Calculations on a Lattice L Spanned by a Set of Conformal Vectors

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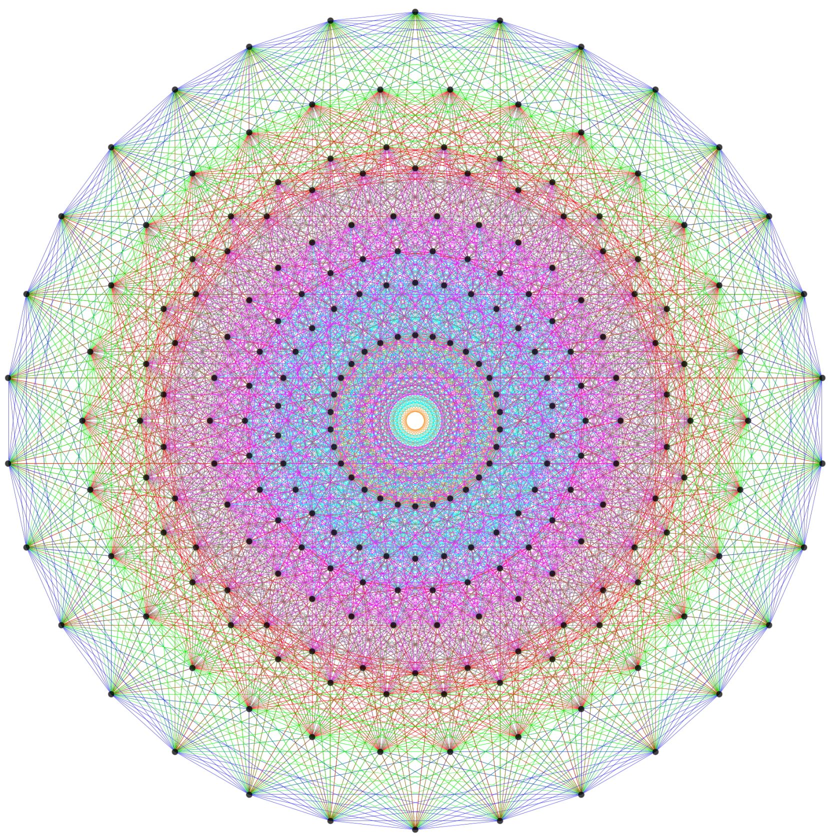

Figure 1: The 240 Vertices of the above polytope represent the root vectors in the E_8 root lattice system. [4]

Contents

1 Overview

This paper outlines a procedure for calculating the fundamental invariants of a scaled version of a lattice L, spanned by a set of conformal vectors. Numerous other calculations related to this lattice are included throughout this paper, a glossary of terms can be found towards its end, and an appendix contains calculations of related pieces of structure for varying dimensions. One can draw similarities between this paper and [2], a paper focusing on the span of idempotents. As these idempotents differ from the conformal vectors appearing in this paper by modest factors of two, one can observe numerous parallels between the objects in [2] and this report.

Later in this document is a discussion of fixed-point lattices spanned by a set of conformal vectors fixed by a single and pairs of involutions. The fundamental invariants calculated for such lattices demonstrate properties of these lattices, and numerous Maple 15 documents related to these calculations can be found on my website, http://qcpages.qc.cuny.edu/ tgaugler100/

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2 Introduction

2.1 Elementary Definitions and Terminology

Definition 1. A **lattice** is a finitely generated free abelian group G with a symmetric bilinear form $f : G \times G \to \mathbb{Q}$. When the bilinear form f for G maps exclusively to the integers, we call G a **integer lattice**. Forwards, we will restrict our attention to lattices whose bilinear form is positive definite.

Definition 2. A **root** of a lattice G is an element $g \in G$ such that $f(g, g) = 2$.

Definition 3. The **root set**, $\Phi(G)$ for a lattice G is defined to be the set $\{g \in G | f(g, g) = 2\}$. The **root sublattice** of G is the $\mathbb{Z}$-span of the set $\Phi(G)$, and if G is equal to the $\mathbb{Z}$-span of its root set, we call G a **root lattice**.

Definition 4. Let G be a lattice. $G_n := \{\lambda \in G | \langle\lambda, \lambda\rangle = 2n\}$. Note that context plays an important role in the use of this notation, as occasionally G_n will also indicate a lattice of degree n.

We introduce some well-understood and commonly used root lattices:

Definition 5. The type A_n integer lattices are generated by the set

$$\{e_i - e_{i+1} | 1 \leq i < n + 1\},$$

where e_i is the i^{th} standard basis vector for $\mathbb{R}^n$. Equivalently, a type A_n lattice is all such elements $\mathbb{Z}e_1 + \mathbb{Z}e_2 + ... + \mathbb{Z}e_{n+1}$ with coordinate sum equal to zero.

Definition 6. The type D_n integer lattices are generated by the set

$$\{e_i - e_{i+1} | 1 \leq i < n\} \cup \{e_{n-1} + e_n\}$$

Equivalently, a type D_n lattice is all such elements of the form $\mathbb{Z}e_1 + \mathbb{Z}e_2 + ... + \mathbb{Z}e_n$ with even coordinate sum.

Definition 7. The E_8 lattice is the $\mathbb{Z}$-span of D_8 and $\frac{1}{2}(1,1,1,1,1,1,1,1)$. The E_6, E_7 lattices are defined up to isometry as the annihilators of any A_2, A_1 sublattice of E_8, respectively. [1](4.2.7)

Remark 1. As mentioned, one model for the E_8 lattice is $D_8 + \mathbb{Z}\frac{1}{2}(1,1,1,1,1,1,1,1)$. Its roots then take the form $\pm e_i \pm e_j$ (of which there are 56), $\pm e_i \mp e_j$ (of which there are 56) and $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2},)$, with evenly many minus signs (of which there are ($\binom{8}{0}$ + $\binom{8}{2}$ + $\binom{8}{4}$+ $\binom{8}{6}$ + $\binom{8}{8}$= 128), giving us a total of 240 roots.

Definition 8. Let G be a lattice of dimension n with symmetric bilinear form $f : G \times G \to \mathbb{Q}$. If f maps exclusively to $2\mathbb{Z}$, we call G an **even lattice**. Notice that if G is of type A_n, D_n, or E_n, scaling the elements of G by $\sqrt{2}$ causes G to become an even lattice. We denote this new lattice $\sqrt{2}G$, or GG.

Definition 9. The **Gram Matrix** R of a lattice with basis $\{\beta_1, \beta_2, ..., \beta_n\}$ is the $n \times n$ matrix $(R_{i,j}) := \langle \beta_i, \beta_j \rangle$.

Definition 10. W is a vector space with a linear transformation t such that $t(x \otimes y) = y \otimes x$ for all $x, y \in w$. $(W \otimes W)^{\pm}$ is the eignespace (of characteristic $\neq 2$) of t for the eigenvalue ± 1 on $W \otimes W$. Then, $W \otimes W = (W \otimes W)^+ \oplus (W \otimes W)^-$. Assume that W has a symmetric bilinear form - then $W \otimes W$ has its own symmetric bilinear form following from this definition and this direct sum is an orthogonal direct sum, which restricts to bilinear forms on both $(W \otimes W)^+$ and $(W \otimes W)^-$.

