Ergodic Theory Motivated by Sarnak's Conjecture in Number Theory

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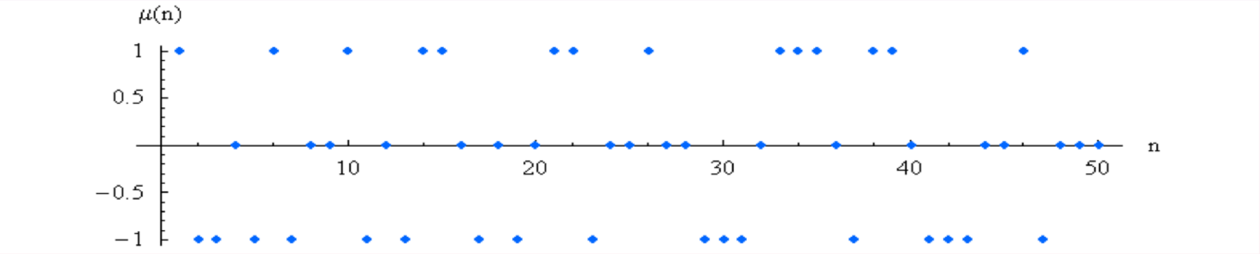
The Möbius Function

The Möbius function is an arithmetic function on the set $\mathbb{N}$ of natural numbers:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r, \text{ a product of distinct prime numbers,} \\ 0 & \text{if } p^2 | n \text{ for some prime number.} \end{cases}$$



Multiplicative

$$\mu(nm) = \mu(n)\mu(m), \quad (n, m) = 1.$$

The Primitive n^{th} Roots of Unity

$$\mu(n) = \sum_{1\leq k\leq n, (k,n)=1} e^{2\pi i \frac{k}{n}}$$

and

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases}$$

Inversion

Suppose $\alpha(n)$ and $\beta(n)$ are two arithmetic functions such that

$$\alpha(n) = \sum_{d|n} \beta(d).$$

$f : X \to X$

$f^{\circ n} : X \to X$

Then

$$\beta(n) = \sum_{d|n} \mu(\frac{n}{d})\alpha(d) = \sum_{d|n} \mu(d)\alpha(\frac{n}{d}).$$

For example, for a flow $\{f^{\circ n} : X \to X\}_{n\in\mathbb{N}}$, let

$$\alpha(n) = \#(\{x \in X \mid f^{\circ n}(x) = x\})$$

$f^{\circ(n-1)}(x)$

and

$$\beta(n) = \#(\{x \in X \mid f^{\circ n}(x) = x, f^{\circ k}(x) \neq x, 1 \leq k \leq n-1\})$$

Riemann Zeta Function

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ $\mathrm{Re}\, s > 1$

be the Riemann zeta function. We have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

The statement (conjecture) that for any $\epsilon > 0$

$$\sum_{n=1}^{N} \mu(n) = O_\epsilon(N^{\frac{1}{2}+\epsilon})$$

is equivalent to the Riemann hypothesis.

Zero Mean

The Möbius function has the zero mean, that is,

$\mathbb{E}\mu = 0$

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) = 0,$$

which is equivalent to the prime number theorem,

$$\pi(x) \sim \frac{x}{\log x} \quad + o(\cdot)$$

where $\pi(x)$ is the number of prime numbers $\leq x$.

$\mu(n)$ in views of dynamics

Consider a compact metric space with the shift on it,

$$\Sigma_3 = \prod_{n=1}^{\infty} \{-1, 0, 1\} = \{v = \cancel{j_1} j_2 \cdots j_n \cdots\}; \sigma_3 : v \to \sigma(v) = j_2 \cdots j_n \cdots .$$

Then $\mu = \mu(1)\mu(2)\cdots\mu(n)\cdots$ is a point in Σ_3 and

$\sigma_3 : \Lambda \Rightarrow \Lambda$

$$\sigma_3 : \Lambda = \overline{\{\sigma_3^{\circ n}(\mu) \mid n \in \mathbb{N}\}} \to \Lambda$$

compact subset

is the Möbius flow.

Question

Is the Möbius flow simple or complicate?

