mathcal{D}')$ is similar to its image $F_2(G_2(\mathcal{P}', \mathcal{D}')) = (\mathcal{P}', \mathcal{D}')$. It would therefore suffice to show that the map $\phi_{z_0} \circ \phi_{\mathcal{D}'}$ is itself, a similarity (where $\phi_{\mathcal{D}'}$ is defined as above). This result, however follows from the fact that $\phi_{z_0} \circ \phi_{\mathcal{D}'}$ fixes ∞ . Thus, up to the equivalence of similarity transformations, the map F_2 is bijective.

□

4. UNBOUNDED PACKINGS AND DESCARTES QUADRUPLES

Letting \mathcal{U} be defined as before, we define the map $F_3 : \mathcal{U} \rightarrow \mathbb{P}\mathcal{D}$, where $\mathbb{P}\mathcal{D}$ denotes the set of non-zero real Descartes quadruples $\mathbf{q} \in \mathbb{R}^4$ up to scaling by some $\lambda \in \mathbb{R}^*$:

$$F_3(\mathcal{P}_u, \mathcal{D}) = \mathbf{q},$$

where \mathbf{q} denotes the Descartes quadruple that is the curvatures of the Descartes configuration \mathcal{D} . It is not hard to see that this map is injective. Elements of \mathcal{U} are after all, defined up to similarity, leaving the curvatures of the circles from the packing fixed or dilating them by some $\lambda \in \mathbb{R}^*$. Since an element of $\mathbb{P}\mathcal{D}$ is unique up to scaling by a real number, this relation is preserved. Along with the fact that two dissimilar Apollonian packings cannot have the same Descartes quadruple, this fact guarantees injectivity.

4.1. The Reduction Algorithm and a Correspondence between \mathcal{A}_∞ and \mathcal{U} .

Given the composition of maps $F_2 \circ F_1$, one may ask whether there is a direct way to obtain unbounded packings in \mathcal{U} from infinite sequences in \mathcal{A}_∞ . It can be shown that given a quadruple \mathbf{q} in a bounded packing P , there exists a unique sequence of generators of \mathcal{A} that reduce that quadruple to a root quadruple. From the properties of our maps, we have the following proposition:

Proposition 7. *The element a_∞ in \mathcal{A}_∞ is the reduction algorithm for a unique \mathbf{q} in $\mathbb{P}\mathcal{D}$; in particular, $F_3 \circ F_2 \circ F_1(a_\infty) = \mathbf{q}$. This Descartes quadruple corresponds, via F_3 , to a unique unbounded packing with marked quadruple in \mathcal{U} up to similarity.*

Proof. Recall that we have the following sequence of maps:

$$\mathcal{A}_\infty \xrightarrow{F_1} \Lambda(\mathcal{P}; \mathcal{D}) \xrightarrow{F_2} \mathcal{U} \xrightarrow{F_3} \mathbb{P}\mathcal{D}$$

Given some element \mathbf{a} in \mathcal{A}_∞ , we can associate to that element a sequence of circles associated with the elements $S_{i_1}S_{i_2}\dots$ that approach a residual point r in a packing P with a quadruple \mathbf{q} associated to some Descartes configuration D . By drawing this correspondence, we can think of the image in \mathbf{a} under F_1 as the residual point associated with the circles that approach it in \mathcal{A}_∞ . Under F_2 , the residual point r is moved to ∞ . The map F_2 also takes the sequence

$$S_{i_1}S_{i_2}\dots \mapsto \phi(C_1)\phi(C_2)\dots$$

Where C_1, C_2, \dots are the circles associated with the sequence of generators. This sequence has to approach the image of $\phi(r)$, which as we know is at ∞ . From this, we note that we have constructed a sequence of generators that approach the point ∞ in some unbounded packing in the set \mathcal{U} . Since the reduction algorithm for such an unbounded packing approach infinity as well, and since the reduction algorithm for any packing is unique, we conclude that we have arrived at the reduction algorithm for some unbounded packing in \mathcal{U} , whose Descartes quadruple will be the image of $F_3 \circ F_2 \circ F_1(\mathbf{a})$. This follows from the \mathcal{A} -equivariance of $F_3 \circ F_2 \circ F_1$.

