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Kobayashi's and Teichmüller's Metrics and Bers Complex Manifold Structure on Circle Diffeomorphisms

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Abstract Given a modulus of continuity ω, we consider the Teichmüller space $\mathcal{TC}^{1+\omega}$ as the space of all orientation-preserving circle diffeomorphisms whose derivatives are ω-continuous functions modulo the space of Möbius transformations preserving the unit disk. We study several distortion properties for diffeomorphisms and quasisymmetric homeomorphisms. Using these distortion properties, we give the Bers complex manifold structure on the Teichmüller space $\mathcal{TC}^{1+H}$ as the union of $\mathcal{TC}^{1+\alpha}$ over all $0 < \alpha \leq 1$, which turns out to be the largest space in the Teichmüller space of C^1 orientation-preserving circle diffeomorphisms on which we can assign such a structure. Furthermore, we prove that with the Bers complex manifold structure on $\mathcal{TC}^{1+H}$, Kobayashi's metric and Teichmüller's metric coincide.

Keywords Bers complex manifold structure, circle diffeomorphism, modulus of continuity, quasisymmetric circle homeomorphism, Teichmüller space, Kobayashi's metric, Teichmüller's metric

MR(2010) Subject Classification 37F99, 32H02

1 Introduction

In 1960s, Bers [4] gave a complex manifold structure on Teichmüller spaces of Riemann Surfaces by using Bers' embedding, which embeds a Teichmüller space into a bounded domain in the space of quadratic differentials on a disk. Here a Teichmüller space can be thought as a quotient space of the space of Beltrami coefficients on a fixed Riemann surface. Then there is a natural Teichmüller metric induced from the space of Beltrami coefficients. For the Bers complex manifold structure on a Teichmüller space, there is Kobayashi's metric. Royden and Gardiner proved in [8, 25] that these two metrics coincide. In particular, this holds for the universal Teichmüller space which is the quotient space of the space of all quasisymmetric circle homeomorphisms modulo the space of all Möbius transformations preserving the unit disk. Using the Beurling–Ahlfors extension [2], one can show that the universal Teichmüller space is a quotient space of the space of all Beltrami coefficients on the unit disk. Bers' embedding

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embeds the universal Teichmüller space into a bounded domain in the space of all holomorphic functions on the outside of the unit disk. Therefore, for the Bers complex manifold structure on the universal Teichmüller space, Kobayashi's metric and Teichmüller's mertic coincide. Later, Gardiner and Sullivan [13] showed that one can give the Bers complex manifold on the universal asymptotically conformal Teichmüller space. Note that the universal asymptotically conformal Teichmüller space is the quotient space of the space of all symmetric circle homeomorphisms modulo the space of all Möbius transformations preserving the unit disk. Still using the Beurling–Ahlfors extension, one can show that it is a quotient space of the space of all Beltrami coefficients on the unit disk asymptotically to zero on the boundary of the unit disk. Then it has been showed in [6, 14] that for the Bers complex manifold structure on the universal asymptotically conformal Teichmüller space, Kobayashi's metric and Teichmüller's mertic coincide. In [22], Nag showed that the C^∞ Teichmüller space which is the quotient space of the space of all C^∞ orientation-preserving circle diffeomorphisms modulo the space of all Möbius transformations preserving the unit disk can be given the Bers complex manifold structure. It becomes a natural question for a long time (refer to [22, 23]) that could we give the Bers complex manifold structure on the C^1 Teichmüller space which is the quotient space of all C^1 orientation-preserving circle diffeomorphisms modulo the space of all Möbius transformations preserving the unit disk? This question is important in our study of a complex manifold structure on the space of geometric Gibbs theory in [16] and originally it was written as part of that paper. Due to its own interest in Teichmüller theory, we write it as a separate sequel paper in our research in the geometric Gibbs theory. In this paper, we study this question. More precise, we study the $C^{1+\omega}$ Teichmüller space which is the quotient space of the space of all $C^{1+\omega}$ orientation-preserving circle diffeomorphisms modulo the space of all Möbius transformations preserving the unit disk, where ω is a given modulus of continuity. We study several distortion properties for diffeomorphisms and quasisymmetric homeomorphisms in §2. For example, there is the sharpest estimation of quasisymmetric distortion (Lemma 2.1) in §2. In §3, we define several Teichmüller spaces and introduce Bers' embedding. By using these distortion properties, in §4, we give the Bers complex manifold structure on $\mathcal{C}^{1+H} = \bigcup_{0<\alpha\leq 1} \mathcal{C}^{1+\alpha}$ (Theorem 4.2). Moreover, in §5, we prove that with the Bers complex manifold structure on $\mathcal{C}^{1+H}$, Kobayashi's metric and Teichmüller's mertic coincide (Theorem 5.1).

2 Quasisymmetric Distortion and Differentiability

We first discuss in this section a very general quasisymmetric distortion result with the sharpest estimation and several results as applications such as differentiability, asymptotical conformability, existence of dual derivative, and symmetric rigidity.

Suppose $h : [0, 1] \to [0, 1]$ is a homeomorphism with $h(0) = 0$ and $h(1) = 1$. An increasing function $\epsilon(t) \geq 0$, $t > 0$, is called a quasisymmetric distortion function for h if

$$\mathrm{e}^{-\varepsilon(t)} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq \mathrm{e}^{\varepsilon(t)} \tag{2.1}$$

for all $x \in [0, 1]$ and all $t > 0$ with $x + t, x - t \in [0, 1]$. When $\varepsilon(t)$ is a bounded function, let $M = \sup_{t>0} \varepsilon(t)$, then we call h an M-quasisymmertric homeomorphism. In addition, if $\varepsilon(t) \to 0^+$ as $t \to 0^+$, then we call h a symmetric homeomorphism. It is not hard to check that if $\varepsilon(t) \equiv 0$, then $h(x) \equiv x$, the identity.

If h is a C^1-diffeomorphism, the derivative $h'(x) > 0$ is a continuous function on $[0, 1]$. As the common understanding, $h'(0)$ and $h'(1)$ mean one-side derivatives. Suppose $\omega(t) \geq 0$, $t \geq 0$, is an increasing bounded continuous function with $\omega(0) = 0$. We call such a function a modulus of continuity. We say $h'(x)$ is an ω-continuous function if there is a constant $C > 0$ such that

$$\sup_{x,y\in[0,1],|x-y|\leq t} |\log h'(x) - \log h'(y)| \leq C\omega(t), \quad \forall t \geq 0. \tag{2.2}$$

One can check that if $h'(x)$ is an ω-continuous function, we can take $\varepsilon(t) = C\omega(t)$, then h is a symmetric homeomorphism with quasisymmetric distortion function $\epsilon(t)$.