The complexity of the Möbius flow

flow

The complexity of a dynamical system can be measured by a number called the entropy h.

$$\begin{tikzcd} \Sigma_3 \arrow[d, "\iota"] & \Lambda \arrow[d] \\ \Sigma_2 & \iota(\Lambda) \end{tikzcd}$$

Consider another compact metric space and the shift on it,

$$\begin{tikzcd} \Lambda \arrow[r, "\sigma_3"] \arrow[d, "\iota"] & \Lambda \arrow[d, "\iota"] \\ \iota(\Lambda) \arrow[r, "\sigma_2"] & \iota(\Lambda) \end{tikzcd}$$

$$\Sigma_2 = \prod_{n=1}^{\infty}\{0,1\} = \{w = i_1 i_2 \cdots i_n \cdots\}; \sigma_2 : w \to \sigma(w) = i_2 \cdots i_n \cdots .$$

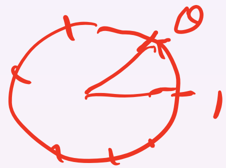
Let $w = \iota(v) = i_1 i_2 \cdots i_n \cdots = j_1^2 j_2^2 \cdots j_n^2 \cdots$. The point $\iota(\mu) \in \Sigma_2$ records all square free natural numbers, which has the density $\frac{6}{\pi^2}$ in $\mathbb{N}$. This implies that the entropy $h(\sigma_2|\iota(\Lambda))$ is $\frac{6}{\pi^2}\log 2$. Thus the entropy $h(\sigma_3|\Lambda)$ is positive since $\sigma_2 : \iota(\Lambda) \to \iota(\Lambda)$ is a factor of $\sigma_3 : \Lambda \to \Lambda$. This says that the Möbius flow is complicate and random.

Oscillation

The Möbius function not only has the zero mean but also is oscillating (Davenport, 1937),

$\mathbb{E}(\mu e^{2\pi i \theta}) = 0$

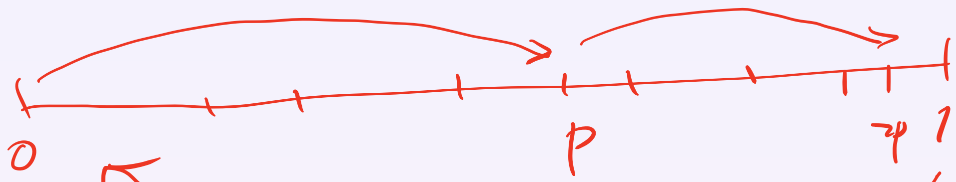
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) e^{2\pi i \theta n} = 0, \quad \forall\, 0 \le \theta < 1.$$



Here $R_\theta(z) = e^{2\pi i \theta} z$ from the unit circle T into itself is the rigid rotation of angle θ. This says that μ and the flow $\{R_\theta^{\circ n}\}_{n\in\mathbb{N}}$ are linearly disjoint, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) R_\theta^{\circ n}(1) = 0, \quad \forall\, 0 \le \theta < 1.$$

Dynamics of Rigid Rotations



1) The flow $\{R_\theta^{\circ n}\}_{n\in\mathbb{N}}$ has zero entropy.
1) The flow $\{R_\theta^{\circ n}\}_{n\in\mathbb{N}}$ is equicontinuous.
2) If $\theta = p/q$ with $(p,q)=1$ is a rational number, then R_θ is periodic, that is, $R_\theta^{\circ q} = Id$.
3) If θ is irrational number, then $\{R_\theta^{\circ n}(z)\}_{n\in\mathbb{N}}$ is dense on T for any z and, moreover,
   i) the flow $\{R_\theta^{\circ n}\}_{n\in\mathbb{N}}$ is uniquely ergodic.
   ii) $\{R_\theta^{\circ n}(z)\}_{n\in\mathbb{N}}$ is equidistributed on T for any z (Weyl, 1910).

$\rightarrow \{n\theta \pmod 1\}$ on $[0,1]$

irrational

Sarnak's conjecture

$f: X \to X$, piece-wise continuous

continuous

$f^{\circ n}: X \to X$

Suppose the flow $\{f^{\circ n}: X \to X\}_{n\in\mathbb{N}}$ on a compact metric space X has zero entropy . Then for any complex-valued continuous function $\phi: X \to \mathbb{C}$ and any $x \in X$, the Möbius function $\mu(n)$ is linearly disjoint with the observation $\{\phi(f^n(x))\}_{n\in\mathbb{N}}$, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\phi(f^n(x)) = 0.$$

Sarnak's conjecture and Equicontinuous Flows

A flow $\{f^{\circ n} : X \to X\}_{n\in\mathbb{N}}$ is equicontinuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$, $d(f^{\circ n}(x), f^{\circ n}(y)) < \epsilon$ for all $n \geq 0$.

