Slices of Parameter Spaces of Meromorphic Functions with Two Asymptotic Values

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This talk is based on joint work with Tao Chen and Linda Keen

Suppose f is a holomorphic map of the complex plane $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We call the semi-group $\{f^n\}_{n=0}^{\infty}$ a complex dynamical system. We want to the future of $f^n(z)$ for a point z as n tends to ∞. For examples, periodic cycle (attracting, repelling, rational indifference, and irrational indifference), Fatou set, and Julia set.

Family of Complex Dynamical Systems

Suppose $\{f_\lambda\}_{\lambda\in\Lambda}$ is a family of holomorphic maps over a parameter space Λ. We want to know those parameters such that dynamical systems around each of them have the same pattern (stable behavior) and those parameters such that patterns of dynamical systems change with a small perturbation at each of them (bifurcation behavior).

The Quadratic Family

One example is the family of quadratic polynomials

$$q_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

In particular, the family on the real line

$$q_t(x) = x^2 + t, \quad -2 \leq t \leq 0.$$

It has only one critical point 0 in $\mathbb{C}$ and

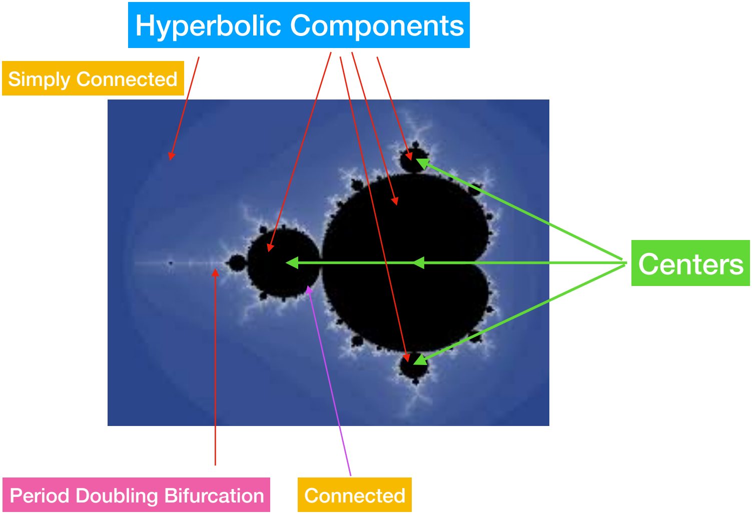
$$M = \{c \in \mathbb{C} \mid \{q_c^n(0)\}_{n=0}^{\infty} \text{ is bounded}\}$$

is called the Mandelbrot set. The bifurcation locus is ∂M.

Hyperbolic components: components of $(\partial M)^c$ such that $\{q_c^n(0)\}_{n=0}^{\infty}$ tends to an attracting cycle.

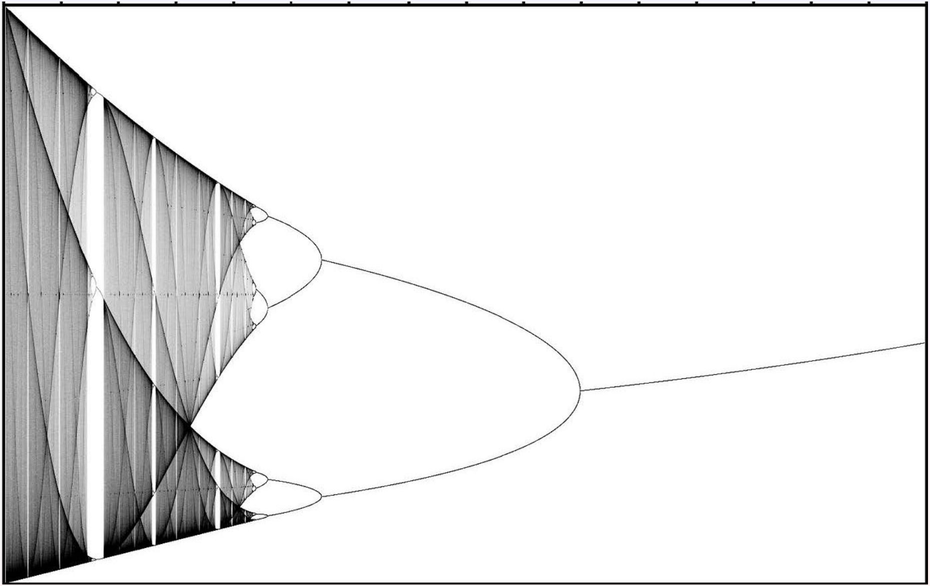
Center: those c such that $\{q_c^n(0)\}_{n=0}^{\infty}$ are periodic cycles.

