A Function Model for the Teichmüller Space of a Closed Hyperbolic Riemann Surface

Yunping Jiang

Queens College and Graduate Center
City University of New York

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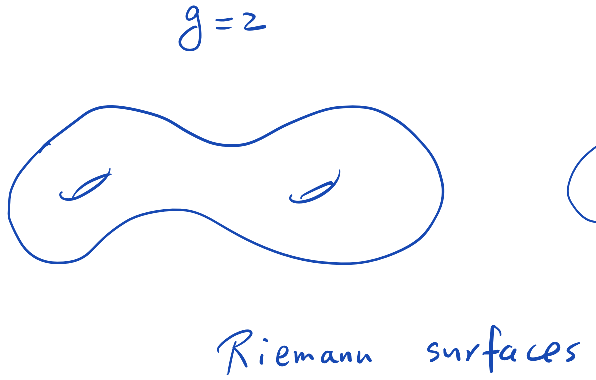
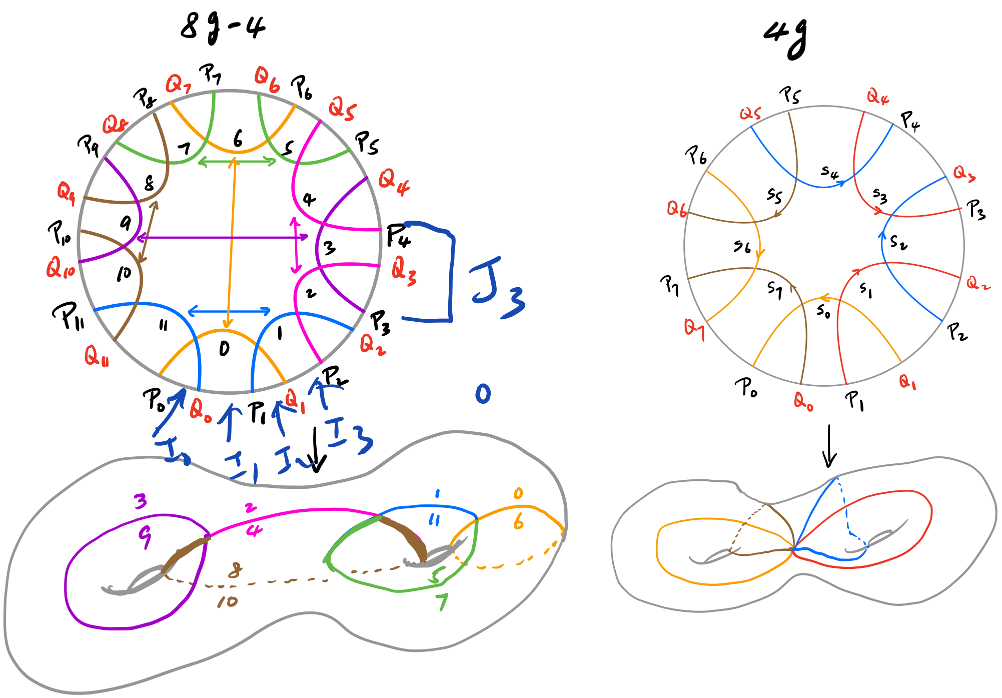
A closed hyperbolic Riemann surface

Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the hyperbolic disk. A closed hyperbolic Riemann surface R can be viewed as $R = \Delta/\Gamma$ where Γ is a co-compact Fuchsian group Γ on Δ. The unit circle $\mathbb{T}$ is the Euclidean boundary of Δ. The genus $g > 1$ is a topological invariant.



Fundamental polygons

A closed Riemann surface of g can be viewed as a $(8g-4)$ (or $4g$) hyperbolic polygons F_g inside the unit disk.