It must also be noted that reduction algorithm in \mathcal{A}_∞ is unique for the Descartes quadruple $F_3 \circ F_2 \circ F_1(\mathbf{a})$. If there exists a Descartes quadruple in $\mathbb{P}\mathcal{D}$ with more than one reduction algorithm, we know from the bijectivity of $F_2 \circ F_1$ that the reduction algorithms corresponding to the image in $\mathbb{P}\mathcal{D}$ must be the same. □

One result of this is that we now know that there must exist some component-free map between elements of \mathcal{A}_∞ and \mathcal{U} that is independent of the Apollonian packing P and Descartes

configuration D . More specifically, we can associate an element of \mathcal{A}_∞ with the unbounded packing in \mathcal{U} where the sequence in \mathcal{A}_∞ acts as the reduction algorithm on a Descartes quadruple associated with the packing \mathcal{U} . Since each unbounded packing corresponds with exactly one reduction algorithm, we claim that this correspondence is bijective on the elements in \mathcal{A}_∞ where at least three generators appear infinitely many times. This can also be demonstrated by noting that $F_1 : \mathcal{A}_\infty \rightarrow \Lambda(\mathcal{P}; \mathcal{D})$ is bijective on the elements in \mathcal{A}_∞ where at least three generators appear infinitely many times, and that it is precisely the image of those elements under F_1 that the also bijective map $F_2 : \Lambda(\mathcal{P}; \mathcal{D}) \rightarrow \mathcal{U}$ sends to either half-plane or full-plane packings. The composition of these bijective maps is also bijective. This then proves that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Lambda(\mathcal{P}; \mathcal{D}) & & \\
 & \nearrow^{(F_1)_{(\mathcal{P}, \mathcal{D})}} & \uparrow \varphi & \searrow^{F_2} & \\
 \mathcal{A}_\infty & & & & \mathcal{U} \\
 & \searrow_{(F_1)_{(\mathcal{P}', \mathcal{D}')}} & \downarrow & \nearrow_{F_2} & \\
 & & \Lambda(\mathcal{P}', \mathcal{D}') & &
 \end{array}$$

And demonstrates a correspondence $\mathcal{C} : \mathcal{A}_\infty \rightarrow \mathcal{U}$ independent of some packing P with a Descartes configuration D , allowing us to make the following addition to our maps:

$$\begin{array}{ccccc}
 \mathcal{A}_\infty & \xrightarrow{F_1} & \Lambda(\mathcal{P}; \mathcal{D}) & \xrightarrow{F_2} & \mathcal{U} \\
 & & \searrow & \nearrow & \\
 & & & & \mathcal{C}
 \end{array}$$

5. ACTION OF THE APOLLONIAN GROUP AND EQUIVARIANCE OF THE MAPS

There is a right regular action of the Apollonian group \mathcal{A} on the set \mathcal{A}_∞ . More precisely, given any Apollonian sequence $\dots S_{i_3} S_{i_2} S_{i_1}$ in \mathcal{A}_∞ , and any a in \mathcal{A} , we let $a(\dots S_{i_3} S_{i_2} S_{i_1}) = \dots S_{i_3} S_{i_2} S_{i_1} a^{-1}$.

Fix an Apollonian packing \mathcal{P} and one of its Descartes configurations \mathcal{D} . To each element of \mathcal{A} , there now corresponds a unique Möbius transformation ϕ_a . In terms of the generators S_i for $i \in \{1, 2, 3, 4\}$ of the group, the corresponding Möbius transformation is inversion with respect to the dual, i.e. an inversion in the circle passing through the three tangency points of the three circles of \mathcal{D} not including the i th. (insert citation here).

