A Study in Ergodic Theory Motivated by Problems in Number Theory

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Dynamical Systems

Suppose X is a space and $f : X \to X$ is a map. We use f^n to denote the n^{th} iteration of f for any integer $n \geq 1$. We use f^0 to mean the identity map from X into itself. We call the semi-group

$$\{f^n\}_{n=0}^{\infty}$$

a dynamical system.

If f is invertible, then we have the inverse f^{-1}. Then for any negative integer n we have $f^n = (f^{-1})^{-n}$. In this case, we have a group

$$\{f^n\}_{n=-\infty}^{\infty}$$

Orbits

For a point $x \in X$, we use $Orb^{+}(x) = \{f^n(x)\}_{n=0}^{\infty}$ to denote the forward orbit of x.

If f is invertible, we use $Orb(x) = \{f^n(x)\}_{n\in\mathbb{Z}}$ to denote the full orbit of x, where $Orb^{-}(x) = \{f^n(x)\}_{n=-\infty}^{0}$ is called the backward orbit of x.

A fundamental problem in dynamical systems is to understand the "future", that is, the limiting behavior of the forward orbit. If f is invertible, we would like to understand the "history", that is the limiting behavior of the backward orbit.

Ergodic Theory

In ergodic theory, suppose $(X, \mathcal{B}, \nu)$ is a probability measure space and $f : X \to X$ is a measure-preserving map, that is, $\nu(f^{-1}(A)) = \nu(A)$ for any $A \in \mathcal{B}$. We consider the Birkhoff sum

$$B_n(\phi)(x) = \frac{1}{N} \sum_{n=1}^{\infty} \phi(f^n x), \quad n = 1, 2, \ldots,$$

for a measurable function ϕ and $x \in X$.

Birkhoff Ergodic Theorem

Let $\mathcal{C}$ be the σ-algebra of all f-invariant subsets. We say that f is ergodic if for any $A \in \mathcal{B}$ satisfying $f^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A) = 1$. This is equivalent to say that $\mathcal{C} = \{\emptyset, X\}$ is a trivial σ-algebra.

Theorem

If $E(|\phi|) = \int_X |f| d\nu < \infty$, then

$$\lim_{n\to\infty} B_n(\phi)(x) = E(\phi|\mathcal{C}) \quad \textit{for a.e. } x \in X.$$

In particular, if f is ergodic, then

$$\lim_{n\to\infty} B_n(\phi)(x) = \int_X f d\mu \quad \textit{for a.e. } x \in X.$$

Number Theory

In number theory, suppose X is a compact metric space and $f : X \to X$ be a (piecewise) continuous map. We consider the weighted Birkhoff sum and call it the Sarnak sum,

$$S_n(\phi)(x) = \frac{1}{N} \sum_{n=1}^{\infty} \mu(n) \phi(f^n x)$$

for a continuous function ϕ and $x \in X$, where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ for } r \text{ distinct prime numbers } p_i; \\ 0 & \text{if } p^2 | n \text{ for a prime number } p. \end{cases}$$

is the Möbius function.

Sarnak's Conjecture

Let $\mathcal{C}(X)$ be the space of all continuous functions on X.

Conjecture

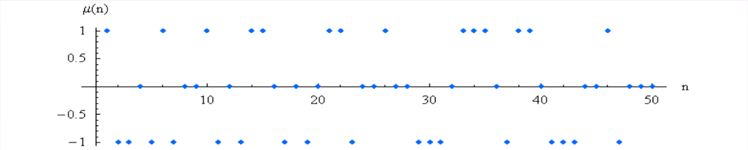
Suppose the topological entropy of f is zero. Then for any $\phi \in \mathcal{C}(X)$,

$$\lim_{n\to\infty} S_n(\phi)(x) = \lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(n)\phi(f^n x) = 0, \quad \forall \ x \in X.$$

The Möbius Function

The Möbius function is an arithmetic function on the set of natural numbers:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r, \text{ a product of distinct prime numbers,} \\ 0 & \text{if } p^2 | n \text{ for some prime number.} \end{cases}$$