The Mandelbrot Set



Complexity of dynamics increases through hyperbolic components by bifurcation

The Bifurcation Diagram for The Quadratic Family



The Tangent Family

In this talk, I would like to talk on our recent work on the tangent family

$$T_\rho(z) = i\rho \tan(z), \quad \rho \in \mathbb{C}^*.$$

In particular, the family on the imaginary line

$$T_t(z) = it \tan(z), \quad 0 < t \leq \pi.$$

Quadratic Polynomial vs Tangent Function

Q1 A quadratic polynomial is a branched cover of finite degree.

Q2 It has a critical value $v = f(0)$ and ∞ is also a critical value.

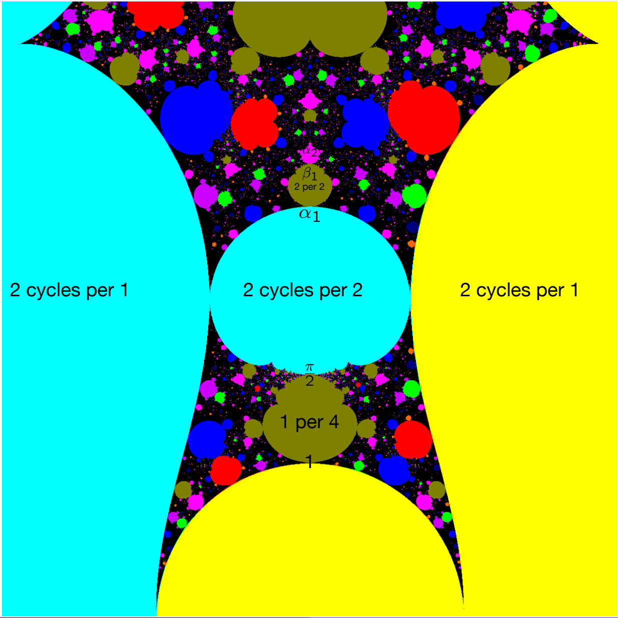
Q3 The post-critical orbit $\{q_c^n(v)\}_{n=0}^{\infty}$ controls the dynamics of the quadratic polynomial.

T1 A tangent function is a branched cover of infinite degree.

T2 It has a symmetric pair of asymptotical values $\{-\rho, \rho\}$ and an essential singularity at ∞.

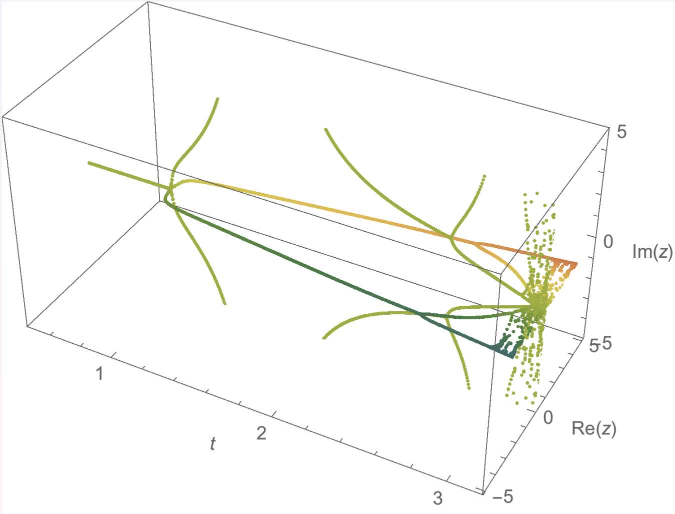
T3 The post-singular orbit $\{T_\rho^n(\rho)\}_{n=1}^{\infty}$ controls the dynamics of the tangent function.

