Global Graphs of the Metric Entropy of SRB Measures

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Topological Entropy

Suppose X is a compact Hausdorff topological space and $f : X \to X$ is a continuous map. Suppose $\mathcal{C}$ is a cover of X by open sets. Then there is a smallest number $\#(\mathcal{C})$ of elements of $\mathcal{C}$ that cover X. Let $H(\mathcal{C}) = \log \#(\mathcal{C})$ and

$$H(f, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{C} \vee f^{-1}\mathcal{C} \vee \cdots \vee f^{-n+1}\mathcal{C}).$$

The topological entropy of f is by definition

$$h(f) = \sup_{\mathcal{C}} H(f, \mathcal{C}).$$

The topological entropy $h(f)$ is a topological conjugacy invariant.

Metric Entropy

Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $f : X \to X$ is a measure-preserving map, that is, $\mu(f^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{B}$. Suppose $\mathcal{P} = \{P_1, \cdots, P_m\}$ is a measurable partition of X. Let

$$H_\mu(\mathcal{P}) = -\sum_{k=1}^{m} \mu(P_k) \log \mu(P_k)$$

and

$$h_\mu(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{-n+1}\mathcal{P}).$$

The metric entropy or the Kolmogorov-Sinai entropy or the measure-theoretic entropy of μ for f is by definition

$$h_\mu(f) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P}).$$

It is know that when X is a compact Hausdorff topological space

$$h(f) = \sup_{\mu} h_\mu(f).$$

The Metric Entropy of SRB Measure

Suppose M is a (compact) C^r Riemannian manifold for $2 \leq r \leq \infty$, $dimM \geq 1$, and $f : M \to M$ is a C^r dynamical system with an (irreducible) hyperbolic attractor Λ, in particular, an Anosov dynamical system. Suppose μ_f is the SRB measure of f (or called the physical measure for f) on Λ. The entropy $h_{\mu_f}(f)$ is called the metric entropy or the Kolmogorov-Sinai entropy of SRB measure.

The metric entropy $h_{\mu_f}(f)$ is a smooth conjugacy invariant.

Topological Conjugacy Class and Topological Entropy $h(f)$

Suppose $\mathcal{A}^r(M)$ is a connected component of the space of all topological conjugacy classes of C^r dynamical systems with hyperbolic attractors. Then

$$h(f) : \mathcal{A}^r(M) \to (0, \infty).$$

is a constant function.

Smooth Conjugacy Class and Metric Entropy $h_{\mu_f}(f)$

The metric entropy $h_{\mu_f}(f) : \mathcal{A}^r(M) \to (0, \infty)$ is not a constant function but it is constant on every smooth conjugacy class. Thus the metric entropy $h_{\mu_f}(f)$ defines a continuous (even smooth) function on the space of all smooth conjugacy classes

$$\mathcal{SA}^r(M) = \{\tau = [f] \mid f \in \mathcal{A}^r(M)\}$$

and then we have a continuous function

$$f \in \tau \in \mathcal{SA}^r(M) \longrightarrow h_{srb}(\tau) = h_{\mu_f}(f) \in (0, \infty).$$

Topological Entropy $h(f)$ vs Metric Entropy $h_{\mu_f}(f)$

- The topological entropy $h(f)$ measures complexity.
- The metric entropy $h_{\mu_f}(f)$ measures levels of complexity.

Question

Which value can be taken by $h_{srb}(f)$ on $\mathcal{SA}^r(M)$? Equivalently, what is the infimium of h_{srb} on $\mathcal{SA}^r(M)$?

Furthermore, since the metric entropy $h_{\mu_f}(f)$ is the sum of all positive Lyapunov exponents of f, we have

Question

Which values can be taken by Lyapunov exponents on $\mathcal{SA}^r(M)$?

Infimum of the Metric Entropy $h_{\mu_f}(f)$

Suppose the topological transitivity in the rest of the talk.

Theorem (Hu-Jiang-J 2008 DCDS)

For any $f_0 \in \mathcal{A}^r(M)$, *we can find a* C^r *path*

$$P = \{f_t \in \mathcal{A}^r(M) \mid 0 \leq t < \infty\}$$

such that $h_{\mu_{f_t}}(f_t) \to 0$ *as* $t \to \infty$. *Thus the infimum of* $h_{srb}(\tau)$ *on* $S\mathcal{A}^r(M)$ *is* 0 *and* $h_{srb}(\tau)$ *takes any value in* $(0, h_{\mu_{f_0}}(f_0)]$.