A partition on the unit circle

The sides of the $(8g-4)$ fundamental polygons F_g are labeled as $0 \le i \le 8g-5$ counter-clockwise. Each side i is paired with another side $p(i)$ by $\gamma_i \in \Gamma$, where

$$p(i) = \begin{cases} 4g-2-i \pmod{8g-4} & \text{if } i \text{ is even;} \\ -i \pmod{8g-4} & \text{if } i \text{ is odd.} \end{cases}$$

Each side can be extended to $\mathbb{T}$ with endpoints (arranged counter-clock-wisely),

$$P_0, Q_0, P_1, Q_1, \cdots, P_{8g-5}, Q_{8g-5}.$$

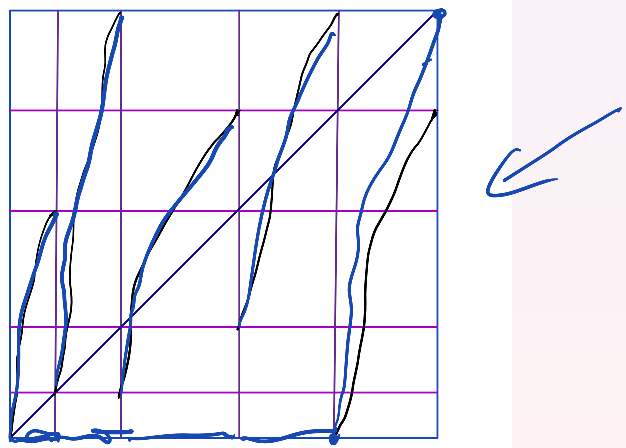
These points give a partition of $\mathbb{T}$ into $16g-8$ intervals

$$\eta = \{I_{2i} = [P_i, Q_i], I_{2i+1} = [Q_i, P_{i+1 \pmod{8g-5}}] \mid 0 \le i \le 8g-5\}.$$

Markov one-dimensional dynamical system

Let $J_i = [P_i, P_{i+1 \pmod{8g-5}})$ for $0 \leq i \leq 8g-5$. Define a piece-wise smooth map $f = f_R : \mathbb{T} \to \mathbb{T}$ as $f(x) = \gamma_i(x)$ for any $x \in J_i$. It is a Markov and expanding one-dimensional dynamical system.

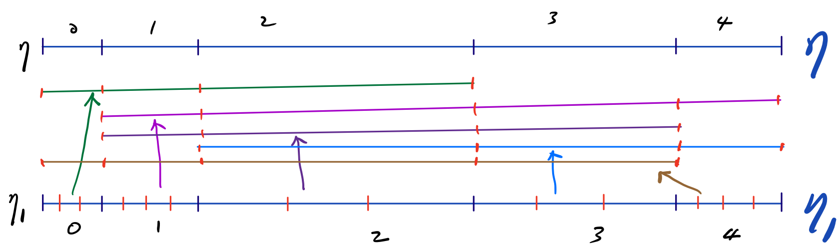
- Markov: each $f|\mathring{I}_i$ is 1-1 and $f(I_i)$ is the union of some intervals in η.
- Expanding: there are two constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \mathbb{T}$ such that $f'(f^i(x))$ are defined for all $0 \leq i \leq n-1$.



A Markov one-dimensional dynamical system

Sequence of nested partitions

Pull-back the partition η, we can get a sequence of nested partitions $\eta_n = f^{-n}\eta$ for all $n \geq 0$.



Any point $\{x\} = \cap_{n=1}^{\infty} I_n(x)$ for a sequence of nested intervals $I_n(x) \in \eta_n$.