We first prove a quasisymmetric distortion result, Lemma 2.1. This kind of results has been contained in other places, for examples, [10, 16, 17]. *The new point in* Lemma 2.1 *is that we give the sharpest quasisymmetric distortion bound $M - 1$ if the map is M-quasisymmetric.*

Let $I_{0,0} = [0, 1]$. For each integer $n > 0$, we cut $[0, 1]$ into 2^n equal-sized intervals,

$$I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \quad 0 \leq k \leq 2^n - 1. \tag{2.3}$$

Then we have that

$$I_{n-1,k} = I_{n,2k} \cup I_{n,2k+1}, \quad \forall 0 \leq k \leq 2^{n-1} - 1. \tag{2.4}$$

From (2.1), we have that

$$\frac{1}{1 + e^{\varepsilon(\frac{1}{2^n})}} |h(I_{n-1,k})| \leq |h(I_{n,2k})|, \quad |h(I_{n,2k+1})| \leq \frac{1}{1 + e^{-\varepsilon(\frac{1}{2^n})}} |h(I_{n-1,k})|.$$

This implies that

$$\prod_{i=1}^{n} \frac{1}{1 + e^{\varepsilon(\frac{1}{2^i})}} \leq |h(I_{n,k})| \leq \prod_{i=1}^{n} \frac{1}{1 + e^{-\varepsilon(\frac{1}{2^i})}}, \quad \forall 0 \leq i \leq 2^n - 1. \tag{2.5}$$

Moreover, by using the fact that $|I_{n,k}| = 1/2^n$, we have a more precise estimation, whenever $I_{n,k} \subset I_{m,l}$,

$$\left(\prod_{i=m+1}^{n} \frac{2}{1 + e^{\varepsilon(\frac{1}{2^i})}}\right) \frac{|h(I_{m,l})|}{|I_{m,l}|} \leq \frac{|h(I_{n,k})|}{|I_{n,k}|} \leq \left(\prod_{i=m+1}^{n} \frac{2}{1 + e^{-\varepsilon(\frac{1}{2^i})}}\right) \frac{|h(I_{m,l})|}{|I_{m,l}|}. \tag{2.6}$$

Then we have the following lemma.

Lemma 2.1 (Quasisymmetric Distortion) *Suppose $\varepsilon(t)$ is a bounded quasisymmetric distortion function for a homeomorphism $h : [0, 1] \to [0, 1]$. Let*

$$M = \sup_{t>0} e^{\varepsilon(t)} \geq 1.$$

That is, h is an M-quasisymmetric homeomorphism. Then we have that

$$|h(x) - x| \leq M - 1, \quad \forall x \in [0, 1].$$

The bound $M - 1$ is the sharpest estimation.

Proof Remember that $I_{n,k} = [k/2^n, (k + 1)/2^n]$. From (2.5), we have that

$$\left(\frac{1}{1 + M}\right)^n \leq |h(I_{n,k})| \leq \left(\frac{M}{1 + M}\right)^n.$$

This implies that

$$\left(\frac{1}{1+M}\right)^n + h\left(\frac{k}{2^n}\right) \le h\left(\frac{k+1}{2^n}\right) \le \left(\frac{M}{1+M}\right)^n + h\left(\frac{k}{2^n}\right).$$

And for any even integer $k = 2i$,

$$-g_n(M) + \left(h\left(\frac{i}{2^{n-1}}\right) - \frac{i}{2^{n-1}}\right) \le h\left(\frac{k+1}{2^n}\right) - \frac{k+1}{2^n} \le \left(h\left(\frac{i}{2^{n-1}}\right) - \frac{i}{2^{n-1}}\right) + f_n(M),$$

where

$$f_n(M) = \left(\frac{M}{M+1}\right)^n - \frac{1}{2^n} \quad \text{and} \quad g_n(M) = \frac{1}{2^n} - \left(\frac{1}{M+1}\right)^n.$$

Consider $h_n(M) = f_n(M) - g_n(M)$ as a function of $M \ge 1$. We have that $h_n(1) = 0$ and

$$h_n'(M) = \frac{n(M^{n-1}-1)}{(M+1)^{n+1}} \ge 0, \quad M \ge 1.$$

This implies that $f_n(M) \ge g_n(M)$ for all $M \ge 1$. Thus we get that for any $n \ge 1$,

$$\max_{0\le k\le 2^n} \left|h\left(\frac{k}{2^n}\right) - \frac{k}{2^n}\right| \le \max_{0\le k\le 2^{n-1}} \left|h\left(\frac{k}{2^{n-1}}\right) - \frac{k}{2^{n-1}}\right| + f_n(M).$$

This implies that

$$\max_{0\le k\le 2^n} \left|h\left(\frac{k}{2^n}\right) - \frac{k}{2^n}\right| \le \sum_{m=1}^{n} f_m(M) = M - 1 + \frac{1}{2^n} - M\left(\frac{M}{1+M}\right)^n \le M-1.$$

So we have that

$$\sup_{n\ge 0} \max_{0\le k\le 2^n} \left|h\left(\frac{k}{2^n}\right) - \frac{k}{2^n}\right| \le M-1, \quad \forall n \ge 0.$$

Since the dyadic points $\{k/2^n\}_{n\ge 0, 0\le k\le 2^n}$ are dense in the unit interval $[0,1]$ and since h is continuous, we get

$$|h(x) - x| \le M - 1, \quad \forall x \in [0,1].$$

For $M > 1$, let $0 < A < 1 - 1/M - 1/M^2$ and define

$$h(x) = \begin{cases} Mx, & 0 \le x \le \dfrac{1}{M^2}; \\ x - \dfrac{1}{M^2} + \dfrac{1}{M}, & \dfrac{1}{M^2} \le x \le \dfrac{1}{M^2} + A; \\ \dfrac{1-A-\frac{1}{M}}{1-A-\frac{1}{M^2}}\left(x - A - \dfrac{1}{M^2}\right) + A + \dfrac{1}{M}, & \dfrac{1}{M^2} + A \le x \le 1. \end{cases}$$

Since $1/M < (1-A-1/M)/(1-A-1/M^2) < M$, h is M-quasisymmetric. One can check that $|h(1/M^2) - 1/M^2| = (M-1)(1/M^2)$. As $M \to 1$, we have shown that $M-1$ is the sharpest possible estimation. This completes the proof. □

Lemma 2.1 actually says the following important fact: *For all M-quasisymmetric orientation–preserving homeomorphisms of* $[0,1]$, *their deviations from identity are always controlled by* $M-1$.

From (2.6), we observe that ratios

$$\left\{ \log\left(\frac{|h(I_{n,k})|}{|I_{n,k}|}\right)\right\}$$

may form a Cauchy sequence if we can endorse an appropriate topology on the set of natural numbers and if

$$\prod_{i=1}^{\infty} \frac{1+e^{\varepsilon(\frac{1}{2^i})}}{2}$$

is a convergent infinite product.

Consider the doubling map $f_0(x) = 2x \pmod 1$. It has two inverse branches

$$g_0(x) = \frac{x}{2} \quad \text{and} \quad g_1(x) = \frac{x+1}{2}.$$

For any finite string $w_n = i_0 i_1 \cdots i_{n-1}$ of 0's and 1's of length $n \geq 1$, define

$$g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}$$

and

$$I_{w_n} = g_{w_n}([0,1]).$$

Let $k = i_0 2^{n-1} + i_1 2^{n-2} + \cdots + 2i_{n-2} + i_{n-1}$. Then we have that $I_{n,k} = I_{w_n}$. We have that

$$I_{w_n} \subset I_{w_{n-1}}.$$

Note that $I_{w_{n-1}} = I_{n-1,l}$ for $l = i_0 2^{n-2} + i_1 2^{n-3} + \cdots + i_{n-2}$. Thus (2.6) becomes

$$\left(\prod_{i=m+1}^{n} \frac{2}{1+e^{\varepsilon(\frac{1}{2^i})}}\right) \frac{|h(I_{w_m})|}{|I_{w_m}|} \leq \frac{|h(I_{w_n})|}{|I_{w_n}|} \leq \left(\prod_{k=m+1}^{n} \frac{2}{1+e^{-\varepsilon(\frac{1}{2^k})}}\right) \frac{|h(I_{w_m})|}{|I_{w_m}|} \tag{2.7}$$

for all $1 \leq m < n$, $w_n = w_m i_m \cdots i_{n-1}$.