Definition 11. $S^2(W) = W \otimes W/(W \otimes W)^-$, and let P be the orthogonal projection of $W \otimes W$ to $(W \otimes W)^+$. It follows from our definition of W that the kernal of P is $(W \otimes W)^-$. Hence, we identify $S^2(W)$ with $(W \otimes W)^+$ by mapping $x \otimes y + (W \otimes W)^- \mapsto P(x \otimes y) = \frac{1}{2}(x \otimes y + y \otimes x) = xy$. This identification and bilinear form on $(W \otimes W)^+$ gives us a bilinear form on $S^2(W)$: given $ab, cd \in S^2(W)$, $\langle ab, cd \rangle = \frac{1}{2}(\langle a, c \rangle \langle b, d \rangle + \langle a, d \rangle \langle b, c \rangle)$

Remark 2. Let Q be a lattice with basis $\{\beta_1, ..., \beta_n\}$. A basis for $S^2(Q)$ is the set $\{\beta_i \beta_j | 1 \leq i \leq j \leq n\}$.

3 Calculating the Smith invariants of a $S^2(Q)$, where Q is a lattice of type A_n or D_n

The Theory of Smith Normal Form: Given an $n \times n$ matrix M with integer entries, there exist invertible integer matrices P, Q such that $PMQ = D$ where $D = diagonal(d_1, d_2, ..., d_n)$, and $d_1 | d_2$,$d_2 | d_3, .., d_{n-1} | d_n$. One can weaken ‘integer entries’ to ‘entries in a PID’ to achieve a more general result.

Proof. [1](10), [3](115) □

3.1 Calculating the Smith invariants of $S^2(\sqrt{2}A_n)$

Take A_n to have a basis as described in definition (5). One basis for $S^2(\sqrt{2}A_n)$ is the set

$$\{(\sqrt{2}e_i - \sqrt{2}e_{i+1})(\sqrt{2}e_j - \sqrt{2}e_{j+1}) | 1 \leq i \leq j \leq n+1\}.$$

This set has cardinality $\binom{n+1}{2}$ for $\sqrt{2}A_n$. For convenience, we index the members of this basis as the set $\{\beta_1, \beta_2, ..., \beta_{\binom{n+1}{2}}\}$. The Gram matrix R for $S^2(\sqrt{2}A_n)$ then follows from calculating

$$(R_{i,j}) = \langle \beta_i, \beta_j \rangle$$

with the inner product given in definition (11).

Example. Take $Q = \sqrt{2}A_4$. The Gram matrix for $S^2(\sqrt{2}A_4)$ is:

$$\begin{bmatrix} 8 & -4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5 & -2 & 0 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 & 1 & -2 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 8 & -4 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & -4 & 5 & -2 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & -2 & 4 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -4 & 2 & 8 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & -4 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 8 \end{bmatrix}$$

which has Smith invariants

$$(1, 1, 1, 1, 1, 1, 10, 10, 10, 50)$$

3.2 Calculating the Smith invariants of $S^2(\sqrt{2}D_n)$

Take D_n to have the basis as described in definition (6). Similar to the case where Q is a lattice of type $\sqrt{2}A_n$, one basis for $S^2(\sqrt{2}D_n)$ is the set $\mathbf{B} = B_1 \cup B_2$;

$$B_1 := \{(\sqrt{2}e_i - \sqrt{2}e_{i+1})(\sqrt{2}e_j - \sqrt{2}e_{j+1}) | i \le j, 1 \le i < n, 1 \le j < n\}$$
$$B_2 := \{(\sqrt{2}e_i - \sqrt{2}e_{i+1})(\sqrt{2}e_{n-1} + \sqrt{2}e_n) | 1 \le i < n\}.$$

Once again, calculating the Gram matrix M for $S^2(\sqrt{2}D_n)$ is reduced to a simple calculation; $(R_{i,j}) = \langle \beta_i, \beta_j \rangle$ for $\beta_i, \beta_j \in \mathbf{B}$ and inner product defined as in definition (11).

Example. Take $Q = D_4$. The Gram matrix for $S^2(\sqrt{2}D_4)$ is:

$$\begin{bmatrix} 8 & -4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5 & -2 & -2 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 4 & 2 & 0 & -2 & 0 & 0 & 0 \\ 2 & -4 & 2 & 2 & 8 & -4 & -4 & 2 & 2 & 2 \\ 0 & 1 & -2 & 0 & -4 & 5 & 1 & -4 & -2 & 0 \\ 0 & 1 & 0 & -2 & -4 & 1 & 5 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 2 & -4 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -4 & 0 & 0 & 8 \end{bmatrix}$$

which has Smith invariants

$$(1, 2, 2, 2, 2, 2, 2, 4, 8, 8)$$

4 Calculating the invariants of the lattice L

Henceforth in this section, Q is a positive definite input lattice.

4.1 Definitions and Terminology

Definition 12. L is the lattice formed by $\mathbb{Z}$- span of a set of type 1 and type 2 conformal vectors. Type one conformal vectors take the form:

$$e_\lambda^\pm := \frac{1}{16}\lambda^2 \pm \frac{1}{4}(e^\lambda + e^{-\lambda}) \tag{4.1}$$

where λ is a norm four vector in a lattice Q. Type-2 conformal vectors take the form

$$e_{J,\varphi} := \frac{1}{16}\omega_J + \frac{1}{32}\sum_{\sigma\in J(4)} \varphi(\sigma)e^\sigma \tag{4.2}$$

Where $M = EE_8$, $J(4) = \{\lambda \in EE_8 | \langle\lambda, \lambda\rangle = 4\}$, and $\varphi \in Hom(EE_8, \{\pm 1\})$. We also have that

$$\omega_J = \frac{1}{4}\sum u_i^2 \tag{4.3}$$

where $\{u_i\}$ is an orthogonal basis for a lattice F containing the lattice L.

Definition 13. Given a lattice Q, T is the lattice formed by the $\mathbb{Z}$-span of terms $e^\lambda + e^{-\lambda}$ where $\lambda \in Q_2/\{\pm 1\}$. We call $v_\lambda := e^\lambda + e^{-\lambda}$, and remark that $v_\lambda = v_{-\lambda}$.

Remark 3. The terms $e^{\pm\lambda}$ where $\lambda \in Q_2/\{\pm 1\}$ for some lattice Q have the following property:

$$\langle e^{\alpha}, e^{\beta} \rangle = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ otherwise} \end{cases} \qquad \alpha, \beta \in Q_2/\{\pm 1\}$$

Definition 14. K is the lattice defined to be the orthogonal direct sum between the symmetric square of a lattice Q and the $\mathbb{Z}$ span of terms $v_\lambda : \lambda \in W$; $K := S^2(Q) \perp T$. For the cases we consider in this paper, K is of interest because it contains a scaled copy of the lattice L.

Definition 15. We call F the lattice containing an integral scaling of the lattice L; $\alpha L \subset F | \alpha \in \mathbb{Z}$. Our choice of F is based on convenience.

Definition 16. D is a vector space over $\mathbb{C}$, defined as:

$$D := S^2(W) \perp \mathbb{C}[Q], \tag{4.4}$$

When T is the lattice arriving from the integer span of the terms $e^{\lambda} + e^{-\lambda}$ where λ is a norm-4 vector in an input lattice Q, we remark that $T \subset \mathbb{C}[Q]$, where $\mathbb{C}[Q]$ is a vector space with an orthonormal basis consisting of elements of the form e^{λ} for $\lambda \in Q$.