Letting $z_0 \in \Lambda(\mathcal{P}; \mathcal{D})$ be a residual point, we see that under the identification of \mathcal{A} with a group of Möbius transformations having the above properties, \mathcal{A} will act on $\Lambda(\mathcal{P}; \mathcal{D})$ as

follows:

$$a(z_0) = \phi_{a^{-1}}(z_0)$$

Whether or not this satisfies the properties of a group action is not immediately clear.

Proposition 8. *The action of \mathcal{A} on $\Lambda(\mathcal{P}; \mathcal{D})$ defined by $a(z_0) = \phi_{a^{-1}}(z_0)$ is indeed a group action.*

Proof. Given any a, b in \mathcal{A} and any z_0 in $\Lambda(\mathcal{P}; \mathcal{D})$, we show that the defined action satisfies $ab(z_0) = a(b(z_0))$. It will suffice to show the following:

$$\begin{aligned} ab(z_0)t &= \phi_{b^{-1}a^{-1}}(z_0) \\ &= \phi_{a^{-1}} \circ \phi_{b^{-1}}(z_0) \end{aligned}$$

This will follow from the fact that given a Descartes quadruple \mathcal{D} ,

$$\begin{aligned} \phi_{a^{-1}} \circ \phi_{b^{-1}}(\mathcal{D}) &= \phi_{a^{-1}}(b^{-1}\mathcal{D}) = b^{-1}(\phi_{a^{-1}}(\mathcal{D})) = b^{-1}a^{-1}(\mathcal{D}) \\ &= \phi_{b^{-1}a^{-1}}(\mathcal{D}) \end{aligned}$$

And thus, $ab(z_0) = \phi_{b^{-1}a^{-1}}(z_0) = \phi_{a^{-1}} \circ \phi_{b^{-1}}(z_0) = a(b(z_0))$ □

One can then ask how the group \mathcal{A} acts on the set \mathcal{U} . An a in \mathcal{A} takes an unbounded packing to itself, an action which corresponds to the group's aforementioned action on $\Lambda(\mathcal{P}; \mathcal{D})$, but in the process changes the marked configuration \mathcal{D}' to $a\mathcal{D}'$, i.e.

$$a(\mathcal{P}', \mathcal{D}') = (\mathcal{P}', a\mathcal{D}').$$

Finally, we see that \mathcal{A} acts on the set $\mathbb{P}\mathcal{D}$ in the most obvious way. It permutes between the various Descartes quadruples \mathbf{q} associated to an unbounded packing via the map F_3 .

One may ask whether the maps F_1 , F_2 and F_3 are \mathcal{A} -equivariant, i.e, whether or not they preserve the group action.

Proposition 9. *The map F_1 is \mathcal{A} -equivariant.*

Proof. Let $\dots S_{i_3} S_{i_2} S_{i_1}$ be a sequence in \mathcal{A}_∞ . Given a packing $(\mathcal{P}, \mathcal{D})$ the circle associated to some finite sequence of generators $S_{i_n} \dots S_{i_1}$ is $S_{i_n} \dots S_{i_1}(\mathcal{D}) / S_{i_{n-1}} \dots S_{i_1}(\mathcal{D})$. Then,

$$\begin{aligned} F_1(a(\dots S_{i_2} S_{i_1})) &= F_1(\dots S_{i_2} S_{i_1} a^{-1}) \\ &= \lim_{n \rightarrow \infty} S_{i_n} \dots S_{i_1} a^{-1}(\mathcal{D}) \setminus S_{i_{n-1}} \dots S_{i_1} a^{-1}(\mathcal{D}) \\ &= \lim_{n \rightarrow \infty} S_{i_n} \dots S_{i_1} \phi_{a^{-1}}(\mathcal{D}) \setminus S_{i_{n-1}} \dots S_{i_1} \phi_{a^{-1}}(\mathcal{D}) \\ &= \phi_{a^{-1}}(\lim_{n \rightarrow \infty} S_{i_n} \dots S_{i_1} \mathcal{D} \setminus S_{i_{n-1}} \dots S_{i_1} \phi(\mathcal{D})) \\ &= a(F_1(\dots S_{i_2} S_{i_1})) \end{aligned}$$