Now we consider the symbolic space

$$\Sigma = \prod_0^{\infty} \{0,1\} = \{w = i_0 i_1 \cdots i_n \cdots \mid i_n \in \{0,1\}\}$$

with the product topology. The product topology can be induced by the metric

$$d(w,w') = \sum_{n=1}^{\infty} \frac{|i_{n-1} - i'_{n-1}|}{2^n}$$

where $w = i_0 i_1 \cdots i_n \cdots, w' = i'_0 i'_1 \cdots i'_n \cdots \in \Sigma$.

For any point $w = i_0 i_1 \cdots i_n \cdots \in \Sigma$, let $w_n = i_0 i_1 \cdots i_n$, then we have that

$$\cdots \subset I_{w_n} \subset I_{w_{n-1}} \subset I_{w_1} \subset [0,1].$$

Thus

$$\bigcap_{n=1}^{\infty} I_{w_n} = \{x_w\}$$

since $|I_{w_n}| = 1/2^n$. Let

$$\pi(w) = x_w : \Sigma \to [0,1].$$

Then π is a continuous map and 1–1 except for a countable subset A of points $w = w_n 000 \cdots$, $w_n 111 \cdots$ for all finite strings $w_n = i_0 \cdots i_{n-1}$. On the set A, π is 2–1, that is,

$$\pi(w_n 000 \cdots) = \pi(w_n 111 \cdots).$$

Lemma 2.2 *If* $\prod_{i=1}^{\infty}(1+\mathrm{e}^{\varepsilon(1/2^i)})/2$ *is a convergent infinite product, then for every* $w \in \Sigma$,

$$\left\{ \log\left(\frac{|h(I_{w_n})|}{|I_{w_n}|}\right)\right\}_{n=1}^{\infty} \tag{2.8}$$

is a Cauchy sequence and

$$\phi(w) = \lim_{n\to\infty} \log\left(\frac{|h(I_{w_n})|}{|I_{w_n}|}\right) : \Sigma \to \mathbb{R} \tag{2.9}$$

defines a continuous function. Moreover,

$$\phi(w_n 000\cdots) = \phi(w_n 111\cdots) \tag{2.10}$$

on A.

Proof Since $\prod_{i=1}^{\infty}(1+\mathrm{e}^{\varepsilon(1/2^i)})/2$ is convergent, we have that

$$\sum_{i=m+1}^{n} \log\left(\frac{2}{1+\mathrm{e}^{-\varepsilon(\frac{1}{2^i})}}\right) \le \sum_{i=m+1}^{n} \log\left(\frac{1+\mathrm{e}^{\varepsilon(\frac{1}{2^i})}}{2}\right) \to 0 \quad \text{as } n \ge m \to \infty.$$

Then (2.7) implies the sequence (2.8) is a Cauchy sequence. Thus the limit

$$\phi(w) = \lim_{n\to\infty} \log\left(\frac{|h(I_{w_n})|}{|I_{w_n}|}\right)$$

exists. To show $\phi(w)$ is a continuous function, we assume $w, w' \in \Sigma$ with $w_m = w'_m$. Then we have $I_{w_m} = I_{w'_m}$. For any $n > m$, we have (2.7) for both $|h(I_{w_n})|/|I_{w_n}|$ and $|h(I_{w'_n})|/|I_{w'_n}|$. This implies that

$$|\phi(w) - \phi(w')| \le \sum_{i=m+1}^{\infty} \log\left(\frac{1+\mathrm{e}^{\varepsilon(\frac{1}{2^i})}}{2}\right) \to 0 \quad \text{as } m \to \infty.$$

To show (2.10), for any $m > 0$, we consider the interval

$$J_m = I_{w_n \underbrace{0\cdots0}_{m}} \cup I_{w_n \underbrace{1\cdots1}_{m}},$$

then we have

$$\begin{aligned}\left(\prod_{i=n+m+1}^{n+k} \frac{2}{1+\mathrm{e}^{\varepsilon(\frac{1}{2^i})}}\right)\frac{|h(J_m)|}{|J_m|} &\le \frac{|h(I_{w_n \underbrace{0\cdots0}_{k}})|}{|I_{w_n \underbrace{0\cdots0}_{k}}|}, \frac{|h(I_{w_n \underbrace{1\cdots1}_{k}})|}{|I_{w_n \underbrace{1\cdots1}_{k}}|} \\ &\le \left(\prod_{i=m+1}^{n+k} \frac{2}{1+\mathrm{e}^{-\varepsilon(\frac{1}{2^i})}}\right)\frac{|h(J_m)|}{|J_m|}, \quad \forall k > m.\end{aligned} \tag{2.11}$$

Similarly,

$$\left\{ \log\left(\frac{|h(J_m)|}{|J_m|}\right)\right\}_{n=1}^{\infty}$$

is a Cauchy sequence and

$$\phi(w_n 000\cdots) = \phi(w_n \cdots 111\cdots) = \lim_{m\to\infty} \log\left(\frac{|h(J_m)|}{|J_m|}\right). \qquad \square$$

Since Σ is a compact metric space, $\phi(w)$ in Lemma 2.2 is a bounded continuous function. Furthermore, $\mathrm{e}^{\phi(\pi^{-1}(x))}$ is a positive continuous bounded function on $[0, 1]$.

Corollary 2.3 *If $\prod_{i=1}^{\infty}(1+\mathrm{e}^{\varepsilon(1/2^i)})/2$ is a convergent infinite product, then h is a C^1-diffeomorphism.*

Proof For any $x\in[0,1]$, let $w\in\Sigma$ such that $\pi(w)=x$. For any $t>0$, we have the largest integer $n>0$ such that $I_{w_n}\supset[x+t,x]$. We know that $n\to\infty$ as $t\to 0$. Consider $[x+t,x]$ as a union of all dyadic intervals $I_{w_m}\subset[x+t,x]\subset I_{w_n}$. Then we have that

$$\begin{aligned}\Big(\prod_{i=m+1}^{\infty}\frac{2}{1+\mathrm{e}^{\varepsilon(\frac{1}{2^i})}}\Big)\frac{|h(I_{w_n})|}{|I_{w_n}|}&\le\frac{h(x+t)-h(x)}{t}\\&=\frac{\sum_{w_m}|h(I_{w_m})|}{\sum_{w_m}|I_{w_m}|}\\&\le\Big(\prod_{i=n+1}^{\infty}\frac{2}{1+\mathrm{e}^{-\varepsilon(\frac{1}{2^i})}}\Big)\frac{|h(I_{w_n})|}{|I_{w_n}|}.\end{aligned}\tag{2.12}$$