4.2 An Overview of the Procedure

We want to study the case where we have a lattice Q in a vector space W. This leads us to define the piece of structure:

$$D := S^2(W) \perp \mathbb{C}[Q]. \tag{4.5}$$

a vector space over $\mathbb{C}$. When T is the lattice arriving from the integer span of the terms $e^{\lambda} + e^{-\lambda}$ where λ is a norm-4 vector in an input lattice Q, we remark that $T \subset \mathbb{C}[Q]$, where $\mathbb{C}[Q]$ is a vector space with an orthonormal basis consisting of elements of the form e^{λ} for $\lambda \in Q$. D contains the conformal vectors that form a basis for L, where L is the lattice spanned by a set of type-1 and type-2 conformal vectors arising from norm-4 vectors in Q (hence, D contains the lattice L).

Given an input lattice Q, we define the lattice K as follows:

$$K := S^2(Q) \perp T \tag{4.6}$$

where T is the $\mathbb{Z}$-span of terms $v_\lambda | \lambda \in Q_2$. In comparing K and L when Q is the EE_8 even lattice, we have that $32L \subseteq K$.

In analyzing the lattice L, we write its basis elements in terms of the orthogonal basis elements of a lattice F (notice that L is contained in the lattice F) contained in the vector space D, and use the theory of Smith Normal form (after some rescaling of L such that its Gram matrix is integral) to calculate its fundamental invariants.

4.3 Finding a basis of a submodule of a free module

Let F be a free abelian group of rank n. Realizing the elements of F as row vectors of the form $\mathbb{Z}^{1\times n}$, we take a finite set S of row vectors in F. Letting E be their span, we calculate a basis of E through the theory of Smith Normal Form: Let A be the integer matrix whose rows are the elements of S. Since elementary row operations preserve the row space of a matrix, E is the row space of $PA = DQ^{-1}$ where P, Q are invertible integer matrices and $PAQ = D$, a diagonal matrix $diagonal(d_1, d_2, ..., d_n)$ satisfying the conditions of Smith normal form. Let $R_1, R_2, ..., R_n$ be the rows of Q^{-1}. We conclude that the nonzero vectors $d_i R_i$ are a basis for E. From this, it follows that the Gram matrix for E is the upper-left block of $(PA) \cdot H \cdot (PA)^t$, where H is the gram matrix of F.

Example. As a simple example, take F to be $\mathbb{R}^6$, with a basis $\{e_1, e_2, .., e_6\}$. Take S to be the set consisting of the following vectors:

$$(5,1,2,3,3,5), (4,2,7,0,7,4), (1,2,1,1,3,7), (2,2,0,1,4,3)$$
$$(4,5,1,2,6,7), (1,5,3,0,4,2), (7,0,5,4,5,6)(3,6,2,5,4,6)$$

and call the space they span E. Arranging them in a matrix A, we find invertible integer matrices P, Q such that $PAQ = D$, where D is a diagonal matrix $diagonal(d_1, d_2, ..., d_n)$ where $d_1 | d_2, ..., d_{n_1} | d_n$:

$$PAQ = \begin{bmatrix} -284 & 3 & -59 & -185 & 244 & -110 & 132 & 16 \\ -372 & 0 & -75 & -238 & 315 & -138 & 176 & 19 \\ -48 & 1 & -10 & -31 & 41 & -19 & 22 & 3 \\ -82 & 1 & -17 & -53 & 70 & -32 & 38 & 5 \\ -393 & 0 & -79 & -252 & 333 & -146 & 186 & 20 \\ -485 & -3 & -96 & -310 & 409 & -176 & 232 & 23 \\ -201 & 5 & -43 & -133 & 175 & -82 & 91 & 13 \\ -412 & 0 & -83 & -264 & 349 & -153 & 195 & 21 \end{bmatrix} \begin{bmatrix} 5 & 1 & 2 & 3 & 3 & 5 \\ 4 & 2 & 7 & 0 & 7 & 4 \\ 1 & 2 & 1 & 1 & 3 & 7 \\ 2 & 2 & 0 & 1 & 4 & 3 \\ 4 & 5 & 1 & 2 & 6 & 7 \\ 1 & 5 & 3 & 0 & 4 & 2 \\ 7 & 0 & 5 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = D$$

A basis B for E is then consists of the nonzero $d_i R_i$, where R_i is the i^{th} row of Q^{-1}:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

Notice that a Gram matrix for F is I_6. It follows from our previous remarks that a Gram matrix for E is then

$$(DQ^{-1}) \cdot H \cdot (DQ^{-1})^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -2 & 0 & 2 & 8 \end{bmatrix}$$

5 Calculating the Fundamental invariants of L when Q has no EE_8 sublattice

5.1 Calculating the Fundamental Invariants for L when $Q = \sqrt{2}A_n$

Remark 4. If Q has no EE_8 sublattice, L is spanned entirely by type-1 conformal vectors.

Remark 5. If Q is a rank n lattice, then $rank(L) = \binom{n+1}{2} + \frac{1}{2}|\Phi(Q)|$.

Remark 6. If $Q = A_n$, then $rank(L) = |\Phi|$, and the type-1 conformal vectors form a basis for L.

Proof. We assume that L is of dimension $\binom{n+1}{2} + \frac{|\Phi(A_n)|}{2}$, and spanned by type-1 and type-2 conformal vectors. It follows from a property of the A_n root lattice that $\binom{n+1}{2} = \frac{|\Phi(A_n)|}{2}$. It is clear that the set $\{\frac{1}{16}\lambda^2 \pm \frac{1}{4}v_\lambda\}$ is of cardinality $2\binom{n+1}{2}$ when $\lambda \in \Phi(A_n)/\{\pm 1\}$. This set is a linearly independent set of type-1 conformal vectors, forming a basis for L. ☐

Reiterating: when Q is of type $\sqrt{2}A_n$, a basis for L is the set

$$\left\{\frac{1}{16}\lambda^2 \pm \frac{1}{4}(e^\lambda + e^{-\lambda})|\lambda \text{ is a norm-4 vector in } \sqrt{2}A_n/\{\pm 1\}\right\}.$$

Definition 17. Given $\alpha, \beta \in Q$, we have that $\langle \alpha^2, \beta^2 \rangle = 2\langle \alpha, \beta \rangle^2$

Definition 18. $\langle e^\pm_\lambda, v_\alpha \rangle = 0 \;\; \forall \alpha, \lambda \in Q,$

$$\langle e^\pm_\lambda, e^\pm_\alpha \rangle = \begin{cases} 0 \text{ if } \langle \lambda, \alpha \rangle = 0 \\ \frac{1}{4} \text{ if } \lambda = \alpha \\ \frac{1}{32} \text{ if } \langle \lambda, \alpha \rangle = \pm 2 \end{cases} \qquad \langle e^\pm_\lambda, e^\mp_\alpha \rangle = \begin{cases} 0 \text{ if } \langle \lambda, \alpha = 0 \rangle \\ 0 \text{ if } \lambda = \alpha \\ \frac{1}{32} \text{ if } \langle \lambda, \alpha \rangle = \pm 2 \end{cases}$$

Definition 19. When $Q = \sqrt{2}A_n$, take a basis for the lattice F to be the set

$$\{e_i e_j | 1 \le i \le j \le n+1\} \cup \{v_\lambda | \lambda \in \Phi(A_n)/\{\pm 1\}\}.$$

This is an orthogonal basis, with Gram matrix H, calculated with the inner product in definition (17) and remark (3). Notice that $\langle e_i e_j, e_i e_j \rangle = 2$ if $i = j$, and 1 otherwise. From this it follows that H is the diagonal matrix with all 2's on its diagonal with the exception of entries corresponding to basis elements of the form $e_i e_j | i \ne j$; the values of such entries are 1. We represent the matrix H with the following notation:

$$H = diagonal(2_{e_1 e_1}, 1_{e_1 e_2}, ..., 1_{e_n, e_{n+1}}, 2_{e_{n+1}, e_{n+1}}, 2_1, ..., 2_{|\Phi(A_n)|/2}).$$

As per the definition of the A_n root lattice, the norm-4 vectors of $\sqrt{2}A_n$ are the roots of $A_n/\{\pm 1\}$ scaled by $\sqrt{2}$. We can assume they all take the form $\sqrt{2}e_i - \sqrt{2}e_j | 1 \le i < j \le n+1$ for $\sqrt{2}A_n$.