Multiplicative

$$\mu(nm) = \mu(n)\mu(m), \quad (n, m) = 1.$$

The Primitive n^{th} Roots of Unity

$$\mu(n) = \sum_{1 \le k \le n, (k,n)=1} e^{2\pi i \frac{k}{n}}$$

and

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases}$$

Dirichlet convolution

Suppose $\phi(n)$ and $\psi(n)$ are two two arithmetic functions. The Dirichlet convolution is defined as

$$(\phi * \psi)(n) = \sum_{d|n} \phi(d)\psi(\frac{n}{d}) = \sum_{ab=n} \phi(a)\psi(b).$$

We have

$$(\phi * \psi) * \alpha = \phi * (\psi * \alpha) \quad (\textit{associative});$$

$$\phi * \psi = \psi * \phi \quad (\textit{commutative});$$

$$\phi * (\psi + \alpha) = \phi * \psi + \phi * \alpha \quad (\textit{distribution});$$

$$\epsilon(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (\textit{identity}),$$

that is, $\phi * \epsilon = \epsilon * \phi = \phi$ for any arithmetic function.

Inverse

For any arithmetic function ϕ with $\phi(1) \neq 0$, it has the inverse ϕ^{-1}, that is, $\phi * \phi^{-1} = \phi^{-1} * \phi = \epsilon$.

The Möbius function μ is the inverse of the arithmetic function $\mathbf{1}(n) = 1$ for all $n \in \mathbb{N}$, that is,

$$(\mu * \mathbf{1})(n) = \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases} = \epsilon(n)$$

Inversion

Suppose $\phi(n)$ and $\psi(n)$ are two arithmetic functions such that

$$\phi(n) = \sum_{d|n} \psi(d).$$

Then

$$\psi(n) = \sum_{d|n} \mu(\frac{n}{d})\phi(d) = \sum_{d|n} \mu(d)\phi(\frac{n}{d}).$$

An Application to Dynamical Systems

For a dynamical system $f : X \to X$, let

$$F_n = \#(\{x \in X \mid f^n(x) = x\})$$

and

$$P_n = \#(\{x \in X \mid f^n(x) = x, f^k(x) \neq x, 1 \leq k \leq n-1\})$$

Then we have

$$F_n = \sum_{d|n} P_n.$$

This implies that

$$P_n = \sum_{d|n} \mu(d) F_{\frac{n}{d}}.$$

Riemann Zeta Function

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

be the Riemann zeta function. We have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

The statement for any $\epsilon > 0$

$$\sum_{n=1}^{N} \mu(n) = O_\epsilon(N^{\frac{1}{2}+\epsilon})$$

is equivalent to the Riemann hypothesis.

Zero Mean

Theorem

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) = 0$$

This theorem is an equivalent statement to the prime number theorem as follows.

Theorem

$$\pi(x) \sim \frac{x}{\log x}$$

where $\pi(x)$ *is the number of prime numbers* $\leq x$.

Oscillation

Theorem (Davenport)

For any $0 \leq \theta < 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) e^{2\pi i \theta n} = 0.$$

In the proof of this theorem, we have

$$\sup_{0\leq\theta<1} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) e^{2\pi i n\theta} \right| \leq B \frac{N}{\log^A N}, \ \ \forall\, N \geq 1.$$

for $A > 1$ and some $B > 0$.

Log-Uniform Oscillating Sequence

Adapted this estimation, we define a more general log-uniform oscillating sequence. Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is a *log-uniform oscillating sequence* if there are two constants $A > 1$ and $B > 0$ such that

$$\sup_{0\leq\theta<1} \Big| \sum_{n=1}^{N} c_n e^{2\pi i n\theta} \Big| \leq B \frac{N}{\log^A N}, \quad \forall\, N \geq 2,$$

with the control condition

$$\sum_{n=1}^{N} |c_n|^\lambda = O(N), \quad \text{for some } \lambda > 1. \tag{1}$$

An Ergodic Theorem

Theorem

Suppose $(X, \mathcal{B}, \nu)$ *is a Borel probability measurable space and* $f : X \to X$ *is an automorphism. Suppose* $\mathbf{c} = (c_n)$ *is a* log-uniform oscillating sequence*. Then for any* $\phi \in L^1(X, \mathcal{B}, \nu)$*, we have that*

$$\lim_{n\to\infty} S_{n,\mathbf{c}}(f)(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n x) = 0, \ \nu\text{-a.e. } x \in X.$$