Note that in the last estimation, we use the additive formula, that is, if $a_i,b_i>0$ are two sequences of positive real numbers and if $C^{-1}\le a_i/b_i<C$ for all i and for a constant $C>0$, then $C^{-1}\le(\sum_i a_i)/(\sum_i b_i)\le C$. Thus we have a similar estimate for $(h(x)-h(x-t))/t$. This combining with (2.10) implies that $h'(x)=\mathrm{e}^{\phi(w)}>0$ exists and is a continuous function on $[0,1]$. Since $[0,1]$ is a compact space, we have that $(h^{-1})'$ is also a continuous function on $[0,1]$. So h is a C^1-diffeomorphism. □

Lemma 2.4 *The infinite product $\prod_{i=1}^{\infty}(1+\mathrm{e}^{\varepsilon(1/2^i)})/2$ is convergent if and only if*

$$\int_0^1\frac{\varepsilon(t)}{t}dt<\infty.\tag{2.13}$$

Proof First, we have that

$$\begin{aligned}\int_0^{\frac12}\frac{\varepsilon(t)}{t}dt&=\sum_{i=1}^{\infty}\int_{\frac{1}{2^{i+1}}}^{\frac{1}{2^i}}\frac{\varepsilon(t)}{t}dt\\&\le\sum_{i=1}^{\infty}\varepsilon\Big(\frac{1}{2^i}\Big)\int_{\frac{1}{2^{i+1}}}^{\frac{1}{2^i}}\frac{1}{t}dt\\&\le(\log 2)\sum_{i=1}^{\infty}\varepsilon\Big(\frac{1}{2^i}\Big).\end{aligned}\tag{2.14}$$

Second, we prove that

$$\sum_{i=1}^{\infty}\varepsilon\Big(\frac{1}{2^i}\Big)\le\int_0^{\infty}\varepsilon\Big(\frac{1}{2^x}\Big)dx=\frac{1}{\log 2}\int_0^1\frac{\varepsilon(t)}{t}dt.$$

In the last equality, we use the change of variable $t=1/2^x$. So we have that $\sum_{i=1}^{\infty}\varepsilon(1/2^i)$ is convergent if and only if $\int_0^1\varepsilon(t)/tdt<\infty$. But the infinite product $\prod_{i=1}^{\infty}(1+\mathrm{e}^{\varepsilon(\frac{1}{2^i})})/2$ is convergent if and only if the infinite series $\sum_{i=1}^{\infty}\varepsilon(1/2^i)$ is convergent. We proved the lemma. □

We call a positive function satisfying (2.13) a Dini function. Define

$$\widetilde{\varepsilon}(t)=\int_0^t\frac{\varepsilon(s)}{s}ds,\quad\forall t>0,\tag{2.15}$$

if $\varepsilon(t)$ is a Dini modulus of continuity. Then $\widetilde{\varepsilon}(t)$ is also a modulus of continuity. Finally, we got a result due originally to Carlson [5] as follows.

Corollary 2.5 *If $\varepsilon(t)$ is a Dini modulus of continuity, then h is a C^1-diffeomorphism. Moreover, $h'(x)$ is an $\widetilde{\varepsilon}$-continuous function.*

In general $\widetilde{\varepsilon}(t)$ may not be a Dini modulus of continuity anymore. In order $h'(x)$ is a Dini function again, we need $\varepsilon(t)$ satisfying

$$\int_0^1 \frac{\varepsilon(t)\log t}{t} dt < \infty. \tag{2.16}$$

When $\varepsilon(t) = Ct^\alpha$ for some constants $0 < \alpha \leq 1$ and $C > 0$, then one can check that $\widetilde{\varepsilon}(t) = (C/\alpha)t^\alpha$. In this case, we call $h'(x)$ an α-Hölder continuous function. When $\alpha = 1$, it is also called a Lipschitz continuous function. We call h $C^{1+\alpha}$ if its derivative h' is α-Hölder continuous.

Corollary 2.6 *If $\varepsilon(t) = Ct^\alpha$, then h is a $C^{1+\alpha}$-diffeomorphism.*

In general, a quasisymmetric homeomorphism is not differentiable and the same is true for a symmetric homeomorphism. Thus if we consider the map $f(x) = h \circ f_0 \circ h^{-1}$ where $f_0(x) = 2x$ (mod 1), then it is not differentiable in general. However, from the dynamical systems point of view, we can study the dual derivative of f. We can prove that the dual derivative is always the constant function 2 when h is a symmetric homeomorphism. A general continuous dual derivative for some quasisymmetric homeomorphism has been studied in our sequel paper [16]. Moreover, general L^1 dual derivatives for all quasisymmetric homeomorphisms such that f preserves the Lebesgue measure have been studied in our sequel paper [15] by using martingale theory. In this section, we give a proof of the dual derivative is 2 when h is symmetric. The reader who is interested in the more general theory in this direction can go to [15, 16]. We would like to note that a general dual derivative is a highly non-trivial function, that is, as long as it is piece-wise constants, then it must be global constant and thus h must be symmetric (see Lemma 2.7), furthermore, if f also preserves the Lebesgue measure, it must be the identity (see Lemma 2.8).

We first give a dual topology on the symbolic space as follows. For any $w_n = i_0 i_1 \cdots i_{n-2} i_{n-1}$, we relabel it as $w_n^* = j_{n-1}j_{n-2}\cdots j_1 j_0$, where $j_{n-1} = i_0, \ldots, j_0 = i_{n-1}$. In this way, we say two w_n^* and $w_n^{*'} = j'_{n-1}j'_{n-2}\cdots j'_1 j'_0$ are m-close if $j_0 = j'_0, \ldots, j_{m-1} = j'_{m-1}$. Remember that previously, we say w_n and w'_n are m-close if $i_0 = i'_0, \ldots, i_{m-1} = i'_{m-1}$. Thus we consider the dual symbolic space

$$\Sigma^* = \prod_{\infty}^{0}\{0,1\} = \{w^* = \cdots j_{n-1}\cdots j_1 j_0 | j_{n-1} \in \{0,1\}, n = 1, 2, \ldots\}$$

with the metric

$$d(w^*, w^{*'}) = \sum_{n=1}^{\infty} \frac{|j_{n-1} - j'_{n-1}|}{2^n} = |I_{w_n^*}|,$$

where $n \geq 1$ is the largest integer such that $w_n^* = w_n^{*'}$. Given any $w^* \in \Sigma^*$, let $w^* = \cdots w_n^*$. The symbolic dynamical system under this dual topology is

$$\sigma^*(w^*) = \cdots j_{n-1}\cdots j_1 : \Sigma^* \to \Sigma^*.$$

Similarly, we have $\sigma^*(w_n^*) = j_{n-1}\cdots j_1$. It is clearly a continuous map. A quasisymmetric homeomorphism h gives another metric on Σ^* as that

$$d_h(w^*, w^{*'}) = |I_{w_n^*}|_h = |h(I_{w_n^*})|$$

for any $w^*, w^{*'} \in \Sigma^*$, where $n \geq 1$ is the largest integer such that $w_n^* = w_n^{*'}$. Under this metric, σ^* is again a continuous map. Moreover, we have that

Lemma 2.7 (Constant Dual Derivative [16]) *Suppose h is a symmetric homeomorphism. Then for any $w^* \in \Sigma^*$, the sequence*

$$\left\{ \log\left(\frac{|h(I_{(\sigma^*(w^*))_{n-1}})|}{|h(I_{w_n^*})|}\right) \right\}_{n=1}^{\infty} \tag{2.17}$$

is a Cauchy sequence and the limit

$$\phi^*(w^*) = \lim_{n\to\infty} \log\left(\frac{|h(I_{(\sigma^*(w^*))_{n-1}})|}{|h(I_{w_n^*})|}\right) = \log 2. \tag{2.18}$$

We call

$$\frac{d_h\sigma^*}{d_h}(w^*) = \mathrm{e}^{\phi^*(w^*)}$$

the dual derivative of σ^* under the metric $d_h(\cdot,\cdot)$.