The “Mandelbrot Set” for the Tangent Family



Bifurcation Diagram for the Tangent Family

$T_t(z) = it\tan(z)$, $0 < t \leq \pi$.



First Few Steps in Bifurcation Diagram

When t increases in $(0, \pi]$, we observed that

1. One period 1 cycle breaks into one period 4 cycle (through rational indifference bifurcation);
2. One period 4 cycle breaks into two period 2 cycles (through a virtual cycle parameter);
3. Period doubling: Two period 2 cycles bifurcate to two period 4 cycles (through parabolic cycle with multiplier -1);
4. Virtual cycle parameter: Two period 4 cycles become two period 4 virtual cycles;
5. Period merging: Two period 4 virtual cycles merge into one period 8 cycle.

Bifurcation Diagram

Theorem (Chen-J-Keen, 2018)

Let t_1 be the only one period doubling bifurcation parameter. In general, we have
$0 < t_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n < \cdots < \pi$,

1. *Virtual Cycle Parameter: At the parameter β_n, $n = 1, 2, \cdots$, T_{β_n} has two period 2^{n+1} virtual cycles;*
2. *Period Merging: After that for $\beta_n < t < \alpha_{n+1}$, two period 2^{n+1} virtual cycles merge into one period 2^{n+2} attracting cycle.*
3. *Cycle Doubling: at α_{n+1}, $T_{\alpha_{n+1}}$ has one period 2^{n+2} parabolic cycle (with multiplier 1) and after that for $\alpha_{n+1} < t < \beta_{n+1}$, it doubles into two period 2^{n+2} attracting cycles.*

Asymptotic Value

Suppose f is a meromorphic function and ∞ is an essential singularity. A point $v \in \mathbb{C}$ s called an asymptotic value if there is a continuous curve $\gamma(t), 0 \leq t < \infty$, such that

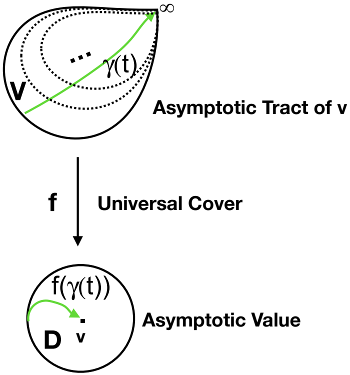
$$\lim_{t \to \infty} \gamma(t) = \infty$$

and

$$\lim_{t \to \infty} f(\gamma(t)) = v.$$

Asymptotic Tract

For each isolated asymptotic value v, we have a disk D centered at v and a simply connected unbounded domain V such that $f : V \to D \setminus \{v\}$ is a universal cover. That is $f|V$ is like e^z topologically on a left-half plane. Here V is called the asymptotic tract for v.



Virtual Cycle and Virtual Cycle Parameter

If an asymptotic value v is also a pole or pre-pole, then there is an $n \geq 2$ such that $f^{n-1}(v) = \infty$. Let $\alpha(t)$ be any component of $f^{-(n-1)}(\gamma(t))$ in D, then

$$\lim_{t \to \infty} \alpha(t) = v \quad \text{and} \quad \lim_{t \to \infty} f^n(\alpha(t)) = v.$$

In this way, we can treat

$$\{v, f(v), \cdots, f^{n-2}(v), \infty\}$$

as a period n cycle. We call it a (period n) virtual cycle. The corresponding parameter in a family $\{f_\lambda\}_{\lambda \in \Lambda}$ is called a (period n) virtual cycle parameter.