The Limiting Map on the Boundary of $\mathcal{A}^r(M)$

Corollary (Hu-Jiang-J 2008 DCDS)

Any $f_0 \in \mathcal{A}^r(M)$ is C^r homotopic to an almost hyperbolic dynamical system $f_\infty = \lim_{t\to\infty} f_t \in \partial\mathcal{A}^r(M)$. And on its almost hyperbolic attractor Λ_{f_∞}, f_∞ admits an infinite and σ-finite SRB measure.

One-Dimensional Volume-Preserving Case

Consider $M = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let m be the Lebesgue probability measure on $\mathbb{T}$. Let $E^r(\mathbb{T})$ be the space of all C^r, $2 \leq r \leq \infty$, orientation-preserving circle expanding endomorphisms of degree $d \geq 2$.

Theorem (J 2008 arXiv, Hu-Jiang-J 2017 DCDS, J 2020 Sci. China Math.)

There is a C^1-path $P = \{f_t\}_{0 \leq t < \infty}$ in $E^r(\mathbb{T})$ such that each map f_t preserves the Lebesgue measure m and $h_m(f_0) = \log d$ (the toplogical entropy) and $\lim_{t\to\infty} h_m(f_t) = 0$. Therefore, the infimum of $h_{srb}(\tau)$ on $SE^r(\mathbb{T})$ is 0 and $h_{srb}(\tau)$ takes any value in $(0, \log d]$. Moreover, $f_t(z) \to z$ as $t \to 0^+$ for $z \neq 1 \in \mathbb{T}$.

Higher-Dimensional Volume-Preserving Case

Theorem (Hu-Jiang-J 2017 DCDS)

Suppose $\mathcal{A}^r_{vol}(M)$, $dim M \geq 2$, is the space of all volume-preserving Anosov C^r diffeomorphisms (or all volume-preserving C^r Anosov endomorphism (i.e., expanding maps)). Let m be the volume form on M. For any $f_0 \in \mathcal{A}^r_{vol}(M)$, we can find a C^r path

$$P = \{f_t \in \mathcal{A}^r_{vol}(M) \mid 0 \leq t < \infty\}$$

such that $h_m(f_t) \to 0$ as $t \to \infty$. Thus the infimum of $h_{srb}(\tau)$ on $\mathcal{S}\mathcal{A}^r_{vol}(M)$ is 0 and $h_{srb}(\tau)$ takes any value in $(0, h_m(f_0))$.

In this theorem, $\lim_{t\to\infty} f_t$ is not very clear in general.

Boundaries of $\mathcal{A}^r(M)$ and $\mathcal{SA}^r(M)$

Question

How can we understand the boundary $\partial\mathcal{A}^r(M)$ (or the boundary $\partial\mathcal{A}^r_{vol}(M)$) and the boundary $\partial\mathcal{SA}^r(M)$ (or the boundary $\partial\mathcal{SA}^r_{vol}(M)$)?

Notice that to understand the boundary, we should first have a metric on the space.

Question

Which value in the interval $[0, h]$ can be taken by the limit of $h_{\mu_f}(f)$ as f approaches to the boundary $\partial\mathcal{A}^r(M)$ (or $\partial\mathcal{A}^r_{vol}(M)$)? And which value in the interval $[0, h]$ can be taken by the limit of $h_{srb}(\tau)$ as τ approaches to the boundary $\partial\mathcal{SA}^r(M)$ (or $\partial\mathcal{SA}^r_{vol}(M)$)?

Blaschke Products

Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. Consider a Blaschke product of degree $d \geq 2$

$$B(z) = e^{2\pi i\alpha} \prod_{n=1}^{d} \frac{z - a_n}{1 - \overline{a}_n z}, \quad 0 \leq |a_n| < 1, \;\; n = 1, \cdots, d, \; 0 \leq \alpha < 1.$$

Assume $B(1) = 1$.