$f:$

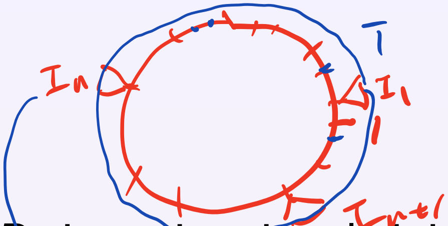
Sarnak's conjecture holds for all equicontinuous flows.

The flow $\{f^{\circ n} : T \to T\}_{n\in\mathbb{N}}$ for an orientation-preserving circle homeomorphism f with an irrational rotation number θ is equicontinuous if and only if f is topologically conjugate to the rigid rotation R_θ, that is, there is a circle homeomorphism $h : T \to T$ such that

$$h \circ f = R_\theta \circ h.$$

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "h"] & T \arrow[d, "h"] \\ T \arrow[r, "R_\theta"] & T \end{tikzcd}$$

Non-Equicontinuous Circle Homeomorphisms



R_θ

$h(f)=0$

Suppose R_θ is an irrational rigid rotation. Consider the orbit $\{R_\theta^{\circ n}(1)\}_{n\in\mathbb{N}}$ and a sequence of pairwise disjoint intervals $\{I_n\}_{n\in\mathbb{N}}$ on T with $\sum_{n=1}^{\infty}|I_n| \leq 1$. Enlarge each point $R_\theta^{\circ n}(1)$ to the interval I_n, we reconstruct the unit circle T and define a circle homeomorphism f such that $f : I_n \to I_{n+1}$ as an increasing linear map mapping endpoints to endpoints and $f = R_\theta$ on $T \setminus (\cup_{n=1}^{\infty} I_n)$. We call these circle homeomorphisms Denjoy counter-examples. Every Denjoy counter-example is non-equicontinuous and has zero entropy.

Our Purpose

We would like to understand the oscillating properties presented in the Möbius function $\mu(n)$ as well as other arithmetic functions and classify all zero entropy flows such that the linear disjointness happens in ergodic theory which can be applied back to number theory.

Log-Uniform Oscillating Sequence

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is a *log-uniform oscillating sequence* if there are two constants $A > 1$ and $B > 0$ such that

$$\sup_{0\le\theta<1}\left|\sum_{n=1}^{N} c_n e^{2\pi i n\theta}\right| \le B\frac{N}{\log^A N}, \quad \forall\, N \ge 2,$$

with the control condition

$$\sum_{n=1}^{N} |c_n|^\lambda = O(N), \quad \text{for some } \lambda > 1. \tag{1}$$

The Möbius function $\mu(n)$ is an example of log-uniform oscillating sequences due to Davenport.

An Ergodic Theorem

Theorem

Suppose $(X, \mathcal{B}, \nu)$ is a Borel probability measurable space and $f : X \to X$ is an automorphism. Suppose $\mathbf{c} = (c_n)$ is a log-uniform oscillating sequence*. Then for any $\phi \in L^1(X, \mathcal{B}, \nu)$, we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n(x)) = 0, \ \nu\text{-a.e. } x \in X.$$

improve to every point

See Sarnak, Lecture Notes; J, Nonlinearity

Definition

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is an *oscillating sequence* if for any $0 \le \theta < 1$,

$\mathbb{E}(\vec{c}\, e^{2\pi i\theta}) = 0$

$$\lim_{N\to 0} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i n\theta} = 0$$

with the control condition (1).

See Fan-J, ETDS, 2018.

The Möbius function $\mu(n)$ is an example of oscillating sequences due to Davenport.