See Sarnak, Lecture Notes; J, Nonlinearity

Invariant Measures

Suppose X is a compact metric space and $\mathcal{B}$ is the σ-algebra of all Borel subsets in X. Let $M(X)$ be the space of all Borel measures on $(X, \mathcal{B})$ equipped with the weak $*$-topology, that is, $\nu_n \to \nu$ if and only if

$$< \phi, \nu_n >= \int_X \phi d\nu_n \to < \phi, \nu >= \int_X \phi d\nu, \ \ \forall \, \phi \in \mathcal{C}(X).$$

Then $M(X)$ is a compact and metrizable space.
Suppose $f : X \to X$ is a measurable map. The push-forward operator $f_* : M(X) \to M(X)$ is defined as $f_*\nu(A) = \nu(f^{-1}(A))$. A measure ν is called an f-invariant if $f_*\nu = \nu$. Let M_f be the space of all f-invariant probability measures in $M(X)$.

Non-Empty and Convexity

Lemma

The space $M_f(X)$ is non-empty and convex.

Proof.

Take any probability measure $\nu_0 \in M(X)$, let

$$\nu_N = \frac{1}{N}\sum_{n=0}^{N-1} f_*^n \nu_0, \quad N = 0, 1, \ldots.$$

$$f_*\nu_N = f_*(\frac{1}{N}\sum_{n=0}^{N-1} f_*^n \nu_0) = \frac{1}{N}\sum_{n=0}^{N-1} f_*^{n+1}\nu_0 = \frac{1}{N}(f_*^N \nu_0 - \nu_0) + \nu_N$$

For a convergent subsequence $\nu_{N_k} \to \nu$, we have $f_*\nu = \nu$. This says that M_f is non-empty.

Convex: $f_*\nu_t = f_*((1-t)\nu_0 + t\nu_1) = (1-t)f_*\nu_0 + tf_*\nu_1 = (1-t)\nu_0 + t\nu_1 = \nu_t$. □

Unique Ergodicity

A measure $\nu \in M_f(X)$ is ergodic if it is an extremal point. We say f is uniquely ergodic if $M_f(X)$ is a singleton. From the Birkhoff ergodic theorem, we have that

Corollary

Suppose f is uniquely ergodic and $M_f(X) = \{\nu\}$. Then for any continuous function $\phi \in \mathcal{C}(X)$, we have that

$$\lim_{n\to\infty} B_n(\phi)(x) = \lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} \phi(f^n x) = \int_X f d\nu, \quad \forall\, x \in X.$$

Minimal Dynamical Systems

Suppose X is a compact metric space and $f : X \to X$ is a continuous map. We say f is minimal if for any $x \in X$,

$$\overline{\{f^n x\}_{n=0}^{\infty}} = X.$$

A problem

The zero topological entropy, the unique ergodicity, and minimality are three different concepts in dynamical systems and ergodic theory. A dynamical system having one of these properties indicates that it has certain simplicity, but it can still be difficult to study and not fully understood.

Problem

In Sarnak's Conjecture, if we replace the assumption that the topological entropy of f is zero by either that f is uniquely ergodic or that f is minimal, what can we say?

Some References

Here are some research papers of mine with my collaborators in this direction:

- Y. Jiang, Zero entropy continuous interval maps and MMLS-MMA property. Nonlinearity 31 (2018) 2689–2702. https://doi.org/10.1088/1361-6544/aab593.
- A. Fan and Y. Jiang, Oscillating sequences, MMA and MMLS flows and Sarnak's conjecture. Ergod. Th. & Dynam. Sys., August 2018, Vol. 38, no. 5, 1709–1744. https://doi.org/10.1017/etds.2016.121.
- S. Akiyama and Y. Jiang, Higher order oscillation and uniform distribution . Uniform Distribution Theory, Volume 14 (2019), no. 1, 1-10. https://doi.org/10.2478/udt-2019-0001.
- Y. Jiang, Orders of oscillation motivated by Sarnak's conjecture. Proceedings of AMS, Volume 147, Number 7, July 2019, 3075-3085. https://doi.org/10.1090/proc/14487.

Some References

Here are some papers from other people which we would like to read carefully next semester.