Remark 7.

$$(\sqrt{2}e_i - \sqrt{2}e_j)^2 = \left(\sqrt{2}(e_i - e_j) \otimes \sqrt{2}(e_i - e_j)\right) = (2e_i e_i + 2e_j e_j - 4e_i e_j)$$

Which can easily be expressed as a row vector using the basis described for F. Below is a more illustrative example.

Example. We calculate the fundamental invariants for L when $A = \sqrt{2}A_3$. The elements of F are 1×16 vectors of the form

$$(e_1e_1, e_1e_2, e_1e_3, e_1e_4, e_2e_2, e_2e_3, e_2e_4, e_3e_3, e_3e_4, e_4e_4, v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3}, v_{\lambda_4}, v_{\lambda_5}, v_{\lambda_6})$$

Writing all type-1 conformal vectors $e_\lambda^\pm$ for norm-4 vectors $\lambda \in \sqrt{2}A_3/\{\pm 1\}$ in terms of the basis of F, we compile a matrix of the form:

$$M_{\sqrt{2}A_3} = \begin{bmatrix} 1/8 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & -1/4 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/8 & -1/4 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/8 & 0 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & -1/4 & 1/8 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 1/8 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & -1/4 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 & -1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/8 & -1/4 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/8 & 0 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & -1/4 & 1/8 & 0 & 0 & 0 & 0 & 0 & -1/4 \end{bmatrix}$$

Scaling this matrix by 8, we find invertible integer matrices P and Q such that $P(8M_{\sqrt{2}A_3})Q = D$ where D is a diagonal matrix satisfying the conditions outlined in theorem (3). Taking H to be as defined as in definition (19), the Gram matrix for $8L$ is then:

$$(P(8M_{\sqrt{2}A_3}))H(P(8M_{\sqrt{2}A_3}))^t = \begin{bmatrix} 16 & 2 & 2 & -52 & 32 & -32 & -16 & 80 & -160 & 160 & -80 & 16 \\ 2 & 16 & 2 & -52 & 20 & -32 & -16 & 64 & -96 & 64 & -16 & 0 \\ 2 & 2 & 16 & -52 & 8 & -20 & -16 & 48 & -48 & 16 & 0 & 0 \\ -52 & -52 & -52 & 540 & -60 & 340 & 240 & -816 & 1232 & -960 & 384 & -64 \\ 32 & 20 & 8 & -60 & 396 & 52 & 144 & -144 & -48 & 144 & -80 & 16 \\ -32 & -32 & -20 & 340 & 52 & 268 & 208 & -592 & 880 & -704 & 288 & -48 \\ -16 & -16 & -16 & 240 & 144 & 208 & 192 & -480 & 640 & -480 & 192 & -32 \\ 80 & 64 & 48 & -816 & -144 & -592 & -480 & 1760 & -2720 & 2208 & -928 & 160 \\ -160 & -96 & -48 & 1232 & -48 & 880 & 640 & -2720 & 4672 & -4064 & 1792 & -320 \\ 160 & 64 & 16 & -960 & 144 & -704 & -480 & 2208 & -4064 & 3744 & -1728 & 320 \\ -80 & -16 & 0 & 384 & -80 & 288 & 192 & -928 & 1792 & -1728 & 832 & -160 \\ 16 & 0 & 0 & -64 & 16 & -48 & -32 & 160 & -320 & 320 & -160 & 32 \end{bmatrix}$$

with fundamental invariants

$$(2, 2, 4, 16, 16, 16, 16, 16, 16, 64, 64, 128)$$

5.2 Calculating the Fundamental Invariants for L when $Q = \sqrt{2}D_n$

When $Q = \sqrt{2}D_n$, take a basis for F to be the set $\{e_ie_j | 1 \leq i \leq n\} \cup \{v_\lambda | \frac{\lambda}{\sqrt{2}} \in \Phi(D_n)/\{\pm 1\}\}$. This is an orthogonal basis, with Gram matrix H calculated with the inner product in definition (18),

$$H = diagonal(2_{e_1 e_1}, 1_{e_1 e_2}, ..., 1_{e_{n-1} e_n}, 2_{e_n, e_n}, 2_1, ..., 2_{|\Phi(D_n)|/2}).$$

As in the case where $Q = \sqrt{2}A_n$, H is the diagonal matrix with all 2's in the coordinates corresponding to basis elements of the form $e_i e_j | i = j$ and v_λ, and 1's elsewhere.

Following in the procedure for analyzing L when $Q = \sqrt{2}A_n$, we write all $e_\lambda^\pm$ in terms of the basis of F, and use the theory of Smith normal form to calculate a Gram matrix for αL where $\alpha \in \mathbb{Z}$. We may assume that all norm-4 vectors in $\sqrt{2}D_n/\{\pm 1\}$ take the form $\sqrt{2}e_i - \sqrt{2}e_j | 1 \leq i < j \leq n$ or $\sqrt{2}e_i + \sqrt{2}e_j | 1 \leq i < j \leq n$.

Remark 8.

$$(\sqrt{2}e_i + \sqrt{2}e_j)^2 = \left(\sqrt{2}(e_i + e_j) \otimes \sqrt{2}(e_i + e_j)\right) = (2e_i e_i + 2e_j e_j + 4e_i e_j)\,,$$

Which can easily be expressed as a row vector using the basis described for F.

Example. When $Q = \sqrt{2}D_3$, we realize the elements of F as 1×12 vectors of the form

$$(e_1 e_1, e_1 e_2, e_1 e_3, e_2 e_2, e_2 e_3, e_3 e_3, v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3}, v_{\lambda_4}, v_{\lambda_5}, v_{\lambda_6})$$

Writing all type-1 conformal vectors $e_\lambda^\pm$ for $\lambda \in \sqrt{2}D_3$ in terms of the basis of F, we compile a matrix $M_{\sqrt{2}D_3}$ of the form:

$$M_{\sqrt{2}D_3} = \begin{bmatrix} 1/8 & 1/4 & 0 & 1/8 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 1/4 & 0 & 0 & 1/8 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/8 & 1/4 & 1/8 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 1/8 & 1/4 & 0 & 1/8 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 1/4 & 0 & 0 & 1/8 & 0 & -1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/8 & 1/4 & 1/8 & 0 & 0 & -1/4 & 0 & 0 & 0 \\ 1/8 & -1/4 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 1/8 & 0 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 & -1/4 & 1/8 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 1/8 & -1/4 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 \\ 1/8 & 0 & -1/4 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 1/8 & -1/4 & 1/8 & 0 & 0 & 0 & 0 & 0 & -1/4 \end{bmatrix} \tag{5.1}$$