Expanding Blaschke Products

Recall that $\mathbb{T} = \partial\Delta$ is the circle. A Blaschke product B is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$|(B^n)'(z)| \geq C\lambda^n, \quad \forall z \in \mathbb{T} \quad \text{and} \quad n \geq 0.$$

The Denjoy-Wolff Theorem

Theorem

Suppose $f : \Delta \to \Delta$ is an analytic map and is not an elliptic Möbius transformation. Then either

(1) *f has a fixed point $p \in \Delta$ such that $f^n(z) \to p$ as $n \to \infty$ for any $z \in \Delta$ and f maps the closure of any hyperbolic disk centered at p into this disk; or*

(2) *there is a point $p \in \partial\Delta$ such that $f^n(z) \to p$ as $n \to \infty$ for any $z \in \Delta$ and f maps any horodisk tangent at p into itself.*

Corollary

A Blaschke product B is expanding if and only if it has a unique fixed point $p_B \in \Delta$.

SRB Measures for Expanding Blaschke Products

Recall that m is the Lebesgue probability measure on $\mathbb{T}$.

Proposition

An expanding Blaschke product preserves the Lebesgue measure m if and only if it fixes 0.

Corollary

Every expanding Blaschke product B has the SRB-measure $\mu_B = (M_B)_* m$ where

$$M_B(z) = \frac{\frac{1-p_B}{1-\overline{p}_B} z + p_B}{1 + \frac{1-p_B}{1-\overline{p}_B}\overline{p}_B z} \quad \textit{and} \quad M_B^{-1}(z) = \frac{1-\overline{p}_B}{1-p_B}\frac{z - p_B}{1-\overline{p}_B z}.$$

The Space of Smooth Conjugacy Classes

Let $\mathcal{B}(d)$ be the space of all expanding Blaschke products of degree $d \geq 2$. Then for any two maps in $\mathcal{B}(d)$ we have a a unique orientation-preserving homeomorphism $h = h_{B_1 B_2}$ fixing 1 such that

$$B_1 \circ h = h \circ B_2.$$

We say B_1 and B_2 are smoothly conjugate if h is a diffeomorphism (this assumption implies that h is a Möbius transformation). Let $S\mathcal{B}(d)$ denote the space of all smooth conjugacy classes $\tau = [B]$. We design a basepoint point $\tau_0 = [q_d]$ for $q_0(z) = z^d$.

A Metric on $S\mathcal{B}(d)$

For any $\tau = [B] \in S\mathcal{B}(d)$, let

$$B_\tau = M_B^{-1} \circ B \circ M_B = a_\tau z \prod_{n=1}^{d-1} \frac{z - c_{n,\tau}}{1 - \overline{c}_{n,\tau} z}$$

be a representation in τ preserving the Lebesgue measure m. Let $\mathcal{G} = \{e^{2\pi \frac{k}{d-1} i}\}_{0 \le k < d-1}$ be the cyclic group of order $d-1$ and let $\mathcal{P}$ be the permutation group of $\{1, \ldots, d-1\}$. Every smooth conjugacy class can be represented by

$$[\mathbf{c}] = [(c_{1,\tau}, \cdots, c_{d-1,\tau})] \in \mathcal{N}^{d-1} = \Delta^{d-1}/(\mathcal{G} \cup \mathcal{P}).$$

Thus we can consider the space of smooth conjugacy classes as $\mathcal{N}^{d-1}$ equipped with the metric $d(\tau, \tau') = \sum_{n=1}^{d-1} |c_{n,\tau} - c_{n,\tau'}|$. The origin is $\mathbf{0} = [0, \ldots, 0]$. When $d = 2$, it is the unit disk Δ and when $d > 2$, it is an orbifold.

A Smooth Path

Let $a_t = a_\infty - \frac{1}{t+\sqrt{3}}$ for $0 \leq t < \infty$. Then

$$P_1 = \left\{ B_t(z) = \frac{z - a_t i}{1 + a_t iz} \cdot \frac{z + a_t i}{1 - a_t iz} \mid 0 \leq t < \infty \right\}$$

is a smooth path in $\mathcal{B}(d)$. Each $B_t(z)$ is an expanding Blaschke product with a unique fixed point $0 \leq p_t < 1$ and the SRB measure μ_{B_t}. It is clearly that $h_{\mu_{B_0}}(B_0) = \log 2$, the topological entropy. Moreover, we have that

Theorem (J 2021 DCDS)

The metric entropy $h_{\mu_{B_t}}(B_t)$ depends on t real analytically and tends to 0 as $t \to \infty$. Thus the infimum of $h_{srb}(\tau)$ on $S\mathcal{B}(d)$ is 0 and $h_{srb}(\tau)$ takes every value in $(0, \log 2]$.