More Examples and Counterexamples of Oscillating Sequences

$e^{2\pi i(1-\alpha)}$

- The sequence $(e^{2\pi i n\alpha})_{n\in\mathbb{N}}$ for some $0 \le \alpha < 1$ is not an oscillating sequence.
- The sequence $(e^{2\pi i \alpha n \log n})_{n\in\mathbb{N}}$ for any $\alpha > 0$ is an oscillating sequence.
- The sequence $(e^{2\pi i n^2\alpha})_{n\in\mathbb{N}}$ for any rational number α is not an oscillating sequence.
- The sequence $(e^{2\pi i n^2\alpha})_{n\in\mathbb{N}}$ for any irrational number α is an oscillating sequence.

Oscillating Sequence in Arithmetic

Definition

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is an *oscillating sequence* in arithmetic if for any $0 \le \theta < 1$, for any $q \ge 1$ and $1 \le r < q$,

$$\lim_{N\to 0} \frac{1}{N} \sum_{1\le n\le N,\ n=r \pmod q} c_n e^{2\pi i n\theta} = 0$$

with the control condition (1).

An oscillating sequence is an oscillating sequence in arithmetic (see H. Daboussi and H. Delange, J. Lond. Math. Soc., 1982 and Fan-J, ETDS, 2018, Proposition 4).

Linear Disjointness

$x \to fx \to \cdots \to f^n x$

Suppose $f : X \to X$ is a (piece-wise) continuous map from a compact metric space X into itself. For any complex-valued continuous function $\phi : X \to \mathbb{C}$ and any $x \in X$, we call $(\phi(f^{\circ n}x))_{n\in\mathbb{N}}$ an observation.

Definition

We say a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers is linearly disjoint from f if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^{\circ n}x) = 0$$

for all observations $(\phi(f^{\circ n}x))_{n\in\mathbb{N}}$.

Mean Lyapunov Stability

Definition

We say that f is mean-L-stable (briefly, MLS) if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^{\circ n}x, f^{\circ n}y) < \epsilon$ for all $n = 0, 1, 2, \cdots$ except for a subset $E = E_{x,y}$ of natural numbers with $\overline{D}(E) < \epsilon$ Here

$$\overline{D}(E) = \limsup_{n\to\infty} \frac{\sharp(E \cap [1, n])}{n}.$$

S. V. Fomin, Dokl. Akad. Nauk SSSR, 1951.

Minimal Mean Lyapunov Stability

A subset $K \subseteq X$ is said to be minimal if $\overline{\{f^{\circ n}x\}_{n=0}^{\infty}} = K$ for any $x \in K$.

Definition

We say that f is minimal MLS (briefly, MMLS) if for every minimal subset $K \subseteq X$, $f|K$ is MLS.

See Fan-J, ETDS, 2018

Minimal Mean Attractability

X $\forall \varepsilon > 0$

Definition

We say $x \in X$ is mean attracted to K if $\forall\, \epsilon > 0$, $\exists\, z = z_{\epsilon,x} \in K$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d(f^{\circ n}x, f^{\circ n}z) < \epsilon.$$

mean attractor

The basin $\mathrm{B}(K)$ is the set of all points $x \in X$ which are meanly attracted to K. We say that f is minimal mean attractable (briefly, MMA) if $X = \bigcup_K \mathrm{B}(K)$ where K varies among all minimal subsets of X.

See Fan-J, ETDS, 2018

Theorem (Fan-J, ETDS, 2018)

Any oscillating sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ is linearly disjoint from any MMA and MMLS f. More precisely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^{\circ n} x) = 0, \quad \forall \phi \in C(X, \mathbb{C}), \quad \forall\, x \in X.$$

The limit is uniform on each minimal subset.

Since an MMA and MMLS ~~dynamical system~~ flow has zero entropy, our result confirms Sarnak's conjecture for a large class of dynamical systems with zero entropy.

Corollary

Sarnak's conjecture holds for all MMLS and MMA ~~dynamical systems~~ flows.

This corollary generalizes many works from other people on Sarnak's conjecture, for examples, P. Sarnak, Lecture Notes; 2010 ; D. Karagulyan, Ark. Mat.,; 2015, J. Li, P. Oprocha, G. Y. Yang, and T. Zeng, Nonlinearity 2017.

Denjoy Counter-Examples

All Denjoy counter-examples $f : T \to T$ are MMLS and MMA.
And ~~T~~they are not equicontinuous on its minimal subsets.

See Fan-J, ETDS, 2018.