Scaling this matrix by 8, we find invertible integer matrices P and Q such that $P(8M_{\sqrt{2}D_3})Q = D$ where D is a diagonal matrix satisfying the conditions outlined in theorem (3). Taking H to be as defined as in above, the Gram matrix for $8L$ is then

$$((8PM_{\sqrt{2}D_3}))\cdot H\cdot(P(8M_{\sqrt{2}D_3}))^t=\begin{bmatrix} 16 & 2 & 0 & -16 & -16 & 0 & 64 & 16 & 48 & -112 & 0 & -16\\ 2 & 16 & -16 & -32 & -32 & -16 & 176 & -16 & 0 & 80 & 16 & 32\\ 0 & -16 & 32 & 64 & 80 & 64 & -464 & 16 & 0 & -64 & -32 & -64\\ -16 & -32 & 64 & 160 & 208 & 160 & -1152 & 16 & -48 & 0 & -64 & -96\\ -16 & -32 & 80 & 208 & 300 & 280 & -1704 & 16 & -32 & 0 & -80 & -112\\ 0 & -16 & 64 & 160 & 280 & 368 & -1728 & 16 & 48 & -96 & -64 & -96\\ 64 & 176 & -464 & -1152 & -1704 & -1728 & 9936 & -96 & 64 & 192 & 480 & 736\\ 16 & -16 & 16 & 16 & 16 & 16 & -96 & 64 & 96 & -384 & 0 & 0\\ 48 & 0 & 0 & -48 & -32 & 48 & 64 & 96 & 320 & -832 & 0 & 0\\ -112 & 80 & -64 & 0 & 0 & -96 & 192 & -384 & -832 & 2816 & 0 & 0\\ 0 & 16 & -32 & -64 & -80 & -64 & 480 & 0 & 0 & 0 & 64 & 128\\ -16 & 32 & -64 & -96 & -112 & -96 & 736 & 0 & 0 & 0 & 128 & 320 \end{bmatrix}$$

with Smith invariants

$$(2,2,4,16,16,16,16,16,16,64,64,128)$$

6 Calculating L when Q has an EE_8 sublattice

6.1 Calculating L when $Q = EE_8$

Remark 9. Recall that E_8 has a total of 240 roots, and that the norm-4 vectors in EE_8 are the roots of E_8 scaled by $\sqrt{2}$.

In this subsection, F will have the orthogonal basis

$$\{e_ie_j|1\le i\le j\le 8\}\cup\{e^{\alpha}|\alpha\in\Phi(E_8)\}$$

For the sake of convenience, its vectors take the form

$$(e_1e_1,e_1e_2,e_1e_3,....,e_1e_8,e_2e_2,e_2e_3,....,e_7e_8,e_8e_8,e^{\lambda_1},e^{-\lambda_1},.....e^{\lambda_{120}},e^{-\lambda_{120}})$$

Remark 10.

$$\langle e_ie_j,e_ie_j\rangle=\begin{cases}1\text{ if }i\neq j\\ 2\text{ if }i=j\end{cases}\qquad \langle e^{\alpha},e^{\beta}\rangle=\begin{cases}1\text{ if }\alpha=\beta\\ 0\text{ otherwise}\end{cases}\tag{6.1}$$

Unlike the case in which Q is of type A_n or D_n, we have to include type-2 conformal vectors in our basis of L. Recall that type-2 conformal vectors take the form

$$e_{J,\varphi}:=\frac{1}{16}\omega_J+\frac{1}{32}\sum_{\alpha\in J(4)}\varphi(\alpha)e^{\alpha}.\tag{6.2}$$

6.1.1 Representing Conformal Vectors with basis elements of F

Type-1 Conformal Vectors

Elements in F can be realized as $1\times(\binom{9}{2}+240)=1\times 276$ vectors. The entries of these vectors correspond to the following basis elements:

$$(\underbrace{-,\ldots\ldots,-}_{\binom{9}{2}\text{ entries corresponding to }e_ie_j},\underbrace{-,\ldots\ldots,-}_{240\text{ entries corresponding to }e^{\pm\lambda}|\frac{\lambda}{\sqrt{2}}\in\Phi(E_8)/\pm 1})$$

Remark 11. Take the norm-4 vector $\sqrt{2}e_i + \sqrt{2}e_j$ in $\sqrt{2}E_8$. We have that

$$(\sqrt{2}e_i + \sqrt{2}e_j)^2 = (\sqrt{2}e_i + \sqrt{2}e_j) \otimes (\sqrt{2}e_i + \sqrt{2}e_j) = 2(e_ie_i + e_je_j + 2e_ie_j)$$

Similarly, take the norm-4 vector $\sqrt{2}e_i - \sqrt{2}e_j$. We have that

$$(\sqrt{2}e_i - \sqrt{2}e_j)^2 = (\sqrt{2}e_i - \sqrt{2}e_j) \otimes (\sqrt{2}e_i - \sqrt{2}e_j) = 2(e_ie_i + e_je_j - 2e_ie_j)$$

Taking all $\lambda \in \Phi(E_8)/\{\pm 1\}$, we consider the set of type-1 conformal vectors $e^{\pm}_{\sqrt{2}\lambda}$. Combining remark (11) with whatever convention we find most convenient for the placement of the e^{λ} terms in the vectors of F, we note that all type-1 conformal vectors lie in the rational span of the basis of F.

Example. Take the vector $\lambda_1 = \sqrt{2}(e_1 + e_2)$. From remark (11), we, have that

$$\lambda_1^2 = 2(e_1e_1 + e_2e_2 + 2e_1e_2).$$

As a result, we represent the vector $e^{+}_{\lambda_1}$ in terms of the basis of F as follows:

$$\frac{1}{16}\lambda_1^2 + \frac{1}{4}v_{\lambda_1} = \frac{1}{8}e_1e_1 + \frac{1}{8}e_2e_2 + \frac{1}{4}v_{\lambda} = (\underbrace{\frac{1}{8}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, \frac{1}{8}, 0, 0, 0..., 0,}_{\binom{9}{2} \text{ total entries}} \underbrace{\pm\frac{1}{4}, \pm\frac{1}{4}, 0, ..., 0)}_{240 \text{ total entries}}$$

Example. Take the vector $\lambda_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. We have that

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}(e_1e_1 + ... + e_8e_8 + 2e_1e_2 + ... + 2e_7e_8)$$

We represent $\frac{1}{16}\lambda_2^2 + \frac{1}{4}v_{\lambda_2}$ in terms of the basis elements of F as follows:

$$\frac{1}{32}(e_1e_1 + ... + e_8e_8 + 2e_1e_2 + ... + 2e_7e_8) = \left(\frac{1}{32}, \frac{1}{16}, ..., \frac{1}{32}, \frac{1}{16}, ..., \frac{1}{32}, 0, ..., \frac{1}{4}, \frac{1}{4}, ..., 0\right),$$

where $\frac{1}{32}$ is the coefficient of all terms $e_ie_j | i = j$, $\frac{1}{16}$ is the coefficient of the terms $e_ie_j | i \neq j$, and there are two $\frac{1}{4}$ coefficients to represent the $\frac{1}{4}(e^{\lambda_2}) + \frac{1}{4}(e^{-\lambda_2})$ terms of this type-1 conformal vector.