- J. Liu and P. Sarnak, The Möbius function and distal flows. Duke Mathematical Journal, Vol. 164 (2015), No. 7, 1353-1399. https://doi.org/10.1215/00127094-2916213.
- H. el Abdalaoui, Oscillating sequences, Gowers norms and Sarnak's conjecture. https://arxiv.org/pdf/1704.07243v3.pdf
- H. el Abdalaoui, J. Ku laga-Przymus, M. Lemańczyk, and T. de la Rue, The Chowla and the Sarnak conjectures from ergodic theory point of view. https://arxiv.org/pdf/1410.1673.pdf
- H. el Abdalaoui, On Veech's proof of Sarnak's theorem on the Möbius flow. https://arxiv.org/pdf/1711.06326v1.pdf

The End

Thanks!

More on our work and work from other people

If there are times left, I would like to talk our work and work from other people recently in this direction in the papers I listed in the previous two slides.

Oscillating Sequence

Definition

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is an *oscillating sequence* if for any $0 \leq \theta < 1$,

$$\lim_{N\to 0} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i n\theta} = 0$$

with the control condition (1).

See Fan-J, ETDS

The Möbius sequence is an example of oscillating sequences due to Davenport's theorem.

More Examples and Counterexamples of Oscillating Sequences

- The sequence $(e^{2\pi i n\alpha})_{n\in\mathbb{N}}$ for some $0 \leq \alpha < 1$ is not an oscillating sequence.
- The sequence $(e^{2\pi i \alpha n \log n})_{n\in\mathbb{N}}$ for any $\alpha > 0$ is an oscillating sequence.
- The sequence $(e^{2\pi i n^2\alpha})_{n\in\mathbb{N}}$ for any rational number α is not an oscillating sequence.
- The sequence $(e^{2\pi i n^2\alpha})_{n\in\mathbb{N}}$ for any irrational number α is an oscillating sequence.

See Fan-J, ETDS and more examples.

Oscillating Sequence in Arithmetic

Definition

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that $\mathbf{c}$ is an *oscillating sequence* in arithmetic if for any $0 \leq \theta < 1$, for any $q \geq 1$ and $1 \leq r < q$,

$$\lim_{N\to 0} \frac{1}{N} \sum_{1\leq n\leq N,\ n=r \pmod q} c_n e^{2\pi i n\theta} = 0$$

with the control condition (1).

An oscillating sequence is an oscillating sequence in arithmetic (see H. Daboussi and H. Delange, J. Lond. Math. Soc. and Fan-J, ETDS, Proposition 4).

Linear Disjointness

Suppose $f : X \to X$ is a (piece-wise) continuous map from a compact metric space X into itself. For any complex-valued continuous function $\phi : X \to \mathbb{C}$ and any $x \in X$, we call $(\phi(f^n x))_{n\in\mathbb{N}}$ an observation.

Definition

We say a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers is linearly disjoint from f if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n x) = 0$$

for all observations $(\phi(f^n x))_{n\in\mathbb{N}}$.

Mean Lyapunov Stability

Definition

We say that f is mean-L-stable (briefly, MLS) if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n x, f^n y) < \epsilon$ for all $n = 0, 1, 2, \cdots$ except for a subset $E = E_{x,y}$ of natural numbers with $\overline{D}(E) < \epsilon$ Here

$$\overline{D}(E) = \limsup_{n\to\infty} \frac{\sharp(E \cap [1, n])}{n}.$$

S. V. Fomin, Dokl. Akad. Nauk SSSR

Minimal Mean Lyapunov Stability

A subset $K \subseteq X$ is said to be minimal if $\overline{\{f^n x\}_{n=0}^{\infty}} = K$ for any $x \in K$.

Definition

We say that f is minimal MLS (briefly, MMLS) if for every minimal subset $K \subseteq X$, $f|K$ is MLS.

See Fan-J, ETDS

Minimal Mean Attractability

Definition

We say $x \in X$ is mean attracted to K if $\forall\, \epsilon > 0$, $\exists\, z = z_{\epsilon,x} \in K$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d(f^n x, f^n z) < \epsilon.$$

The basin $\mathrm{B}(K)$ is the set of all points $x \in X$ which are meanly attracted to K. We say that f is minimal mean attractable (briefly, MMA) if $X = \bigcup_K \mathrm{B}(K)$ where K varies among all minimal subsets of X.

See Fan-J, ETDS

MMLA and MMA and Oscillation

Theorem (Fan-J, ETDS)

Any oscillating sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *is linearly disjoint from any MMA and MMLS dynamical system* f. *More precisely,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n x) = 0, \ \ \forall \phi \in C(X, \mathbb{C}), \ \ \forall\, x \in X.$$

The limit is uniform on each minimal subset.