Limiting Map And Limiting Measure

Moreover, $\lim_{t\to\infty} B_t = B_\infty$ and $\lim_{t\to\infty} p_t = 1$ and $\lim_{t\to\infty} \frac{4\pi}{1-p_t^2}\mu_t = \mu_\infty$ all exist, where

$$B_\infty = \frac{z - \frac{i}{\sqrt{3}}}{1 + \frac{iz}{\sqrt{3}}} \cdot \frac{z + \frac{i}{\sqrt{3}} i}{1 - \frac{iz}{\sqrt{3}}}$$

and μ_∞ is an infinite and σ-finite measure on $\mathbb{T}$.

The map B_∞ has a parabolic fixed at 1, that is, $B_\infty(1) = 1$ and $B'_\infty(1) = 1$, and $|B'_\infty(z)| > 1$ for all $z \neq 1 \in \mathbb{T}$. Thus B_∞ is an almost expanding Blaschke product admitting a unique infinite and σ-finite ergodic SRB measure μ_∞ on $\mathbb{T}$.

The Boole Formula

Consider $N(x) = \frac{x-i}{x+i}$ from the real line $\mathbb{R}$ to the circle $\mathbb{T}$. Then

$$F(x) = N^{-1} \circ B_\infty \circ N(x) = x - \frac{1}{x}$$

is the Boole function. Since $(N^{-1})_* \mu_\infty = dx$ is the Lebesgue measure on $\mathbb{R}$, F preserves dx, that is,

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx$$

for any integrable real function f. (It is the Boole formula discovered in 1857 by Boole). Since B_∞ is ergodic wrt μ_∞, F is ergodic wrt dx (a result proved by Adler and Weiss in 1970).

A Smooth Volume-Preserving Path

The smooth path P_1 induces another smooth path in $\mathcal{B}(2)$

$$P_2 = \{\widehat{B}_t(z) = M_{B_t}^{-1} \circ B_t \circ M_{B_t}(z) = z\frac{z - c_t}{1 - c_t z} \mid 0 \leq t < \infty\}.$$

Each map $\widehat{B}_t$ preserves the Lebesgue measure m on $\mathbb{T}$.

Theorem (J 2021 DCDS)

The path P_2 is a smooth path of volume-preserving expanding Blaschke products and for every $z \in \mathbb{T} \setminus \{1\}$,

$$\widehat{B}_t(z) \to z, \;\; t \to \infty$$

Moreover, the metric entropy $h_m(\widehat{B}_t)$ tends to 0 as $t \to \infty$.

The Metric Entropy h_{srb}

The smooth paths P_1 and P_2 project to the space $S\mathcal{B}(2)$ of smooth conjugacy classes as a same smooth path. Furthermore, we can see the global graph of the function $h_{srb}(\tau) : \Delta = S\mathcal{B}(2) \to (0, \infty)$.

Theorem (J 2021 DCDS)

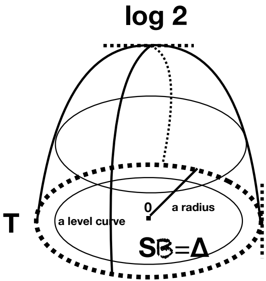
The metric entropy h_{srb} is a smooth (actually, real analytic) function on Δ with the level curves

$$\mathbb{T}_r = \{c \in \Delta \mid 0 \leq |c| = r < 1\}.$$

Moreover, it takes the maximum value $\log 2$ at its unique critical point 0, that is, $h'_{srb}(0) = 0$ and $h_{srb}(0) = \log 2$ is the global maximum value of h_{srb} on Δ. Moreover, $h_{srb}(c) \to 0$ and $h'_{srb}(c) \to -\infty$ as $c \to \mathbb{T} = \partial S\mathcal{B}(2)$. The global graph of h_{srb} on Δ looks like a bell.