Semi Adding Machines

Σ_2

$v = i_0 i_2 i_3 \cdots$

$+1$

$add(v) =$

$i_0 = 0, \quad i_0 + 1 = 1$

$i_0 = 1$

$i_0 + 1 = 01$

All continuous infinitely renormalizable interval maps with zero topological entropy such that they are only semi-conjugate to the adding machine on their strange attractors are MMLS and MMA. They are not equicontinuous on its minimal subsets.

See J, Nonlinearity, 2018.

A Counterexample for MMLS and MMA

Consider an affine distal map

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

for an irrational number $0 < \alpha < 1$. Then the oscillating sequence $\mathbf{c} = (e^{2\pi i n^2 \alpha})_{n\in\mathbb{N}}$ is not linearly disjoint from f. Thus, f is not MMLS and MMA but has zero entropy.

See Fan-J, ETDS, 2018.

Oscillating Sequence of Higher Order

Definition

We call a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers an oscillating sequence of order $d \geq 2$ if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n e^{2\pi i P(n)} = 0$$

for every real coefficient polynomial P of degree $\leq d$ with the control condition (1).

See J, PAMS, 2019.

The Möbius function $\mu(n)$ is an example of an oscillating sequence of order d for all $d \geq 2$ due to Hua. 14605

Definition

We call $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers an oscillating sequence of order $d \geq 2$ in arithmetic if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N, n=r \pmod q} c_n e^{2\pi i P(n)} = 0,$$

for all real coefficient polynomials P of degree $\leq d$ and every pair of integers $0 \leq r < q$ with the control condition (1).

See J, PAMS, 2019

The Möbius function $\mu(n)$ is also an example oscillating sequence of order d for all $d \geq 2$ in arithmetic due to Hua.

Higher Order Oscillating Sequences Other Than the Möbius Function

Theorem (Akiyama-J, UDT, 2019)

Suppose g is a positive C^2 function on $(1, \infty)$ with non-negative first and second derivatives. For a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real number α), sequences

$$\mathbf{c} = \left(e^{2\pi i \alpha \beta^n g(\beta)}\right)_{n \in \mathbb{N}}$$

$\{\beta^n \pmod 1\}$

are oscillating sequence of order d as well as in arithmetic for all $d \geq 2$.

Note that the sequence $\{\alpha\beta^n g(\beta) \pmod 1\}_{n\in\mathbb{N}}$ has positive entropy.

a.e yes, find a concrete β is interesting

Affine Maps of the d-Torus

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-torus. Let $A \in GL(d, \mathbb{Z})$, the space of all $d \times d-$matrices of integer entries with determinants ± 1.

The map $T_{A,\mathbf{a}}\mathbf{x} = A\mathbf{x} + \mathbf{a} : \mathbb{T}^d \to \mathbb{T}^d$ is an affine map, where $\mathbf{x}$ is a variable and $\mathbf{a}$ is a constant point in $\mathbb{T}^d$.

The map $T_{A,\mathbf{0}} = A\mathbf{x} : \mathbb{T}^d \to \mathbb{T}^d$ is an automorphism of $\mathbb{T}^d$.

Affine Distal Maps on the d-Torus

An affine map

$$T_{A,\mathbf{a}}(\mathbf{x}) = A\mathbf{x} + \overset{x^0}{\mathbf{a}} : \mathbb{T}^d \to \mathbb{T}^d$$

is called distal if all eigenvalues of A are 1.

Linear disjointness for Order d and d-Affine Distal Maps

Theorem (J, PAMS, 2019)

Any oscillating sequence of order $d \geq 2$ is linearly disjoint from any affine distal map $T_{A,\mathbf{a}}$ of the d-torus $\mathbb{T}^d$.

Zero Entropy Affine Maps on the d-Torus

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+\alpha \\ y+h(x) \end{pmatrix}$$ continuous $h(x)$?

In order for $T_{A,\mathbf{a}}$ to have zero entropy, the absolute values $|\lambda_i|$ of all eigenvalues λ_i, $1 \le i \le d$, of A must be ≤ 1 due to Sinai. Moreover, every λ_i must be a root of unity due to Kronecker. This says that $T^k_{A,\mathbf{a}}$ is an affine distal map for some $k \ge 1$.