Type 2 conformal Vectors

As iterated in equation (6.2), type-2 conformal vectors take the form

$$e_{J,\varphi} := \frac{1}{16}\omega_J + \frac{1}{32}\sum_{\alpha \in J(4)} \varphi(\alpha)e^{\alpha}$$

where J is the doubly even lattice EE_8, $J(4) = \{\alpha \in J | \langle \alpha, \alpha \rangle = 4\}$, $\varphi \in Hom(EE_8, \{\pm 1\})$, and ω_J is the called **the Virasoro element** of the vertex operator algebra V_J. In this context, we have that

$$\omega = \frac{1}{2}\sum u_i^2 \tag{6.3}$$

where the set $\{u_i\}$ is an orthonormal basis for the vector space W containing EE_8. In our case, it is convenient to allow this set to be the the 8-element set $\{e_1, e_2, ..., e_8\}$.

We can realize any homomorphism $\varphi \in (EE_8, \{\pm 1\})$ as a function of the form $f_y : x \mapsto (-1)^{\langle x,y\rangle}$ where $y \in dual(EE_8) = \frac{1}{2}EE_8$. In this case, there are 256 cosets of EE_8 in $dual(EE_8)$.

Consider the rows of the following matrix:

$$\mathcal{R} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}$$

Taking all sums of the form

$$\alpha_1 R_1 + \alpha_2 R_2 + ... + \alpha_8 R_8 : \alpha_i \in \{0, 1\}$$

T where R_i is the i^{th} row of the matrix $\mathcal{R}$, we compile a set S of 256 unique vectors into a matrix. Each row of this matrix can represent one of the homomorphisms from EE_8 to ± 1 in the following way: given a norm-4 vector λ,

$$\frac{1}{32} \sum_{\varphi \in Hom(EE_8, \{\pm 1\})} \varphi(\lambda) e^{\lambda} = \frac{1}{32} \sum_{\sigma_i \in S} (-1)^{\langle \sigma_i, \lambda\rangle} e^{\lambda}$$

Where $\sigma_i \in S$, and the sum on the right-hand side runs though all 256 elements of S.

Since $\omega_M = \frac{1}{2}\sum u_i^2$ where $u_i \in \{e_1, e_2, ..., e_8\}$, we have that in the rational span of the basis elements of F, $\frac{1}{32}\omega_M$ can be represented as follows:

$$\frac{1}{2 \cdot 16} \sum u_i^2 = (\frac{1}{2 \cdot 16}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2 \cdot 16}, ..., \frac{1}{2 \cdot 16}, 0, ..., 0)$$

where the coefficients of $\frac{1}{2 \cdot 16}$ correspond to the elements $e_i e_j | i = j$ in the vectors of F.

As a result, all 256 type-2 conformal vectors for L when $Q = EE_8$ take the following form in the rational span of F:

$$(\frac{1}{32}, 0, .., 0, \frac{1}{32}, ..0, \frac{1}{32}, 0, ..., 0, \frac{1}{32}, \pm\frac{1}{32}, \pm\frac{1}{32}, ..., \pm\frac{1}{32})$$

where the coefficients of the terms $e_i e_j | i = j$ are $\frac{1}{32}$, and the signs of the terms with coefficients $\pm\frac{1}{32}$ are determined by a representative from the set S.

6.2 Calculating the Gram matrix for L

In calculating the Gram matrix for L, we proceed as we have in previous sections: let M be the matrix whose rows are the $(56 + 56 + 128 + 2^8) = 476$ type-1 and type-2 conformal vectors that span L. As evidenced by our representation of conformal vectors in the rational span of F, M is not integral- let $U = 32M$. U is an integral matrix, ergo it can be manipulated into Smith Normal form. Let the matrix H represent the Gram matrix for F: recall that

$$\langle e_ie_j, e_ie_j\rangle = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ otherwise} \end{cases} \tag{6.4}$$

Also,

$$\langle e^\alpha, e^\beta\rangle = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ otherwise} \end{cases} \tag{6.5}$$

A basis for F is formed through members of the set $\{e_ie_j|1 \le i \le j \le 8\}$, (of which there are $\binom{8+1}{2}$ terms), and terms of the form $e^{\pm\alpha}$ for all $\alpha \in \Phi(E_8)/\{\pm 1\}$ (of which there are 240). From equations (6.4) and (6.5), H is the 276×276 diagonal matrix with all 1's on its diagonal with the exception of the entries corresponding to the norms of $\langle e_ie_j, e_ie_j\rangle|i = j$, whose values are 2. Calculating invertible integer matrices P, Q such that the product PUQ results in a matrix D adhering to Smith normal form, the Gram matrix for $32L$ is the product $PU \cdot H \cdot (PU)^t$, with Smith invariants:

(32, 32, 32, 32, 32, 32, 32, 32, 32, 32, , 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, , 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, , 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 256)

7 Involutions and Fixed Point Sublattices

7.1 Definitions and Terminology

As usual, let L be the lattice spanned by a set of conformal vectors.

Definition 20. Let $e_{\lambda,a}$ denote e^+_λ when $a = 1 \mod 2$, and e^-_λ when $a = 0 \mod 2$. As such, $e'_{\lambda,a} = e_{\lambda,(a+1 \ mod\ 2)}$.

Definition 21. The set CV_{L_Q} is a set of type-1 and type-2 conformal vectors that span L_Q.

Definition 22. The set $\mathcal{I}_{L_Q}$ is the set of involutions we associate with a lattice L_Q.

Definition 23. The set $FV_{t(v)}$ is the set of conformal vectors fixed by an involution $t(v)$.

Remark 12. Given a conformal vector $v \in CV_{L_Q}$, one can construct an involution $t(v) \in \mathcal{I}_{L_Q}$.

We now examine specific and important properties of involutions based on type-1 and type-2 conformal vectors.

Definition 24. For $e^\pm_\lambda, e(\varphi) \in CV_{L_Q}$, Let $t(e^\pm_\lambda)$ indicate an involution based on a type-1 conformal vector, and let $t(e(\varphi))$ represent an involution based on a type-2 conformal vector, where $\varphi \in Hom(Q, \{\pm 1\})$.

Property 1. An involution $t(e^{\pm}_{\alpha})$ fixes type-1 and type-2 conformal vectors under the following conditions:

$$e^{\pm}_{\lambda} \text{ if } \langle \alpha, \lambda \rangle \in \{0, 4, -4\}, \qquad e(\varphi) \text{ if } \langle \alpha, \varphi \rangle \ mod\ 2 = 0$$

where $\varphi \in Hom(Q, \{\pm 1\})$ here is considered a vector in $\frac{1}{2}Q/Q$. Similarly, an involution $t(\varphi)$ fixes type-1 and type-2 conformal vectors under the following conditions:

$$\text{all } e'_{\lambda,\varphi} \qquad e(\phi) \text{ where } \varphi = \phi \text{ or } \varphi\phi \text{ singular}$$

where $e'_{\lambda,\varphi} = e_{\lambda,(\langle \lambda,\varphi \rangle + 1\ mod\ 2)}$.

Property 2. The map $v \to t(v)$ is injective.

Definition 25. The lattice $FCV(t(v))$ is spanned by the vectors in a lattice L_Q fixed by an involution $t(v) \in \mathcal{I}_{L_Q}$.

Definition 26. The lattice $Ann(FCV(t(v)$ is the annihilator of the lattice $FCV(t(v)))$.