The third equality is due to the fact that if a is in A , \mathcal{D} is a fixed Descartes quadruple and if ϕ is a Möbius transformation, then $\phi(aD) = a\phi(D)$. The fourth equality follows from both the bijectivity and continuity of ϕ . Thus, we see that F_1 is indeed A -equivariant.

□

Proposition 10. *The map F_2 is \mathcal{A} -equivariant.*

Proof. As we saw in section 4.1, via the composition of maps $F_2 \circ F_1$, one may associate to each infinite sequence in \mathcal{A}_∞ , an unbounded packing in \mathcal{U} . In particular, this composition of maps is independent of choice of bounded packing and Descartes Quadruple; its image depends solely on the infinite sequences in \mathcal{A}_∞ via the reduction algorithm which corresponds uniquely to each unbounded packing. Acting on such an element of \mathcal{A}_∞ by an element of the Apollonian group, we obtain a different reduction algorithm and hence a different image in \mathcal{U} . Its image, will however, have the same reduction algorithm as the original unbounded packing acted on by a . More precisely, for a_∞ in \mathcal{A}_∞ , $F_2 \circ F_1(a_\infty a)$ will have the same reduction algorithm as $aF_2 \circ F_1(a_\infty)$. It follows from this that the composition of maps $F_2 \circ F_1$ is \mathcal{A} -equivariant. Now, by the above proposition, F_1 is itself \mathcal{A} -equivariant. Hence, F_2 must be \mathcal{A} -equivariant as well.

□

It is an easy consequence of the definition of $F_3 : \mathcal{U} \rightarrow \mathbb{P}\mathcal{D}$ that this map is \mathcal{A} -equivariant as well. In short, F_3 associates a quadruple to a Descartes configuration of a given unbounded packing; permuting between Descartes configurations of a packing is equivalent to permuting between their associated Descartes quadruples.

6. THE IMAGE OF $\overline{\mathcal{A}}$ IN \mathcal{U} ; SELF-SIMILAR UNBOUNDED PACKINGS

Suppose we take the following representation for an element \mathbf{a} in $\overline{\mathcal{A}}$:

$$\mathbf{a} = \dots aaab$$

Where a is some finite sequence of the four generating elements of \mathcal{A} , and b is a similar finite sequence of generators. It is clear that there exist two elements in $\overline{\mathcal{A}}$ that are fixed by $c = b^{-1}ab$:

$$\dots aaab \quad \text{or} \quad \dots a^{-1}a^{-1}a^{-1}b$$

Following the image of this element in $\overline{\mathcal{A}}$ through $F_3 \circ F_2 \circ F_1$, we see that this element corresponds to the Descartes quadruple \mathbf{q} of some unbounded packing. Now, by \mathcal{A} -equivariance of the associated maps, $aq = q \in \mathbb{P}\mathcal{D}$. Thus, $aq = \lambda q$ in the set \mathbb{R}^4 , for some λ in \mathbb{R}^* . In other words, q is an eigenvector of the sequence of generators c . From this we can show that the Descartes quadruple \mathbf{q} corresponds to curvature values of a self-similar unbounded packing- by restricting our definition of self-similar packings to non-trivial similarity relations, it is clear that by relating curvature values in the packing via some constant λ , this unbounded packing is self-similar.

We now prove the following converse to the above result.