Transversality

Suppose λ_0 is a virtual cycle parameter. Let $p = f_{\lambda_0}^{n-2}(\lambda_0)$, a pole. Define $D(\lambda) = f_\lambda^{n-2}(\lambda) - p$. We say f_λ is transversal at λ_0 if $D'(\lambda_0) \neq 0$.

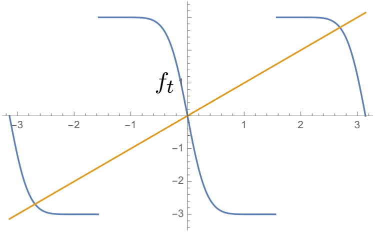
Theorem (Chen-J-Keen, 2018)

For $T_t(z) = it\tan(z)$, $0 < t \leq \pi$, the transversality holds at every virtual cycle parameter.

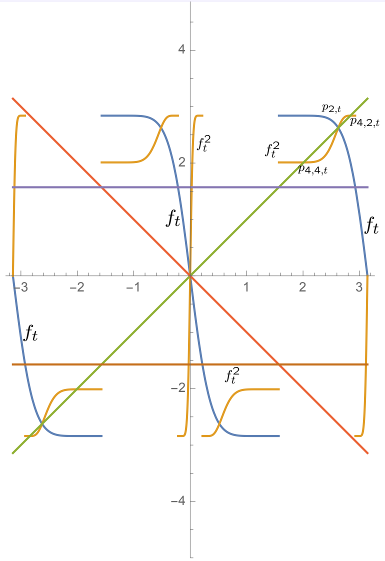
One of the main techniques we used in the proof of this theorem is holomorphic motions.

Graph on the Real Line

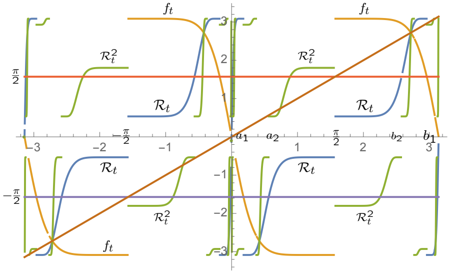
In the proof of the bifurcation diagram, we use renormalization and the transversality theorem. The graph of the function $f_t(x) = T_t^2(x)$ on the real line looks like



The First Renormalization

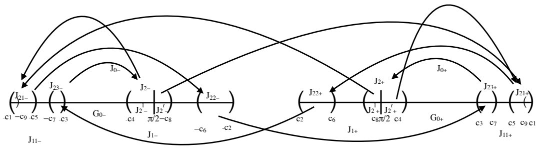


The Second Renormalization



Infinitely Renormalizable Tangent Map

We use the renormalization method and the Transversality Theorem to prove the Bifurcation Diagram Theorem. In the end, let $t_\infty = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n$, we have an infinitely renormalizable tangent map T_{t_∞}. It has two Cantor sets which form an attractor for $T^2_{t_\infty}$ on the real line and $T^2_{t_\infty}$ dances around these two Cantor sets.



The Quadratic Family as A Slice

The quadratic family is a slice in the moduli space of quadratic rational functions. That is, a one complex dimensional subspace.

The moduli space of quadratic rational functions is a two complex dimensional space.

Another Slice

For each $\rho \in \mathbb{C}$, $0 < |\rho| < 1$, the family of quadratic rational functions,

$$\frac{1}{\rho}\Big(z + b + \frac{1}{z}\Big), \quad b \in \mathbb{C}$$

is another slice of the moduli space of quadratic rational functions which has been studied by Goldberg and Keen, 1990.This is a slice where maps have attracting fixed point at ∞ with the multiplier ρ, $|\rho| < 1$.