7.2 Overview

Consider the lattice L_{EE_8}, and it associated set of conformal vectors $CV_{L_{EE_8}}$ (recall that $|CV_{L_{EE_8}}| = 496$). Taking $v \in CV_{L_{EE_8}}$, one can construct an involution $t(v)$ (an isometry of L where $|t(v)| = 2$). One can study the fixed point sublattices arising from the span of the conformal vectors fixed by the isometry $t(v)$ or the span of conformal vectors fixed by two isometries $t(v_1), t(v_2)$. Another object of interest is the annihilator of these fixed-point sublattices; calculations of the Smith invariants of such objects can be found in section 7.4 and 7.6.

7.3 $FCV(t(v)) : t(v) \in \mathcal{I}_{L_{ee_8}}$

Take L_{EE_8}, spanned by a set of 496 conformal vectors. We remark that given an involultion $t(v)$ for $v \in CV_{L_{EE_8}}$, $t(v)$ fixes exactly 256 conformal vectors in $CV_{L_{EE_8}}$. Let $FCV_{t(v)\in EE_8}$ denote the set of conformal vectors fixed by $t(v) \in \mathcal{I}_{L_{EE_8}}$. Constructing a matrix M_{EE_8} as described in previous sections, we denote the matrix whose rows are the conformal vectors fixed by $t(v)$ by $M_{t(v),EE_8}$, and denote the matrix whose rows are conformal vectors not fixed by $t(v)$ by $M'_{t(v),EE_8}$. One can find invertible itneger matrices P and Q such that $P(8 \cdot M_{t(v),EE_8})Q = D$ where D satisfies the conditions outlined in the theory of Smith normal form. Letting H be the Gram matrix for F, we calculate the invariants for $P(8 \cdot M_{t(v),EE_8}) \cdot H \cdot (P(8 \cdot M_{t(v),EE_8}))^t$ to be

32, 32, 32, 32, 32, 32, 32, 32, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 128, 128, 128, 128, 128, 128, 128, 128 , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256,

Claim. Given involutions $t_1(v), t_2(v) \in \mathcal{I}_{L_{EE_8}}$, $FCV(t_1(v)) \cong FCV(t_2(v))$.

It follows from extensive calculations that the Smith Invariants for the lattice $FCV(t(v))$ are the same for any $t(v) \in \mathcal{I}_{L_{EE_8}}$. These invariants are displayed above.

7.4 $FCV(t_1(v), t_2(v)) : t_1(v), t_2(v) \in \mathcal{I}_{L_{ee_8}}$

Definition 27. The lattice $FCV(t_1(v), t_2(v))$ is spanned by the vectors in a lattice L_Q fixed by an the involutions $t_1(v)$ and $t_2(v)$.

Claim. $FCV(t_1(v), t_2(v)) \cong FCV(t_3(v), t_4(v)) : t_1(v), t_2(v), t_3(v), t_4(v) \in \mathcal{I}_{L_{EE_8}}$ if $|FV_{t_1(v)} \cap FV_{t_2(v)}| = |FV_{t_3(v)} \cap FV_{t_4(v)}|$.

We now shift our attention to the fixed-point sublattice of L_{EE_8} spanned by conformal vectors fixed by two involutuions. In calculating this lattice, we do as we have in the past: we construct the matrix $M_{FCV(t_1(v),t_2(v))}$ by removing from M_{EE_8} the conformal vectors fixed by involutions $t_1(v)$ and $t_2(v)$, then calculate its Smith invariants as previously described. We have two casces:

7.4.1 Case 1

When $|FV_{t_1(v)} \cap FV_{t_2(v)}| = 128$, we get fundamental invariants of:

32, 32, 32, 32, 32, 32, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 128, 128 , 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 128, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 256, 512, 512, 512, 512, 512, 512

7.4.2 Case 2

When $|FV_{t_1(v)} \cap FV_{t_2(v)}| = 136$, we get fundamental invariants of:

32, 32, 32, 32, 32, 32, 32, 32, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 128, 128, 128, 128, 128, 128, 128, 128 , 128, 128, 128, 128, 128, 128, 128, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 384, 3840

7.5 Further Directions

One can take the conformal vectors spanning $FCV(t_1(v), t_2(v))$ and examine their kernel. Scaling the vectors in their kernel such that they lie in a scaled copy of L, we would hope to have a convenient annihilator subllatice of $FCV(t_1(v), t_2(v))$. While computationally speaking this problem should be remarkably simple, limitations with Maple 15's memory handling and a lack of time prevented these calculations.

8 Glossary

Notation	Overview	Notable Properties and References
G	Lattice	See definition (1)
f or $\langle -, - \rangle$	Symmetric Bilinear Form	A property to a lattice determined by context. See definition (1)
$\Phi(G)$	Set	The root set of a lattice G is defined in definition (3).
A_n	Lattice	The A_n root lattice is defined in definition (5).
D_n	Lattice	The D_n root lattice is defined in definition (6).
E_n	Lattice	The E_8, E_7, E_7 root lattices are defined in definition (7).
GG_n	Lattice	A definition for the even lattice of a type A, D, E root lattice can be found in definition (8).
W	Vector Space	W is a vector space with a linear transformation t such that $t(x \otimes y) = y \otimes x$ for all $x, y \in w$. $(W \otimes W)^{\pm}$ is the eigenspace of characteristic $\neq 2$ of t for the eigenvalue ± 1 on $W \otimes W$. Then, $W \otimes W = (W \otimes W)^{+} \oplus (W \otimes W)^{-}$. Assume that W has a symmetric bilinear form - then $W \otimes W$ has its own symmetric bilinear form following from this definition and this direct sum is an orthogonal direct sum.
Q	Lattice	Q is the input lattice for L, a discussion of which can be found in section (4)
$S^2(W)$	Vector Space	$S^2(W) = W \otimes W / (W \otimes W)^{-}$, and the kernal of the orthogonal projection P of $W \otimes W$ to $(W \otimes W)^{+}$ is $(W \otimes W)^{-}$. Hence, we identify $S^2(W)$ with $(W \otimes W)^{+}$ by mapping $x \otimes y + (W \otimes W)^{-} \mapsto P(x \otimes y) = \frac{1}{2}(x \otimes y + y \otimes x) = xy$. This identification and bilinear form $(W \otimes W)^{+}$ gives us a bilinear form on $S^2(W)$: given $ab, cd \in S^2(W)$ is $\frac{1}{2}(\langle a, c \rangle \langle b, d \rangle + \langle a, d \rangle \langle b, c \rangle)$
$S^2(Q)$	Lattice	Analogous properties to $S^2(W)$
L	Lattice	L is the span of a set of conformal vectors, see definition (12). The notation L_Q appears sparely throughout the paper, which simply indicates the lattice L resulting from an input lattice Q. Similarly, CV_Q denotes the set of conformal vectors spanning L.
$e_{\lambda}^{\pm}$	Conformal Vector	An element of the spanning set of L. See equation (4.1).
$e_{M,\varphi}$	Conformal Vector	An element of the spanning set of L. See equation (6.2)
ω_M	Virasoro Element	See equation (6.3) and its preceeding paragraph.
v_{λ}	Element of T	See definition (13).
F	Lattice	F contains an integer multiple of L, $\alpha L \subset F \mid \alpha \in \mathbb{Z}$. See definition (15).
T	Lattice	Generated by the $\mathbb{Z}$-span of v_{λ} where $\lambda \in Q_2$ for an input lattice Q. See definition (13).
$\mathbb{C}[Q]$	Vector Space	$\mathbb{C}[Q]$ is a vector space with orthonormal basis $e^{\lambda} \mid \lambda \in Q_2$. $T \subset \mathbb{C}[Q]$ when T is the $\mathbb{Z}$-span of $v_{\lambda} \mid \lambda \in Q_2$.
K	Lattice	Given an input lattice Q contained in a vector space W, $K = S^2(Q) \perp T$. $32L \subset K$ when Q is EE_8. See definition (14).
D	Vector Space	$D = S^2(W) \perp \mathbb{C}[Q]$. D contains the conformal vectors that span L.
$M_{[]}$	Lattice	M is the lattice whose rows are conformal vectors in terms of the basis of F. Context determines exactly what M is, for an example see equation (5.1)