MMLS and MMA and Sarnak's Conjecture

Since an MMA and MMLS dynamical system has zero topological entropy, our result confirms Sarnak's conjecture for a large class of dynamical systems with zero topological entropy.

Corollary

Sarnak's conjecture holds for all MMLS and MMA dynamical systems.

This corollary generalizes many works from other people on Sarnak's conjecture, for examples, P. Sarnak, Lecture Notes; ; D. Karagulyan, Ark. Mat.,; J. Li, P. Oprocha, G. Y. Yang, and T. Zeng, Nonlinearity

Examples: Equicontinuous Dynamical Systems

Any equicontinuous dynamical system is MMLS and MMA. For examples,

- all p-adic polynomials.
- all p-adic rational maps with good reduction.
- all circle homeomorphisms conjugate to rigidity rotations.
- all automorphisms $A\mathbf{x}$ of the 2-torus with zero topological entropy (i.e., all eigenvalues of A have absolute value 1).
- all affine maps $A\mathbf{x} + \mathbf{b}$ of the d-torus such that A is diagonalizable with zero topological entropy.

See Fan-J, ETDS, and more examples.

Examples: Not Globally Equicontinuous but Minimally Equicontinuous

All $(2, 2, \cdots)$-infinitely renormalizable C^3 folding maps of an interval with negative Schwarzian derivatives are MMLS and MMA. They are not equicontinuous on the whole interval but still equicontinuous when they are restricted on their strange attractors.

See Fan-J, ETDS

Examples: Denjoy Counter-Examples in Circle Homeomorphisms

All Denjoy counter-examples $f : S^1 \to S^1$ of the circle are MMLS and MMA. They are not equicontinuous even when they are restricted on their minimal sets.

See Fan-J, ETDS

Examples: Continuous Infinitely Renormalizable Interval Maps with Zero Topological Entropy

All continuous infinitely renormalizable interval maps with zero topological entropy such that they are only semi-conjugate to the adding machine on their strange attractors are MMLS and MMA. They are not equicontinuous even when they are restricted on their strange attractors.

See J, Nonlinearity

A Counterexample for MMLS and MMA

Consider an affine distal map

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

for an irrational number $0 < \alpha < 1$. Then the oscillating sequence $\mathbf{c} = (e^{2\pi i n^2 \alpha})_{n\in\mathbb{N}}$ is not linearly disjoint from f. Thus, f is not MMLS and MMA but has zero topological entropy.

See Fan-J, ETDS

Oscillating Sequence of Higher Order

Definition

We call a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers an oscillating sequence of order $d \geq 2$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P(n)} = 0$$

for every real coefficient polynomial P of degree $\leq d$ with the control condition (1).

See J, PAMS

The Möbius sequence $(\mu(n))_{n\in\mathbb{N}}$ is an example of an oscillating sequence of order d for all $d \geq 2$ due to Hua's theorem.

Oscillating Sequence of Higher Order in Arithmetic

Definition

We call $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers an oscillating sequence of order $d \geq 2$ in arithmetic if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N, n=r \pmod q} c_n e^{2\pi i P(n)} = 0,$$

for all real coefficient polynomials P of degree $\leq d$ and every pair of integers $0 \leq r < q$ with the control condition (1).

See J, PAMS

The Möbius sequence $(\mu(n))_{n\in\mathbb{N}}$ is also an example oscillating sequence of order d for all $d \geq 2$ in arithmetic due to Hua's theorem.

Higher Order Oscillating Sequences Other Than the Möbius Sequence

Theorem (Akiyama-J, UDT)

Suppose g is a positive C^2 function on $(1, \infty)$ with non-negative first and second derivatives. For a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real number α), sequences

$$\mathbf{c} = \left(e^{2\pi i \alpha \beta^n g(\beta)}\right)_{n\in\mathbb{N}}$$

are oscillating sequence of order d as well as in arithmetic for all $d \geq 2$.

Affine Maps of the d-Torus

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-torus. Let $A \in GL(d, \mathbb{Z})$, the space of all $d \times d-$matrices of integer entries with determinants ± 1.