Corollary (J, PAMS, 2019)

Any oscillating sequence of order $d \ge 2$ in arithmetic is linearly disjoint from any zero entropy affine map $T_{A,\mathbf{a}}$ of the d-torus $\mathbb{T}^d$.

We have also study some non-linear skew products on the d-torus.

Corollary (J, PAMS)

Sarnak's conjecture holds for all zero entropy affine maps of the d-torus for any $d \geq 2$.

There are some other works on Sarnak's conjecture for zero entropy affine and nonlinear maps of the *d*-torus, in particular, the 2-torus and the 3-torus. And there are some work on Sarnak's conjecture for flows f with quasi-discrete spectrum. For examples, Liu-Sarnak, Duke J. Math; Z. Wang, Invent.; Huang-Liu-Wang, arXiv:1907.01735; e. H. el Abdalaoui, arXiv1704.07243.

The Thur-Morse Sequence

The Thur-Morse sequence

0 11 01001 $\overline{01} = 10$

$$m = 0110100110010110\ldots$$

has zero entropy. Let $\mathbf{m} = e^{\pi i m} = (m_n)_{n\in\mathbb{N}}$. Konieczny, 2016, arXiv, shows that $\mathbf{m}$ has a small sequence of Gowers norms, that is, for any $d \geq 1$, there exists $c = c(d) > 0$ such that

$$\|\mathbf{m}\|_{U^d[N]} = O(N^{-c}).$$

Using this result, Abdalaoui, 2017, arXiv, shows that $\mathbf{m}$ is an oscillating sequence of order d for all $d \geq 1$. If we take the Thur-Morse sequence as a zero entropy flow and $\mathbf{m}$ as a higher order oscillating sequence and $\phi = e^{\pi i x_1}$ as a function. Then they are not linearly disjoint, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} m_n m_n = 1.$$

Correlation and linear disjointness

Bourgain, Sarnak, and Ziegler, 2013, shows the following criterion for the linear disjointness by using the decay of correlation: Suppose $F, \nu : \mathbb{N} \to \mathbb{C}$ are two arithmetic functions with $|F|, |\nu| \le 1$ and ν is multiplicative. If for any pair of distinct primers numbers p_1, p_2,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(p_1 n)\overline{F(p_2 n)} = 0,$$

then F and ν are linearly disjoint, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(n)F(n) = 0,$$

Chowla's Conjecture and the Decay of Multi-Correlations

The following is an old conjecture (1965) in number theory, which relates to the Riemann hypothesis.

Conjecture

For each choice of $0 = k_0 < k_1 < \cdots < k_r$, $r > 0$, *and each choice of* $i_0, i_1, \cdots i_r \in \{1, 2\}$, *not all* $\mu^{i_j}(n + k_j) = 1$, *we have the decay of the multi-correlation, that is,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu^{i_1}(n + k_1) \cdot \ldots \cdot \mu^{i_r}(n + k_r) = 0.$$

Definition

A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers is said to be a Chowla sequence if the control condition (1) and if for each choice of $0 \le k_1 < \cdots < k_r$, $r > 0$, and each choice of $i_1, \cdots i_r \in \mathbb{N}$ such that not all $c_{n+k_j}^{i_j} = |c_{n+k_j}|$, the decay of the multi-correlation holds, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} \prod_{j=1}^{r} c_{n+k_j}^{i_j} = 0.$$

By using a similar method as Akiyama-J, UDT, we can construct a Chowla sequence in the form $\mathbf{c} = \left(e^{2\pi i \alpha \beta^n g(\beta)}\right)$. a.e. β

Chowla's Conjecture and Sarnak's Conjecture

Chowla's conjecture implies Sarnak's conjecture. See Sarnak, 2010 and H. El Abdalaoui, J. Kulaga-Przymus, M. Lemznczyk, T. de la Rue, arXiv:1410.1673.

Veech (AJM and London Notes) shows that that there is a unique admissible measure on the Möbius flow (Chowla measure). A recent paper, el H. el Abdalaoui, arXiv:1711.06326, says that, from Veech's work with the help of Tao's logarithmic Theorem on logarithmic Sarnak's conjecture, we have that

Sanark's conjecture implies Chowla's conjecture.

(1) ✓

(2)

5(5)

Thanks!