References

[1] Robert L. Griess, Jr., *An Introduction to Groups and Lattices: Finite Groups and Positive Definite Rational Lattices.* International Press, Boston, 2010n

[2] *A Vertex Operator Algebra related to* EE_8 *with Automorphism Group* $O^+(10,2)$. Monster and Lie Algebras, Berlin, 1998

[3] B.Hartley, T.O. Hawkes, *Rings, Modules and Linear Algebra* Spottiswoode, Ballanyne &Co, 1970

[4] Assorted, E_8 *(mathematics)*. Wikipedia. July 18^{th}, 2012.

9 Appendix

9.1 Assorted Calculations

Forwards, the 'Determinant' refers to the determinant of the Gram matrices associated to the indicated lattice.

9.1.1 Calculations on $S^2(\sqrt{2}A_n)$

n	Prime Factorization of Determinant	Sequence of Fundamental Invariants
2	$((2))^2\,((3))^3$	1,6,18
3	$((2))^{11}$	1,2,2,4,4,32
4	$((2))^4\,((5))^5$	1,1,1,1,1,1,10,10,10,50
5	$((2))^{11}\,((3))^6$	1,1,1,1,1,1,2,2,2,2,6,6,6,6,72
6	$((2))^6\,((7))^7$	1,1,1,1,1,1,1,1,1,1,1,1, 1,1,1,14,14,14,14,14,98
7	$((2))^{31}$	1,1,1,1,1,1,1,1,1 ,1,1,1,1,1,1,2,2,2,2,2,2,8,8,8,8,8,8,128

9.1.2 Calculations on $S^2(\sqrt{2}D_n)$

n	Prime Factorization of Determinant	Sequence of Fundamental Invariants
3	$((2))^{11}$	1,2,2,4,4,32
4	$((2))^{14}$	1,2,2,2,2,2,2,4,8,8
5	$((2))^{17}$	1,1,1,1,1,1,2,2,2,2,4,4,4,4,32
6	$((2))^{20}$	1,1,1,1,1,1,2,2,2,2,2,2, 2,2,2,2,2,2,4,8,8
7	$(2)^{23}$	1,1,1,1,1,1,1,1,1 ,1,1,1,1,1,1,2,2,2,2,2,2,4,4,4,4,4,4,32
8	$((2))^{26}$	1,1,1,1,1 ,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,4,8,8

9.1.3 Calculations on L when $Q = A_n$

n	Prime Factorization of Determinant	Sequence of Fundamental Invariants
2	$((2))^{19}\,((3))^3$	2,2,4,32,96,288
3	$((2))^{47}$	2,2,4,16,16,16,16,16,16,64,64,128
4	$((2))^{66}\,((5))^5$	2,2,2,2,4,4,4,4,4,4,32,32 ,32,32,32,32,160,160,160,800
5	$((2))^{106}\,((3))^6$	2,2,2,2,4,4,4,4 ,4,4,8,8,8,8,16,16,32,32,32,32,32,32,32,32,32,96,96,96,96,288
6	$((2))^{141}\,((7))^7$	2, 2 , 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 224, 224, 224, 224, 224, 1568
7	$((2))^{213}$	2, 2, 2, 2, 2, 2 , 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 128, 128, 128, 128, 128, 128, 512

9.1.4 Calculations on L when $Q = D_n$

n	Prime Factorization of Determinant	Sequence of Fundamental Invariants
3	$((2))^{47}$	2, 2, 4, 16, 16, 16, 16, 16, 16, 64, 64, 128
4	$((2))^{82}$	2, 2, 4, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 64
5	$((2))^{127}$	2, 2, 2 , 2, 4, 4, 4, 4, 4, 4, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 64, 64, 64, 64, 128
6	$((2))^{182}$	2, 2, 2, 2 , 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 32, 32, 32, 32, 64
7	$((2))^{247}$	2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 64, 64, 64, 64, 64, 64, 128
8	$((2))^{322}$	2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 64

Theorem 1. *Let L be a rational lattice, and G a sublattice of L of finite index $|L : G|$. Then*

$$det(L)|L : G|^2 = det(G)$$

Proof. [1] □

Remark 13. One might notice that for corresponding L_Q and $S^2(Q)$, the prime factorization of the product of their Smith invariants are similar. These similarities are expected, and follow directly from this theorem.

Remark 14. We have that $S^2(A_2) \cong S^2(D_3)$, so it comes as no surprise that the fundamental invariants for $S^2(\sqrt{2}D_3)$ are the same as the fundamental invariants for $S^2(\sqrt{2}A_2)$.

9.2 Calculating the Fundamental Invariants for L when $Q = \sqrt{2}E_7$ or $Q = \sqrt{2}E_6$

Definition 28. Recall that E_7 andE_6 are defined as the annihilators of any A_1 and A_2 sublattice of E_8, respectively. As such, a set of roots for E_7 when we consider the A_1 sublattice of E_8 with basis $e_7 - e_8$ is as follows:

$$\Phi(E_7) = \Phi(D_6) \cup \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \cup \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \cup \pm(e_7 + e_8)$$

where the roots of the form $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$ have evenly many minus signs. Hence, E_7 has a total of $4(\binom{6}{2})+2(\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6})+2=126$ roots.

Similarly, the roots in E_6 when the A_2 sublattice of E_8 has the basis $\{e_6-e_7, e_7-e_8\}$ are as follows:

$$\Phi(E_6)=\Phi(D_5)\bigcup\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\bigcup\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$$

Where the roots of the form $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$ have evenly many minus signs. Hence, E_6 has $4\binom{5}{2}+2\left(\binom{5}{0}+\binom{5}{2}+\binom{5}{5}\right)=72$ roots.

When $Q=\sqrt{2}E_6$ or $Q=\sqrt{2}E_7$, the type-1 conformal vectors are sufficient to form a basis for L. If $Q=\sqrt{2}E_6$ we restrict our attention to norm-4 vectors of the form $\sqrt{2}(e_i\pm e_j)|1\le i<j\le 5$ and all $\sqrt{2}(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ with evenly many signs. Taking F to be as described in section (6) for $Q=EE_8$, we write the type-1 one conformal vectors of L as described in terms of the basis of F and compile matrices M_{EE_6} and M_{EE_7} with all such vectors. Proceeding as we have in other sections, we calculate the invariants for L when $Q=\sqrt{2}E_7$ and $Q=\sqrt{2}E_7$ as follows:

When $Q=\sqrt{2}E_7$:

2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 32, 32, 32, 32, 64, 256, 512, 512, 512, 512, 1024

When $Q=\sqrt{2}E_6$:

2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 64, 64, 64, 64, 64, 192, 768, 768, 768, 768, 9216