Proof For any $n > m > 0$,

$$f^{n-m} = h \circ f_0^{n-m} \circ h^{-1} : h(I_{(\sigma^*(w^*))_{n-1}}) \to h(I_{(\sigma^*(w^*))_{m-1}}).$$

This map is actually

$$h : I_{(\sigma^*(w^*))_{m-1}} = 2^{n-m} I_{(\sigma^*(w^*))_{n-1}} (\mathrm{mod}\ 1) \to h(I_{(\sigma^*(w^*))_{m-1}}).$$

Let $\alpha_n : I_{(\sigma^*(w^*))_{n-1}} \to I$ and $\beta_m : I_{(\sigma^*(w^*))_{m-1}} \to I$ be two increasing linear homeomorphisms. Then $h_{n,m} = \alpha_n \circ h(2^{m-n}h^{-1}) \circ \beta_m^{-1} : I \to I$ is an $\mathrm{e}^{\varepsilon(1/2^{m-1})}$-quasisymmetric homeomorphism. For

$$0 < t = \frac{|h(I_{w_m^*})|}{|h(I_{(\sigma^*(w^*))_{m-1}})|} < 1,$$

Lemma 2.1 implies

$$|h_{n,m}(t) - t| \leq \mathrm{e}^{\varepsilon(1/2^{m-1})} - 1.$$

This is equivalent to

$$\left| \frac{|h(I_{w_n^*})|}{|h(I_{(\sigma^*(w^*))_{n-1}})|} - \frac{|h(I_{w_m^*})|}{|h(I_{(\sigma^*(w^*))_{m-1}})|} \right| \leq \mathrm{e}^{\varepsilon(1/2^{m-1})} - 1.$$

Since $\varepsilon(1/2^{m-1}) \to 0$ as $m \to \infty$, we see that (2.17) is a Cauchy sequence. Similarly, we also get

$$\left| \frac{|h(I_{w_n^*})|}{|h(I_{(\sigma^*(w^*))_{n-1}})|} - \frac{|I_{w_m^*}|}{|I_{(\sigma^*(w^*))_{m-1}}|} \right| \leq \mathrm{e}^{\varepsilon(1/2^{n-1})} - 1.$$

But $|I_{w_m^*}|/|I_{(\sigma^*(w^*))_{m-1}}| = 1/2$. This implies (2.18). This completes the proof. □

Among all $f = h \circ f_0 \circ h^{-1}$, we are, in particular, interested in those preserve the Lebsegue measure on $[0, 1]$, that is,

$$|f^{-1}(I)| = |I|$$

for all subintervals $I \subset [0, 1]$. By applying Lemma 2.7, we prove that among all $f = h \circ f_0 \circ h^{-1}$ for all symmetric homeomorphisms h, f_0 is the only one preserving the Lebesgue measure. This is a rigidity result. We studied such rigidity problem for $f = h \circ f_0 \circ h^{-1}$ for all quasisymmetric homeomorphisms h in [16].

Lemma 2.8 (Symmetric Rigidity [17]) *Suppose h is a symmetric homeomorphism and suppose $f = h \circ f_0 \circ h^{-1}$ preserves the Lebesgue measure. Then $h(x) \equiv x$ is the identity.*

Proof For any finite w_n^* of length n, then $(\sigma^*)^{-m}(w_n^*) = \{w_m^* w_n^*\}$ for all finite w_m^* of length m. Since f preserves the Lebesgue measure, we have that as $m \to \infty$,

$$\frac{|h(I_{\sigma^*(w_n^*)})|}{|h(I_{w_n^*})|} = \frac{|f^{-m}(h(I_{\sigma^*(w_n^*)}))|}{|f^{-m}(h(I_{w_n^*}))|} = \frac{\sum_{w_m^*} |h(I_{\sigma^*(w_m^* w_n^*)})|}{\sum_{w_m^*} |h(I_{w_m^* w_n^*})|} = \cdots = 2.$$

All boundary points $\{I_{w_n^*}\}$ for all n and all w_n^* are 2-adic numbers $\{k/2^n \mid n \geq 0, 0 \leq k \leq 2^n\}$. Inductively use the above equality, h fixes all 2-adic numbers and all these 2-adic numbers are dense in $[0,1]$ and h is continuous, we get $h = \mathrm{Id}$. This ends the proof. □

Now we consider the extension of h to the diamond domain in the complex plane $\mathbb{C}$

$$D = \left\{ z = x + \mathrm{i}y \;\middle|\; \left| x - \frac{1}{2} \right| + |y| \leq \frac{1}{2} \right\}$$

by using modified Beurling–Ahlfor's formula

$$\begin{aligned} H(z) &= u + \mathrm{i}v \\ &= \frac{1}{2} \int_0^1 (h(x+ty) + h(x-ty))dt + \mathrm{i} \int_0^1 (h(x+ty) - h(x-ty))dt. \end{aligned} \tag{2.19}$$

Note that

$$u = \frac{1}{2} \int_0^1 (h(x+ty) + h(x-ty))dt = \frac{1}{2y} \int_{x-y}^{x+y} h(t)dt$$

and

$$v = \int_0^1 (h(x+ty) - h(x-ty))dt = \frac{1}{y} \left(\int_x^{x+y} h(t)dt - \int_{x-y}^{x} h(t)dt \right).$$

It is clear that $\overline{H(z)} = H(\overline{z})$. The complex dilatation of H is

$$\mu_H = \frac{H_{\overline{z}}}{H_z}.$$

From the calculation in [1], we have that $\|\mu_H\|_\infty \leq k < 1$. Actually, one can make more precise estimation for $k \leq (M^2-1)/(M^2+1)$ where $M = \sup_{t>0} \mathrm{e}^{\varepsilon(t)}$. Thus H is a quasiconformal homeomorphism. We will not provide detailed calculation of this estimation. The reader who is interested in this calculation may refer to [20]. However, we will provide more detailed calculation for the asymptotical behavior of $\mu_H(x+\mathrm{i}y)$ when $y \to 0$ by using Lemma 2.1 when $\epsilon(t) \to 0^+$ as $t \to 0^+$ (see [13] and other related papers for other estimations).