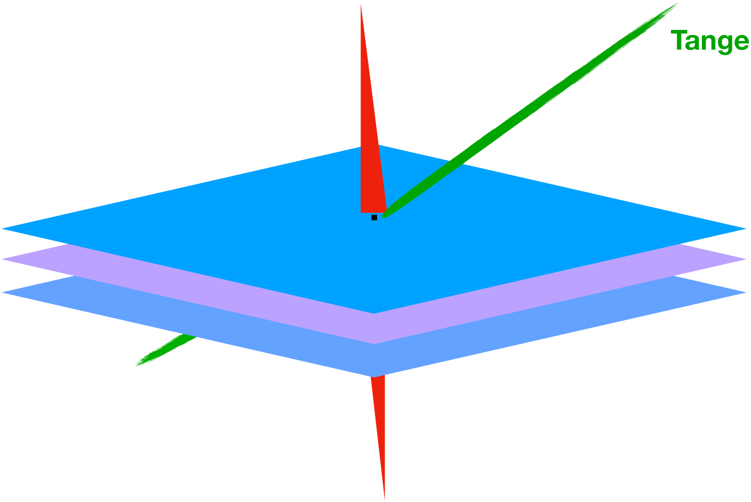
The Tangent Family as A Slice

The tangent family can be thought as a slice (when $\lambda = \rho$) in the two complex dimensional family of meromorphic functions

$$f_{\lambda,\rho}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}} \quad \text{with} \quad \frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho}.$$

Here $\lambda \neq 0$ and $\mu \neq 0$ are asymptotic values and (λ, ρ) are parameters.

Two Complex Dimensional Space



The Slice for A Fixed ρ, $0 < |\rho| < 1$

The next natural slice we have studied is $f_\lambda(z) = f_{\lambda,\rho}(z)$ for a fixed $0 < |\rho| < 1$ and $\lambda \in \mathbb{C} \setminus \{0, \rho/2\}$. Since 0 is an attracting fixed point, it must attracts one or two asymptotic values. We can divide the parameter space $\mathbb{C} \setminus \{0, \rho/2\}$ into a shift locus

$$\mathcal{S} = \{0 \text{ attracts both } \lambda \text{ and } \mu\}$$

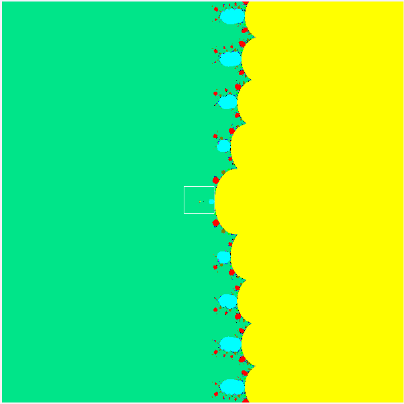
and two "Mandelbrot sets"