Symbolic dynamical system

Let $B = \{0, 1, \cdots, 16g-9\}$. We define a $(16g-8) \times (16g-8)$ matrix $A = (a_{ij})$ as $a_{ij} = 1$ if $f(I_i) \supset I_j$ and 0 otherwise and a symbolic space

I_j

I_i

$$\Sigma_A = \{w = \cancel{i_0} i_1 \cdots i_{n-1} i_n \cdots \mid i_{n-1} \in B, a_{i_{n-1} i_n} = 1, n = 1, 2, \cdots\}.$$

and the shift $\sigma_A(w) = i_1 \cdots i_{n-1} i_n \cdots : \Sigma_A \to \Sigma_A$. Let $w = w_n i_n \cdots \in \Sigma_A$. Each interval of η_n has a unique labelling w_n denoted as I_{w_n}. Then we have that

$I_{iw_n} \xrightarrow{f} I_{w_n}$

$I_{w_n i}$

$$I_{w_n} = \cup_{i \in B, a_{i_{n-1} i} = 1} I_{w_n i} \quad \text{and} \quad f^{-1}(I_{w_n}) = \cup_{i \in B, a_{i i_0} = 1} I_{i w_n}.$$

All points of $\mathbb{T}$ except for those endpoints of I_{w_n} has a unique labelling $\{x_w\} = \cap_{n=1}^{\infty} I_{w_n}$ and

$$f(x_w) = x_{\sigma_A(w)}.$$

Dual symbolic dynamical system

Consider the dual symbolic dynamical system

$$\Sigma_A^* = \{w^* = \cdots j_n i_{n-1} \cdots j_1 \cancel{j_0} \mid j_{n-1} \in B, a_{j_n j_{n-1}} = 1, n = 1, 2, \cdots\}.$$

and the dual shift $\sigma_A^*(w^*) = \cdots j_n j_{n-1} \cdots j_1 : \Sigma_A^* \to \Sigma_A^*$. Let $w^* = \cdots j_n w_n^* \cdots \in \Sigma_A^*$. Then we have $I_{w_n^*} \subset I_{\sigma_A^*(w_n^*)}$. Define

$$s_n(w_n^*) = \frac{|I_{w_n^*}|}{|I_{\sigma_A^*(w_n^*)}|}.$$

$w_n^* = w_n$

$I_{\sigma^*(w_n^*)}$

$I_{w_n^*}$

$I_{\sigma^*(w_n^*)}$

$I_{w_n^*}$

$I_{\sigma_A^*(w_n^*)}$

$I_{w_n^*}$

f

$I_{\sigma_A^*(w_{n+1}^*)}$

$I_{w_{n+1}^*}$

Limiting function

metric space

d

$|s(w^*) - s(w^{*\prime})|$

$\leq C d(w^*, w^{*\prime})^\alpha$

Theorem

For every $w^* \in \Sigma_A^*$, the limit $s(w^*) = \lim_{n\to\infty} s_n(w_n^*)$ exists and defines a Hölder continuous function

$$s = s_R : \Sigma_A^* \to (0,1).$$

$C(\Sigma_A^*)$

$\cup$

S_g ⟵ complex structure

We call s_R the scaling function for R.

Teichmüller space and scaling function

Take a fixed hyperbolic Riemann surface R_0 of g, say its fundamental polygon is symmetric, i.e. the Ford domain F_g. The Teichmüller space $\mathcal{T}_g = \mathcal{T}(R_0)$ is the space of equivalence classes of all marked Riemann surfaces $h_R : R_0 \to R$. Here two marked Riemann surfaces R_1 and R_2 are Teichmüller equivalent if there is a conformal isomorphism $\alpha : R_1 \to R_2$ such that $h_{R_2}^{-1} \circ \alpha \circ h_{R_1}$ is isotopic to the identity.

$R_1 \overset{T}{\sim} R_2 \iff s_{R_1} = s_{R_2}$

Theorem

The scaling functions of two closed hyperbolic Riemann surfaces of the same genus g are the same if and only if these two Riemann surfaces are Teichmüller equivalent.

Thus we have that $s_\tau = s_R$ for all $R \in \tau \in \mathcal{T}_g$.