The map $T_{A,\mathbf{a}}\mathbf{x} = A\mathbf{x} + \mathbf{a} : \mathbb{T}^d \to \mathbb{T}^d$ is an affine map, where $\mathbf{x}$ is a variable and $\mathbf{a}$ is a constant point in $\mathbb{T}^d$.

The map $T_{A,\mathbf{0}} = A\mathbf{x} : \mathbb{T}^d \to \mathbb{T}^d$ is an automorphism of $\mathbb{T}^d$.

Affine Distal Maps on the d-Torus

An affine map

$$T_{A,\mathbf{a}}(\mathbf{x}) = A\mathbf{x} + \mathbf{a} : \mathbb{T}^d \to \mathbb{T}^d$$

is called distal if all eigenvalues of A are 1.

Theorem (J, PAMS)

Any oscillating sequence of order $d \geq 2$ is linearly disjoint from any affine distal map $T_{A,\mathbf{a}}$ of the d-torus $\mathbb{T}^d$.

Linear disjointness for Order d and Zero Topological Entropy d-Affine Maps

In order for $T_{A,\mathbf{a}}$ to have zero topological entropy, the absolute values $|\lambda_i|$ of all eigenvalues λ_i, $1 \leq i \leq d$, of A must be ≤ 1 due to Sinai's theorem. Moreover, every λ_i must be a root of unity due to Kronecker's Lemma. This says that $T_{A,\mathbf{a}}^k$ is an affine distal map for some $k \geq 1$.

Corollary (J, PAMS)

Any oscillating sequence of order $d \geq 2$ in arithmetic is linearly disjoint from any zero topological entropy affine map $T_{A,\mathbf{a}}$ of the d-torus $\mathbb{T}^d$.

Applying the KAM type theorem, the corollary can be generalized to some zero topological entropy nonlinear maps on the d-torus.

Zero Topological Entropy Torus Maps and Sarnak's Conjecture

Corollary (J, PAMS)

Sarnak's conjecture holds for all zero topological entropy affine maps of the d-torus for any $d \geq 2$.

There are some other works on Sarnak's conjecture for zero topological entropy affine and nonlinear maps of the d-torus, in particular, the 2-torus and the 3-torus, for examples, Liu-Sarnak, Duke J. Math; Z. Wang, Invent.; Huang-Liu-Wang, arXiv:1907.01735; e. H. el Abdalaoui, arXiv1704.07243.

Chowla sequences

There is an oscillation sequence of higher order such that it is not linearly disjoint with some class of zero topological entropy dynamical systems. See e. H. el Abdalaoui, arXiv1704.07243.

Definition

A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of complex numbers is said to be a Chowla sequence if for each choice of $0 \leq k_1 < \cdots < k_r$, $r > 0$, and each choice of $i_1, \cdots i_r \in \mathbb{N}$ such that not all $c_{n+k_j}^{i_j} = |c_{n+k_j}|$, the multi-correlation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} \prod_{j=1}^{r} c_{n+k_j}^{i_j} = 0,$$

with the control condition (1).

By using a similar method as Akiyama-J, UDT, we can construct a Chowla sequence in the form $\mathbf{c} = \left(e^{2\pi i\alpha\beta^n g(\beta)}\right)$.

Conjecture

The Möbius sequence $(\mu(n))_{n\in\mathbb{N}}$ *is a Chowla sequence, that is, for each choice of* $0 = k_0 < k_1 < \cdots < k_r$, $r > 0$, *and each choice of* $i_0, i_1, \cdots i_r \in \{1, 2\}$, *not all* $\mu^{i_j}(n + k_j) = 1$, *the multi-correlation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu^{i_1}(n + k_1) \cdot \ldots \cdot \mu^{i_r}(n + k_r) = 0.$$

Chowla's Conjecture and Sarnak's Conjecture

Chowla's conjecture implies Sanark's conjecture. See Sanark, 2010 and H. El Abdalaoui, J. Kulaga-Przymus, M. Lemznczyk, T. de la Rue, arXiv:1410.1673.

Veech's work (AJM and London Notes) on Sarnak's conjecture says that there is a unique admissible measure on the Möbius flow (Chowla measure). A recent paper, el H. el Abdalaoui, arXiv:1711.06326, says that, from Veech's work with the help of Tao's logarithmic Theorem on logarithmic Sarnak's conjecture, we have that

Sanark's conjecture implies Chowla's conjecture.

Thanks!