Since

$$H_z = \frac{1}{2}(u_x + v_y) + \frac{\mathrm{i}}{2}(v_x - u_y) \quad \text{and} \quad H_{\overline{z}} = \frac{1}{2}(u_x - v_y) + \frac{\mathrm{i}}{2}(v_x + u_y),$$

we have that

$$\begin{aligned} |\mu_H| &= \left| \frac{(u_x - v_y) + \mathrm{i}(v_x + u_y)}{(u_x + v_y) + \mathrm{i}(v_x - u_y)} \right| \\ &= \sqrt{\frac{(u_x - v_y)^2 + (v_x + u_y)^2}{(u_x + v_y)^2 + (v_x - u_y)^2}} \end{aligned}$$

$$= \sqrt{\frac{(1-\frac{v_y}{u_x})^2+(\frac{v_x}{u_x})^2(1+\frac{u_y}{v_x})^2}{(1+\frac{v_y}{u_x})^2+(\frac{v_x}{u_x})^2(1-\frac{u_y}{v_x})^2}}. \tag{2.20}$$

Now we estimate ratios of partial derivatives in the last formula. First, we have

$$u_x = \frac{h(x+y)-h(x-y)}{2y}$$

and denote $a(x,y) = (h(x)-h(x-y))/(h(x+y)-h(x))$. Then we get

$$v_x = \frac{h(x+y)+h(x-y)-2h(x)}{y} = 2u_x\frac{1-a(x,y)}{1+a(x,y)}.$$

Since $\mathrm{e}^{-\varepsilon(y)} \le a(x,y) \le \mathrm{e}^{\varepsilon(y)}$, we have that

$$\left|\frac{v_x}{u_x}\right| \le 2\frac{\mathrm{e}^{\varepsilon(y)}-1}{\mathrm{e}^{\varepsilon(y)}+1}. \tag{2.21}$$

Now

$$u_y = -\frac{1}{2y^2}\left(\int_{x-y}^{x+y} h(t)dt\right) + \frac{h(x+y)+h(x-y)}{2y} = \frac{v_x}{2}\left(1-\int_0^1 h_1(t)dt\right),$$

where

$$h_1(t) = \frac{h(x+ty)+h(x-ty)-2h(x)}{h(x+y)+h(x-y)-2h(x)} : [0,1] \to [0,1]$$

is a homeomorphism with $h_1(0) = 0$ and $h_1(1) = 1$. And, similarly,

$$v_y = 2u_x\left(1-\int_0^1 h_2(t)dt\right),$$

where

$$h_2(t) = \frac{h(x+ty)-h(x-ty)}{h(x+y)-h(x-y)} : [0,1] \to [0,1]$$

is also a homeomorphism with $h_2(0) = 0$ and $h_2(1) = 1$. Both h_1 and h_2 are ($M = \mathrm{e}^{\varepsilon(y)}$)-quasisymmetric, thus Lemma 2.1 implies that

$$|h_1(t)-t|, |h_2(t)-t| \le \mathrm{e}^{\varepsilon(y)}-1, \quad \forall t \in [0,1].$$

This implies that

$$\frac{3}{2}-\mathrm{e}^{\varepsilon(y)} \le \left(1-\int_0^1 h_1(t)dt\right), \left(1-\int_0^1 h_2(t)dt\right) \le \mathrm{e}^{\varepsilon(y)}-\frac{1}{2}.$$

We get

$$\frac{1}{2}\left(\frac{3}{2}-\mathrm{e}^{\varepsilon(y)}\right) \le \frac{u_y}{v_x} \le \frac{1}{2}\left(\mathrm{e}^{\varepsilon(y)}-\frac{1}{2}\right)$$

and

$$2\left(\frac{3}{2}-\mathrm{e}^{\varepsilon(y)}\right) \le \frac{v_y}{u_x} \le 2\left(\mathrm{e}^{\varepsilon(y)}-\frac{1}{2}\right).$$

These give us that

$$\left|1-\frac{v_y}{u_x}\right| \le 2(\mathrm{e}^{\varepsilon(y)}-1), \tag{2.22}$$

$$4-2\mathrm{e}^{\varepsilon(y)} \le 1+\frac{v_y}{u_x} \le 2\mathrm{e}^{\varepsilon(y)}, \tag{2.23}$$

$$\frac{7}{4}-\frac{1}{2}\mathrm{e}^{\varepsilon(y)} \le 1+\frac{u_y}{v_x} \le \frac{1}{2}\mathrm{e}^{\varepsilon(y)}+\frac{3}{4}, \tag{2.24}$$

and

$$\frac{5}{4}-\frac{1}{2}\mathrm{e}^{\varepsilon(y)}\leq 1-\frac{u_y}{v_x}\leq \frac{1}{2}\mathrm{e}^{\varepsilon(y)}+\frac{1}{4}. \tag{2.25}$$

From the estimations of (2.21)–(2.25), we finally get that

$$|\mu_H(z)|\leq C_0(\mathrm{e}^{\varepsilon(y)}-1)\leq C_1\varepsilon(y),\quad \forall z=x+\mathrm{i}y\in D, \tag{2.26}$$

where $C_0>0$ and $C_1>0$ are constants. This estimate implies that when h is symmetric, then its extension to D is asymptotically conformal near $[0,1]$. In particular, when h is a C^1-diffeomorphism such that h' is an ω-continuous function, then we can take $\varepsilon(t)=C\omega(t)$ for some constant $C>0$. Therefore, we have the following lemma.

Lemma 2.9 *Suppose h is a C^1-diffeomorphism such that h' is an ω-continuous function. Then we have that*

$$|\mu_H(z)|\leq C\omega(y),\quad \forall z=x+\mathrm{i}y\in D,\quad \text{as } y\to 0^+ \tag{2.27}$$

for some constant $C>0$.

3 Teichmüller Structures on Circle Diffeomorphisms

We denote $\mathbb{C}$ the complex plane and $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ the Riemann sphere. We denote $\mathbb{R}$ the real line. Let $\Delta=\{z\in\mathbb{C}\mid |z|<1\}$ be the unit disk and $\Delta_\infty=\widehat{\mathbb{C}}\setminus\overline{\Delta}$. We use $\Delta^*=\Delta\setminus\{0\}$ to denote the punctured disk and $\Delta_\infty^*=\mathbb{C}\setminus\overline{\Delta}$. We use $T=\{z\in\mathbb{C}\mid |z|=1\}$ to denote the unit circle. It has a universal cover $\pi(x)=\mathrm{e}^{2\pi\mathrm{i}x}:\mathbb{R}\to T$. Let

$$\mathbb{H}=\{z=x+\mathrm{i}y\in\mathbb{C}\mid y>0\}\quad\text{and}\quad \mathbb{L}=\{z=x+\mathrm{i}y\in\mathbb{C}\mid y<0\}$$

be the upper and lower half planes. Then

$$\pi(z)=\mathrm{e}^{2\pi\mathrm{i}z}:\mathbb{H}\to\Delta^*\quad\text{and}\quad \mathbb{L}\to\Delta_\infty^*$$

are both universal covers.

Consider the space $\mathcal{H}$ of all orientation-preserving homeomorphisms of T. Let $\mathcal{M}$ be the subspace of all Möbius transformations in $\mathcal{H}$ preserving the unit disk. Let $\mathcal{TH}=\mathcal{H}/\mathcal{M}$. It can be identified with all $h\in\mathcal{H}$ fixing -1, 1, and i.

For any $h\in\mathcal{TH}$, let $H:\mathbb{R}\to\mathbb{R}$ be the lift of h such that $H(0)=0$. Then H is a strictly increasing continuous function such that $H(x+1)=H(x)+1$ and $H(1/2)=1/2$ and $H(1/4)=1/4$.