$$\mathcal{M}_\lambda = \{0 \text{ attracts only } \mu\}$$

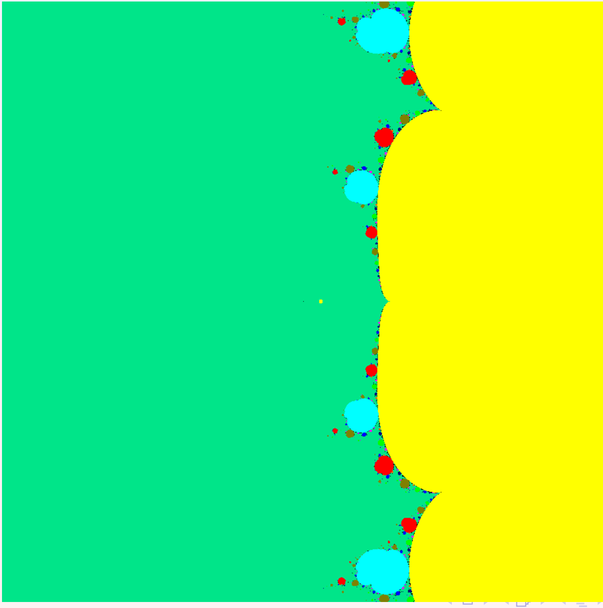
and

$$\mathcal{M}_\mu = \{0 \text{ attracts only } \lambda\}$$

Parameter Space for the Slice of $0 < |\rho| < 1$



Enlarge of the Parameter Space for the Slice of $0 < |\rho| < 1$



Next Enlarge of the Parameter Space for the Slice of $0 < |\rho| < 1$

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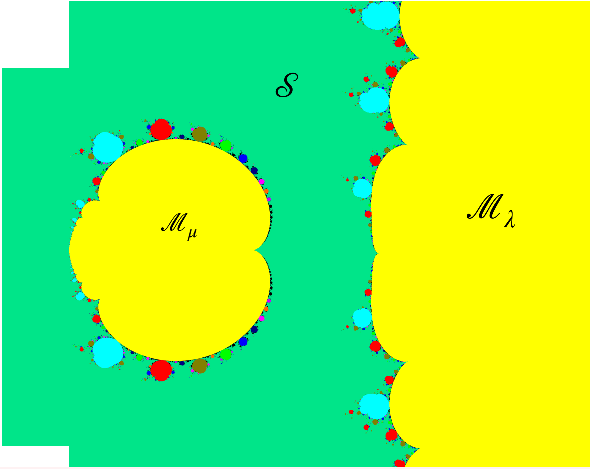


Our main work in this direction is to understand the topological structures of $\mathcal{S}$ and $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$. Furthermore, the combinatorial structures of all stable components and all virtual center parameters.

Main Structure

Theorem (Chen-J-Keen, 2019)

For each $0 < |\rho| < 1$, the shift locus $\mathcal{S}$ is holomorphically isomorphic to a punctured annulus. The puncture is at the origin. The other puncture of the parameter plane, $\rho/2$, is on the boundary of the shift locus. Both the Mandelbrot-like sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ are connected and full.

Combinatorial Structure

Theorem (Chen-J-Keen, 2019)

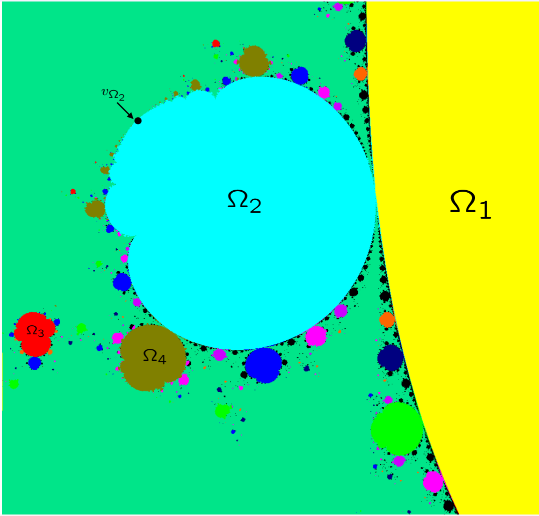
The period n virtual cycle parameters can be mapped to the set of all sequences $\mathbf{k}_n = k_n, k_{n-1,} \ldots, k_1,\ k_i \in \mathbb{Z}$, in such a way that each such period n parameter is an accumulation point in $\mathbb{C}$ of a sequence of period $n+1$ parameters $\mathbf{k}_n, k_{n-1}, \ldots, k_1, k_{0,j}$. This combinatorial description of the virtual cycle parameters (that is, virtual centers) determines combinatorial descriptions of stable (shell) components in $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$.

Common Boundary

Theorem (Chen-J-Keen, 2019)

Every virtual cycle parameter is both a boundary point of a stable component and a boundary point of the shift locus. Furthermore, the dynamics of the family $\{f_\lambda\}$ is transversal at these parameters. Moreover, the set of all virtual cycle parameters is dense in the common boundary of the shift locus $\mathcal{S}$ and the sets $\mathcal{M}_\lambda \cup \mathcal{M}_\mu$.