A function model

Let $\mathcal{S}_g = \{s_R\}$ be the space of scaling functions for all hyperbolic Riemann surfaces R of the same genus $g \geq 2$. We have a bijective map

$$\iota : \mathcal{T}_g \to \mathcal{S}_g; \quad \iota(\tau) = s_\tau$$

We call $\mathcal{S}_g$ a function model for the Teichmüller space $\mathcal{T}_g$.

Teichmüller's metric

$\mathcal{T}_g$ $\qquad$ $\mathcal{S}_g$

$d_{\mathcal{T}}$ $\qquad$ $d_{\mathcal{T}}$

The classical metric defined on $\mathcal{T}$ is Teichmüller's metric $d_{\mathcal{T}}(\cdot, \cdot)$. Note that Teichmüller's metric coincides with Kobayashi's metric when we think $\mathcal{T}_g$ as a complex Banach manifold. Since $\iota : \mathcal{T}_g \to \mathcal{S}_g$ is bijective, we can define the Teichmüller metric on $\mathcal{S}_g$ as

$$d_{\mathcal{T}}(s, s') = d_{\mathcal{T}}(\iota^{-1}(s), \iota^{-1}(s')).$$

The maximum metric

Consider the space $C(\Sigma_A^*)$ of all continuous functions on Σ_A^*. For any s in $C(\Sigma_A^*)$, we have the maximum norm

$$||s|| = \sup_{w^* \in \Sigma_A^*} |s(w^*)|.$$

The maximum metric on $\mathcal{T}_g$ is defined as

$$d_{\max}(\tau, \tau') = ||s_\tau - s_{\tau'}|| \quad \forall\, \tau, \tau' \in \mathcal{T}_g$$

Topology

Is the maximum metric meaningful? For this question, we need first to see if it gives the same topology on $\mathcal{T}_g$ as the one given by Teichmüller's metric.

$T_{S_0} S_g =$

Theorem

The identity map

$$id_{\mathcal{T}} : (\mathcal{T}_g, d_{\mathcal{T}}) \to (\mathcal{T}_g, d_{max})$$

is uniformly continuous map and the identity map

$$id_{\mathcal{T}} : (\mathcal{T}_g, d_{max}) \to (\mathcal{T}_g, d_{\mathcal{T}})$$

is only continuous.

Thus, both Teichmüller's metric and the maximum metric induce the same topology on $\mathcal{T}_g$.

Topological entropy

R. Adler and L. Flatto 1991 BAMS
Abrams, Katok Ugarcovici

For the matrix A, we have that for some positive integer n such that A^n is a positive matrix, i.e., $A^n > 0$. The Perron-Frobenius theorem says that A has a unique maximum positive simple eigenvalue λ_{max}. The number $\log \lambda_{max}$ is the topological entropy $h_{top}(\sigma_A^*)$ of σ_A^* as well as $\mathcal{T}_g$. From the structure of A,

$$\lambda_{max} = 4g - 3 + \sqrt{(4g-3)^2 - 1}.$$

Thus $h_{top}(\sigma_A^*) = \log(4g - 3 + \sqrt{(4g-3)^2 - 1})$.

The topological entropy is a topological invariant and measures the complexity. It is a constant on $\mathcal{S}_g = \mathcal{T}_g$.

Thermodynamical formalism

$\log s$

w_n^*

$[w_n^*] \ni w^* \xrightarrow{\sigma^*} \xrightarrow{\sigma^*} \xrightarrow{\sigma^*} \cdots$

Since $0 < s = s_\tau < 1$ is a Hölder continuous function on Σ_A^*, the classical Gibbs theory says that we have a unique probability measure $\mu = \mu_\tau$ on Σ_A^* such that

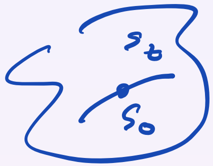
$$C^{-1} \leq \frac{\mu([w_n^*])}{\exp(\underbrace{-Pn}_{=0} + \sum_{i=0}^{n-1} \log s((\sigma_A^*)^i(w^*)))} \leq C$$

for all n-cylinder $[w_n^*]$ and $w^* \in [w_n^*]$, where C is a fixed constant. Here the number $P = P(\log s)$ is called the pressure. In our case $P(\log s) = 0$ for all $s \in \mathcal{S}_g$.