We say h is quasisymmetric if there is a bounded increasing function $\varepsilon(t)>0$ such that

$$\mathrm{e}^{-\varepsilon(t)}\leq\frac{H(x+t)-H(x)}{H(x)-H(x-t)}\leq \mathrm{e}^{\varepsilon(t)},\quad \forall x\in\mathbb{R}, t>0. \tag{3.1}$$

We say h is symmetric if, in additional, $\varepsilon(t)\to 0^+$ as $t\to 0^+$. In the above formula, we only need to check for $x\in[0,1]$. It is also clear that if h is quasisymmetric or symmetric, then $H|[0,1]$ is quasisymmetric or symmetric in the sense of §2.

We use $\mathcal{Q}$ to denote the space of all quasisymmetric h in $\mathcal{H}$ and $\mathcal{TQ}=\mathcal{Q}/\mathcal{M}$. Here $\mathcal{TQ}$ is called the universal Teichmüller space.

We use $\mathcal{S}$ to denote the space of all symmetric h in $\mathcal{Q}$ and $\mathcal{TS}=\mathcal{S}/\mathcal{M}$, which is called the universal asymptotically conformal Teichmüller space.

Given a modulus of continuity ω, we use $\mathcal{C}^{1+\omega}$ to denote the space of all diffeomorphisms h in $\mathcal{S}$ such that H' are ω-continuous functions. We call $\mathcal{TC}^{1+\omega} = \mathcal{C}^{1+\omega}/\mathcal{M}$ the $C^{1+\omega}$ Teichmüller space.

Another space $\mathcal{A}$ is the space of all analytic diffeomorphisms h in $\mathcal{S}$. We call $\mathcal{TA} = \mathcal{A}/\mathcal{M}$ the real analytic Teichmüller space. Thus we have the following sequence of spaces,

$$\mathcal{TA} \subset \mathcal{TC}^{1+\omega} \subset \mathcal{TS} \subset \mathcal{TQ} \subset \mathcal{TH}. \tag{3.2}$$

The Teichmüller structure on $\mathcal{TQ}$ is introduced by identifying it with the Teichmüller space $\mathcal{T}(\Delta)$ of all Riemann surfaces with the basepoint Δ. And the Teichmüller structure on $\mathcal{TA}$, $\mathcal{TC}^{1+\omega}$, or $\mathcal{TS}$, respectively, just inherits from $\mathcal{TQ}$, that is, we just treat them as sub-Teichmüller spaces of the universal Teichmüller space $\mathcal{TQ}$. Let us define it in more details.

Consider the unit ball

$$\mathcal{BM}(\Delta) = \{\mu \in L^\infty(\Delta) \mid \|\mu\|_\infty < 1\}$$

of all measurable complex functions on $\mathbb{C}$ with bounded essential maximum norms. Then one can define the Teichmüller metric on $\mathcal{BM}(\Delta)$ as

$$d_{\mathcal{BM}}(\mu, \nu) = \tanh^{-1}\left\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\right\|_\infty = \frac{1}{2}\log\left(\frac{1+\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty}{1-\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty}\right). \tag{3.3}$$

It induces the same topology as the topology when we think $\mathcal{BM}(\Delta)$ as a subset of $L^\infty(\Delta)$. But the distance between any point μ to the boundary of $\mathcal{BM}(\Delta)$ is infinite.

A measurable function μ on $\mathbb{C}$ is called a Beltrami coefficient if its L^∞-norm $k = \|\mu\|_\infty < 1$. Let $\mathcal{BM}(\mathbb{C})$ be the space of all Beltrami coefficients. It is the unit ball of the space of $L^\infty(\mathbb{C})$. Since $L^\infty(\mathbb{C})$ is an infinite-dimensional complex vector space, $\mathcal{BM}(\mathbb{C})$ is an infinite-dimensional complex manifold.

Given $\mu \in \mathcal{BM}(\mathbb{C})$, the equation

$$w_{\overline{z}} = \mu w_z \tag{3.4}$$

is called the Beltrami equation. A solution w is a quasiconformal self-homeomorphism of $\hat{\mathbb{C}}$ whose quasiconformal dilatation is less than or equal to $1 \le K = (1+k)/(1-k) < \infty$. We have the following important theorem in Teichmüller theory.

Theorem 3.1 (The Measurable Riemann Mapping Theorem) *The Beltrami equation* (3.4) *always has a solution w which is a K-quasiconformal homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$. If we consider a normalized solution w^μ in the meaning that it always fixes three given points -1, 1, and* i*, then it is unique for any given μ, and moreover, depends on $\mu \in \mathcal{BM}(\mathbb{C})$ holomorphically.*

The reader may refer to [11] for the development of this theorem and relating references. Using this theorem, Bers gave a complex manifold structure on $\mathcal{TQ}$ by using Bers' embedding as we will explain now.

For any $\mu \in \mathcal{BM}(\Delta)$, we can extend it to the whole complex plane by defining $\mu(z) = \mu(z^*)^*$ for any $z \in \Delta_\infty$ where z^* is the reflection point of z with respect to the unit circle T, that is, $z\overline{z^*} = 1$. We still denote it as μ. By Theorem 3.1, we have the normalized K-quasiconformal homeomorphism w_μ of $\mathbb{C}$ solving (3.4). Since μ is invariant under the reflection with respect to T, $h = w_\mu|T$ is a self-homeomorphism of T.

Denote

$$\mathrm{Cr}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_4 - z_3)(z_1 - z_2)}$$

as the modified cross-ratio of four distinct points on $\widehat{\mathbb{C}}$. It takes values in $\widehat{\mathbb{C}} \setminus \{-1, 0, \infty\}$. Let $Q(z_1, z_2, z_3, z_4)$ be the quadrilateral with vertices z_1, z_2, z_3, z_4 positively orientated. Since h is K-quasiconformal, we have that

$$\mathrm{mod}(Q(h(z_1), h(z_2), h(z_3), h(z_4)) \leq K\mathrm{mod}(Q(z_1, z_2, z_3, z_4))$$

for any four distinct points.

Now we consider four distinct points z_1, z_2, z_3, z_4 on T positively orientated and the quadrilateral $\Delta(z_1, z_2, z_3, z_4)$. Let $\lambda : (0, \infty) \to (0, \infty)$ be the distortion function defined as

$$\lambda(\mathrm{mod}(\Delta(z_1, z_2, z_3, z_4))) = \mathrm{Cr}(z_1, z_2, z_3, z_4).$$

It is a continuous increasing function satisfying $\lambda(1/t) = 1/\lambda(t)$ for all $t \in (0, \infty)$. So $\lambda(1) = 1$. Consider four distinct points $z_1, z_2, z_3, z_4 \in T$ with $\mathrm{Cr}(z_1, z_2, z_3, z_4) = 1$, then

$$\begin{aligned}\mathrm{Cr}(h(z_1), h(z_2), h(z_3), h(z_4)) &= \lambda(\mathrm{mod}(h(\Delta(z_1, z_2, z_3, z_4)))) \\ &\leq \lambda(K\mathrm{mod}(\Delta(z_1, z_2, z_3, z_4))) \\ &= \lambda(K\lambda^{-1}(Cr(z_1, z_2, z_3, z_4))) \\ &= \lambda(K). \end{aligned} \tag{3.5}$$