Stable Component and Virtual Center

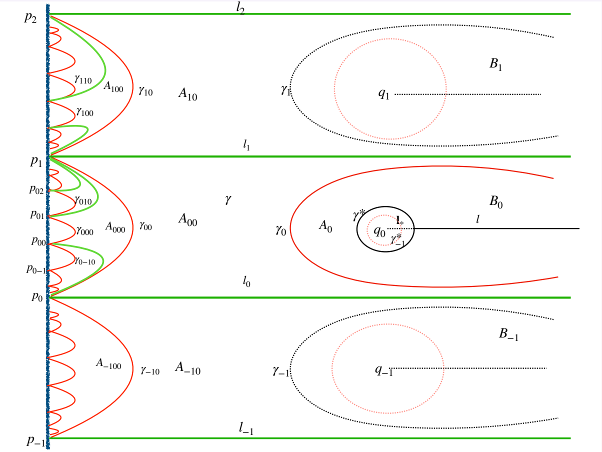


Stable Components of Order 1

There are two stable components Ω_1 of period 1. One is in $\mathcal{M}_\lambda$ and the other is in $\mathcal{M}_\mu$. For the one in $\mathcal{M}_\lambda$, the virtual center is at infinity; however, for the one in $\mathcal{M}_\mu$, the virtual center is at the finite point $\rho/2$ which is a parameter singularity.

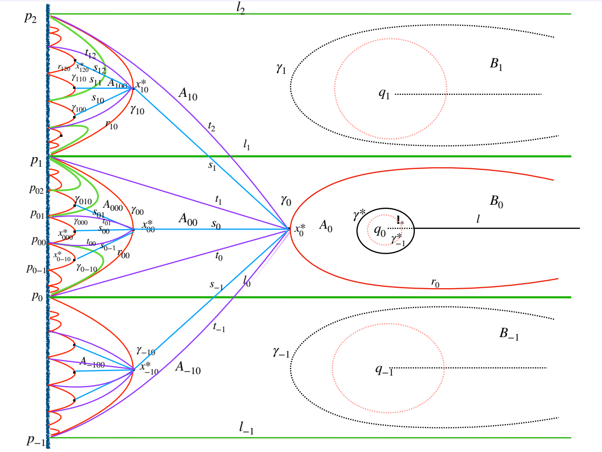
A Model Map

For the slice $0 < |\rho| < 1$, we have a $\lambda_0 \in \Omega_1 \subset \mathcal{M}_\lambda$ such that at the fixed point $q_0 = q(\lambda_0)$ of f_{λ_0}, $f'_{\lambda_0}(q_0) = \rho$. We use the dynamics of f_{λ_0} to study the shift locus.



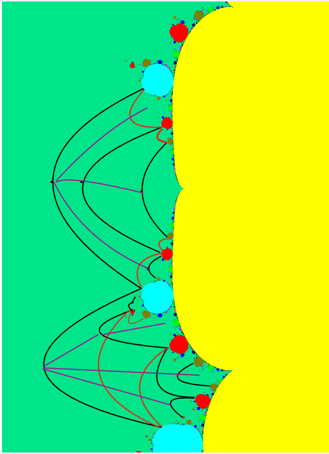
A Model Map and “External Rays”

In the dynamics of the model map, we can see “external rays” landing at all poles and prepoles in the Julia set.



Parameter Space and "External Rays"

The "external rays" in the dynamical plane of the model map can be translated into the parameter space as "external rays" in the shift locus.



Accessible Boundary Points in the Shift Locus

Theorem (Chen-J-Keen, 2020)

The parameters in the bifurcation locus that are virtual centers and the parameters for which the function $f_{\lambda,\rho}$ has a parabolic cycle or for which an asymptotic value is mapped onto a repelling cycle by some iterate of $f_{\lambda,\rho}$ are accessible from inside the shift locus.

Thanks!