Pressure metric

Consider a smooth path $s_t : (-\delta, \delta) \to \mathcal{S}_g$ and let μ_t be the corresponding Gibbs measure. Since the mean

$\mathcal{S}_g$



$$\int_{\Sigma_A^*} \dot{\log} s_0 \, d\mu_0 = P(\dot{\log} s_0) = 0.$$

$$= \frac{d \log s_t}{dt}\Big|_{t=0}$$

$P(\log s_t) = 0$

We can calculate the pressure metric on the tangent space $\underline{T_{s_0}\mathcal{S}_g}$ as

$$\|\dot{\log} s_0\|_P^2 = \frac{var(\dot{\log} s_0, \mu_0)}{-\int_{\Sigma_A^*} \log s_0 \, d\mu_0}.$$

where

$$\underline{var(\dot{\log} s_0, \mu_0)} = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^*} \|\sum_{j=0}^{n-1} \dot{\log} s_0 \circ (\sigma_A^*)^j (w^*)\|^2 d\mu_0$$

is the variance.

The Weil-Petersson metric

The Weil-Petersson metric on $T_{\tau_0}\mathcal{T}_g$ now can be calculated by using the function model as

$$||\dot{\tau_0}||^2_{WP} = \frac{3area(\tau_0)}{4}||\dot{\log s_0}||^2_P$$

Note that $\dot{\tau_0}$ can be represented uniquely by a harmonic Beltrami differential $\mu = \rho^{-2}\overline{\phi}$ where ρ is the hyperbolic metric and ϕ is a holomorphic quadratic differential. The classical way to calculate the Weil-Petersson metric is

$$||\dot{\tau_0}||^2_{WP} = ||\mu||^2_{WP} = \int \rho^2|\mu|^2 = \int \rho^{-2}|\phi|^2.$$

The global graph of the metric entropy function

The metric entropy of σ_A^* for the Gibbs measure μ^* can be calculated as $h_{\mu^*}(\sigma_A^*) = -\int_{\Sigma_A^*} \log s(w^*) d\mu^*$. It defines a continuous (even smooth function)

$$ent(s) = h_{\mu^*}(\sigma_A^*) = \frac{\pi Area(F)}{l_{hyp}(\partial F)} = \frac{\pi^2(4g-4)}{l_{hyp}(\partial F)} : \mathcal{S}_g \to (0, h_0].$$

where

$$h_0 = \frac{\pi^2(4g-4)}{(8g-4)\cosh^{-1}(1+2cos(\frac{\pi}{4g-2}))}.$$

The metric entropy of μ is a Teichmüller equivalence invariant and measures the level of the complexity. The function $ent(s)$ reaches the maximum value h_0 at the base-point $s_0 = \iota(\tau_0)$ and tends to 0 as s goes to the Euclidean boundary $\partial\mathcal{S}_g$.

Characterization problem

$s \in C(\Sigma_A)$

$+$ conditions

$\longleftrightarrow s \in S_g$

For the Teichmüler space, we have that

a) $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$,

b) $\mathcal{T}_g$ is a pseudo-convex domain.

c) $\mathcal{T}_g$ admits a complex manifold structure of $3g-3$,

d) $\mathcal{T}_g$ can be embedded into $\mathbb{C}^{3g-3}$ as a contractible set.

$T_{s_0} S_g$

c) and d) imply that S_g admits a complex manifold structure and contractible. a) says that not every Hölder continuous function on Σ_A^* is a point in $\mathcal{S}_g$.

Problem

Character a function s_τ in $\mathcal{S}_g$.

The end

Thanks!