Theorem 11. *Let $(\mathcal{P}', \mathcal{D}')$ denote a self-similar unbounded packing in \mathcal{U} . Then, $(\mathcal{P}', \mathcal{D}')$ is the image of some element s in $\overline{\mathcal{A}}$ under the map $F_2 \circ F_1$.*

Proof. Let $(\mathcal{P}', \mathcal{D}')$ be a self-similar unbounded packing. Notice that the self-similarity of the packing requires the self-similarity to be in the Apollonian group. The operations of self-similarity all correspond to multiplication of the associated Descartes quadruple \mathbf{q} by some fixed scalar. Since \mathbf{q} is taken to be an element of the projective Descartes variety $\mathbb{P}\mathcal{D}$, the following holds:

$$\lambda q = q,$$

where λ is the fixed scalar operating on \mathbf{q} via the self-similarity. Thus, \mathbf{q} may be viewed as the eigenvector in \mathbb{R}^4 for the eigenvalue λ , which corresponds to some operator leaving the packing associated to \mathbf{q} fixed. In light of the \mathcal{A} -equivariance of the map $F_3 \circ F_2 \circ F_1$, we conclude that this operator corresponds to an element of the Apollonian group leaving a_∞ fixed in A_∞ . Such an action by an element of the Apollonian group can only exist, however, for those a_∞ belonging to $\overline{\mathcal{A}}$. Otherwise, the word corresponding to $a \in \mathcal{A}$ fixing a_∞ would have to be of infinite length, which is impossible. It follows from this that $(\mathcal{P}', \mathcal{D}')$ is the image of some $a_\infty = s$, an eigenvector \mathbf{q} of $\overline{\mathcal{A}}$. \square

Another result of this is with regards to the eigenvectors of elements in $\overline{\mathcal{A}}$. Recall that the sequence of generators $c = b^{-1}ab$ in \mathcal{A} fixes exactly two elements:

$$\dots aaab \quad \text{or} \quad \dots a^{-1}a^{-1}a^{-1}b$$

So that the image of \mathbf{a} in the projective variety $\mathbb{P}\mathcal{D}$ is some eigenvector \mathbf{q} that is fixed by c . Since the eigenvector corresponds to a self-similar packing, and since the curvatures decrease with respect to the action of a on a Descartes quadruple of the packing, we know that this implies that there exists at least one eigenvalue λ such that

$$c \mathbf{q} = \lambda \mathbf{q} \quad \text{and} \quad \lambda \in (0, 1)$$

In fact, we have the following claim:

Proposition 12. *If \mathbf{w} is an eigenvector of c as above with eigenvalue ν , then $w = F_3 \circ F_2 \circ F_1(\dots aab)$ or $w \notin \mathbb{PD}(\mathbb{R})$ or $\nu \notin (0, 1)$.*

Proof. Given an eigenvector \mathbf{w} in \mathbb{PD} , we know that \mathbf{w} corresponds to an Apollonian packing

$$c w = \nu w$$

Where the curvatures are decreasing at each action of c on w . This tells us that the packing follows the reduction algorithm $\dots aab$. Since the reduction algorithm uniquely determines each unbounded packing with a marked configuration, and each Apollonian packing with a Descartes configuration corresponding to w has the same algorithm, we can conclude that $\mathbf{w} = \mathbf{q}$.

□

7. CLOSING REMARKS

With the little time we had in this program, our group managed to successfully construct a correspondence between residual points and unbounded packings. We also arrived at a method with which one can arrive at residual points. Among other things however, we were unable to discover explicit conditions for a quadruple to satisfy in order for it to correspond to an unbounded packing. This would have answered one of our initial motivating questions: *how will the shape of a packing change depending on the placement of the center of circle C_3 , the point z_3 , in $\widehat{\mathbb{C}}$ so that its corresponding circle C_3 will lie tangent to given 2 tangent circles C_1 and C_2 with curvatures 1 and α ?* Nevertheless, the results in this paper point towards various surprising and potentially deep consequences. For instance there exists an almost natural topology on the set \mathcal{A}_∞ , which is preserved via the (continuous) maps F_1, F_2 and F_3 . Such a topological approach to the topic gives us an especially elegant description and consequence of the action of \mathcal{A} on the above sets; it may also be used for other, currently unknown applications. Finally, one can potentially say more about the elements of $\overline{\mathcal{A}} \subset \mathcal{A}$. Since all these elements correspond, via the above maps, to eigenvectors of certain 4×4 matrices, one may look at the eigenvalues acting on them. These eigenvalues, zeros of quartic polynomials bring to the fore the following potential correspondence:

$$\mathcal{A}_\infty \leftrightarrow \mathbb{PD}(\mathcal{Q}(\mu))$$

between the elements in $\overline{\mathcal{A}}$ and the image of such elements in the projective Descartes variety with the added condition of being in the extended quartic field of \mathbb{Q} . This correspondence alone may hold interesting number-theoretic applications to integral Apollonian circle packings.

8. APPENDIX I: TABLES

Included below is a table with the first 13 unique Apollonian group elements ordered by length up to cyclic rotation of the generators and permutation of the indices by the Permutation group S_4 . The type of packing generated by the Apollonian group element has been noted in the table. Both the eigenvalues and the corresponding eigenvectors belong to a shared quartic extension of \mathbb{Q} and have been approximated for the table.

| Type | Length | Apollonian Group element | Eigenvalue | Eigenvector |
|------------|--------|--------------------------|-------------|--|
| Strip | 2 | S_2S_1 | 1 | [1,1,0,0] |
| Half-plane | 3 | $S_3S_2S_1$ | 0.055728092 | [6.854101945, 2.617033985, 1, 0] |
| Half-plane | 4 | $S_2S_3S_2S_1$ | .02943726 | [5.828427077, 2, 1, 0] |
| Full-plane | 4 | $S_4S_3S_2S_1$ | .01433430 | [24.13891204, 8.352410442, 2.890053609, 1] |
| Half-plane | 5 | $S_2S_1S_3S_2S_1$ | .01515499 | [6.561552896, 2.438447188, 1, 0] |
| Full-plane | 5 | $S_4S_1S_3S_2S_1$ | .00684964 | [48.99771, 18.29081102, 6.208048046, 1] |
| Full-plane | 6 | $S_4S_2S_1S_3S_2S_1$ | .0022111 | [149.4152903, 55.95385998, 21.26666935, 1] |
| Half-plane | 6 | $S_2S_3S_1S_3S_2S_1$ | .00515475 | [7.464101628, 3, 1, 0] |
| Full-plane | 6 | $S_4S_3S_1S_3S_2S_1$ | .0028021 | [117.3983197, 46.77931940, 14.76462982, 1] |
| Half-plane | 6 | $S_2S_1S_2S_3S_2S_1$ | .01020514 | [6, 2.101020513, 1, 0] |
| Full-plane | 6 | $S_4S_1S_2S_3S_2S_1$ | .0031057 | [107.6656464, 35.88854801, 17.94427419, 1] |
| Full-plane | 6 | $S_4S_3S_2S_3S_2S_1$ | .0038178 | [88.19916404, 27.00126707, 16.26670899, 1] |
| Full-plane | 6 | $S_2S_4S_2S_3S_2S_1$ | .0038611 | [40.63102908, 13.01970752, 6.374248457, 1] |

We hope to expand this table with future research and provide a counting formula to predict the number of unique packings occur given length n . Currently, we have a formula for n not divisible by 2 or 3:

$$U(n) = \frac{1}{24n} \sum_{q|n} (3^q - 3) \text{ such that all } q \text{ are proper divisors of } n$$

We have run into difficulties accounting for the symmetries that occur when n is divisible by 2 or 3. For future research, we will also be normalizing the eigenvectors so that the sum of the entries is equal to 1.

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