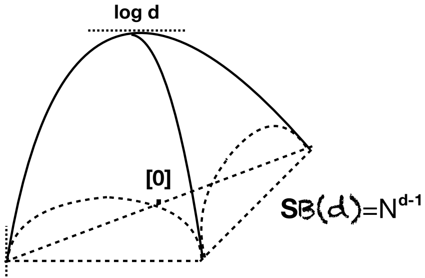
A Global Graph of h_{srb}

For $d = 2$, the global graph looks like



Higher Degree Case

For $d > 2$, the space of smooth conjugacy classes $\mathcal{N}^{d-1}$ is an orbifold. The graph of $h_{srb} : \mathcal{N}^{d-1} \to \mathbb{R}^+$ looks like



Application to the Mandelbrot Set

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Consider the family of quadratic polynomials $q_c(z) = z^2 + c$, $z, c \in \mathbb{C}$. It has only one finite critical point 0. The Mandelbrot set is

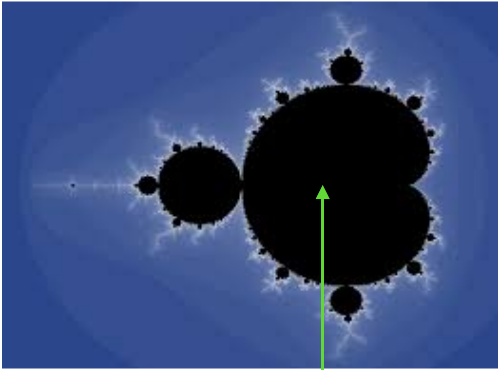
$$\mathcal{M} = \{c \in \mathbb{C} \mid q_c^n(0) \not\to \infty \text{ as } n \to \infty\},$$

which is a compact and connected subset in $\mathbb{C}$. The main cardioid $\mathcal{M}_0$ is, by definition,

$$\mathcal{M}_0 = \{c \in \mathcal{M} \mid q_c^n(0) \to \text{the attractive fixed point of } q_c \text{ as } n \to \infty\},$$

which is a Jordan domain.

The Mandelbrot Set and the Main Cardioid



The Main Cardioid

Let $c \in M_0$.Then q_c has a unique attracting fixed point p_c in $\mathbb{C}$, that is, $q_c(p_c) = p_c$ and the multiplier $\lambda_c = q_c'(p_c)$ has the modulus less than 1, that is

$$|\lambda_c| < 1.$$

The Fatou set (or stable set) is

$$F_c = \{z \in \widehat{\mathbb{C}} \mid q_c^n(z) \to p_c \text{ or } \infty, \text{ as } n \to \infty\}$$

The Julia set (chaotic set) is the complement $J_c = \widehat{\mathbb{C}} \setminus F_c$, which is a Jordan circle (actually, a quasi-circle).

www.youtube.com/watch?v=9lZiZRST8KY

www.youtube.com/watch?v=wbFVAjudzO8

For $c \in M_0$, $q_c|J_c : J_c \to J_c$ is a Markov expanding dynamical system. It has a unique Gibbs measure μ_c on J_c. Let $h_{\mu_c}(q_c|J_c)$ be the metric entropy of its Gibbs measure μ_c. Then we have a function

$$\mathcal{E}_q(c) = h_{\mu_c}(q_c|J_c) : \mathcal{M}_0 \to (0, \log 2].$$

The Global Graph of $h_{\mu_c}(q_c|J_c)$ on the Main Cardioid

Corollary

The function $\mathcal{E}_q(c)$ is a real analytic function with level curves $M_r = \{c \in \mathcal{M}_0 \mid |\lambda(c)| = r\}$ for $0 \leq r < 1$. It is a strictly decreasing function (with respect to the level curves) and takes the maximum value $\log 2$ at its unique critical point 0, that is, $\mathcal{E}_q'(0) = 0$ and $\mathcal{E}_q(0) = \log 2$ is the global maximum value of $\mathcal{E}_q$ on $\mathcal{M}_0$. Moreover,

$$\mathcal{E}_q(c) \to 0 \quad \textit{as} \quad c \to \partial\mathcal{M}_0$$

and along gradients (curves perpendicular to the level curves),

$$\mathcal{E}_q'(c) \to -\infty \quad \textit{as} \quad c \to \partial\mathcal{M}_0.$$

The global graph of $\mathcal{E}_q$ looks like a distorted bell.

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Thanks!