Now take $z_1 = \mathrm{e}^{2\pi\mathrm{i}(x+t)}$, $z_4 = \mathrm{e}^{2\pi\mathrm{i}x}$, $z_3 = \mathrm{e}^{2\pi\mathrm{i}(x-t)}$, and $z_2 = \mathrm{e}^{2\pi\mathrm{i}(x+\pi)}$ for $x \in [0, 1]$ and small $t > 0$. We have $\mathrm{Cr}(z_1, z_2, z_3, z_4) = 1$. Notice that $h(z_1) = \mathrm{e}^{2\pi\mathrm{i}H(x+t)}$, $h(z_4) = \mathrm{e}^{2\pi\mathrm{i}H(x)}$, $h(z_3) = \mathrm{e}^{2\pi\mathrm{i}H(x-t)}$, $h(z_2) = \mathrm{e}^{2\pi\mathrm{i}H(x+\pi)}$. We have a bounded function $\widetilde{\lambda}(t) \geq 1$ such that $\widetilde{\lambda}(t) \to 1$ as $t \to 0$ and such that

$$\frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq \widetilde{\lambda}(t)\mathrm{Cr}(h(z_1), h(z_2), h(z_3), h(z_4)) \leq \widetilde{\lambda}(t)\lambda(K) = \mathrm{e}^{\varepsilon(t)}, \tag{3.6}$$

where $\varepsilon(t) = \log(\widetilde{\lambda}(t)\lambda(K))$ is a bounded positive function. Since h is uniformly continuous on T, we can make this inequality holding for all $t > 0$. Similarly, by taking $z_3 = \mathrm{e}^{2\pi\mathrm{i}(x-t)}$, $z_2 = \mathrm{e}^{2\pi\mathrm{i}x}$, $z_4 = \mathrm{e}^{2\pi\mathrm{i}(x+t)}$, and $z_1 = \mathrm{e}^{2\pi\mathrm{i}(x+\pi)}$ for $x \in [0, 1]$ and small $t > 0$, we get

$$\frac{H(x) - H(x-t)}{H(x+t) - H(x)} \leq \widetilde{\lambda}(t)\mathrm{Cr}(h(z_1), h(z_2), h(z_3), h(z_4)) \leq \widetilde{\lambda}(t)\lambda(K) = \mathrm{e}^{\varepsilon(t)}.$$

Therefore, h is a quasisymmetric homeomorphism of T. Then we have a map

$$\mathcal{P} : \mu \in \mathcal{BM}(\Delta) \to \tau = [w_\mu|T] \in \mathcal{TQ}. \tag{3.7}$$

This map is onto. This can be shown by using the modified Beurling–Alhfors extension given in formula (2.19).

Suppose $\tau = [h] \in \mathcal{TQ}$. Let H be the lift homeomorphism of h to $\mathbb{R}$ with $H(0) = 0$. We still use H to mean the extended homeomorphism defined on the complex plane $\mathbb{C}$ by using formula (2.19). Then H is a K-quasiconformal homeomorphism of $\mathbb{C}$ where $K \leq M^2$, $M = \sup_{t>0} \mathrm{e}^{\varepsilon(t)}$, and $\overline{H(z)} = H(\overline{z})$. Moreover, in our case, $H(z + 1) = H(z) + 1$ for all $z = x + \mathrm{i}y \in \mathbb{C}$. This induces an extended K-quasiconfomal homeomorphism w of h on the Riemann sphere $\widehat{\mathbb{C}}$. Let $\mu = w_{\overline{z}}/w_z$ be the Beltrami coefficient of w, then $\mu \in \mathcal{BM}(\mathbb{C})$. Since

$\mu(z) = \mu(z^*)^*$, we have that $\mu|\Delta$, which we still denote as μ, is in $\mathcal{BM}(\Delta)$ and $[w|T] = [h] = \tau$. We will not provide the detailed calculation of this since they are standard. The map $\mathcal{P}$ in (3.7) is an onto map. We say that $\mu, \nu \in \mathcal{BM}(\Delta)$ are equivalent, denote as $\mu \sim \nu$, if $\mathcal{P}(\mu) = \mathcal{P}(\nu)$. One can check that it is an equivalent relation. Thus we can identify $\mathcal{TQ} = \mathcal{BM}/\sim$, the space of all equivalent classes $[\mu]$ in $\mathcal{BM}(\Delta)$ and $\mathcal{P}$ induces a bijective map, which we still denote as $\mathcal{P}$. Teichmüller's metric on $\mathcal{TQ}$ is, by definition,

$$d_T(\tau, \eta) = \inf\{d_{\mathcal{BM}}(\mu, \nu) \mid \mu, \nu \in \mathcal{BM}(\Delta), \mu \in \tau, \nu \in \eta\}. \tag{3.8}$$

The space $(\mathcal{TQ}, d_T(\cdot, \cdot))$ is called the universal Teichmüller space. We will not give a detailed calculation of this because it is standard. The reader who is interested in it may refer to [1, 20]. Three spaces

$$(\mathcal{TA}, d_T(\cdot, \cdot)), \quad (\mathcal{TC}^{1+\omega}, d_T(\cdot, \cdot)), \quad (\mathcal{TS}, d_T(\cdot, \cdot))$$

are considered as sub-Teichmüller spaces of $(\mathcal{TQ}, d_T(\cdot, \cdot))$. We will give a detailed calculation about the asymptotical behavior of μ near T by following formula (2.26), which we write it as

$$|\mu(z)| \le C\varepsilon\Big(-\frac{\log|z|}{2\pi}\Big), \quad \text{as } |z| \to 1. \tag{3.9}$$

We define three subspaces of $\mathcal{BM}(\Delta)$. Let $A_r = \{z \mid r < |z| < 1\}$ and $C > 0$ means a constant. Define

$$\mathcal{BM}^a(\Delta) = \{\mu \in \mathcal{BM}(\Delta) \mid \mu|A_r \equiv 0 \text{ for some } 0 < r < 1\}, \tag{3.10}$$

$$\mathcal{BM}^\omega(\Delta) = \{\mu \in \mathcal{BM}(\Delta) \mid \|\mu|A_r\|_\infty \le C\omega(1-r),\ \forall 0 < r < 1\} \tag{3.11}$$

$$\mathcal{BM}^0(\Delta) = \{\mu \in \mathcal{BM}(\Delta) \mid \|\mu|A_r\|_\infty \to 0 \text{ as } r \to 1\} \tag{3.12}$$

Inequality (3.9) implies that if $h \in \mathcal{C}^{1+\omega}$, then, for Beurling–Ahlfors extension w, $\mu_w \in \mathcal{BM}^\omega(\Delta)$ and if $h \in \mathcal{S}$, then, for Beurling–Ahlfors extension w, $\mu_w \in \mathcal{BM}^0(\Delta)$.

Suppose $\mu \in \mathcal{BM}^0(\Delta)$. Reflecting it with respect to T, we have a Beltrami cofficient which we still denote it as $\mu \in \mathcal{BM}(\mathbb{C})$. Remember that we use w_μ to denote the normalized solution of (3.4) fixing -1, 1, i. Let

$$A_n = \Big\{z \in \mathbb{C} \;\Big|\; 1 - \frac{1}{n} < |z| < 1 + \frac{1}{n}\Big\}$$

for any integer $n > 1$. Define $\mu_n = \mu$ on $\Delta \setminus A_n$ and $\mu_n = 0$ on A_n for any integer $n \ge 1$. Then we get a sequence of $\{\mu_n\}_{n=1}^\infty$ of Beltrami coefficients in $\mathcal{BM}^a(\mathbb{C})$ with $\mu_n(z) = \mu_n(z^*)^*$. Let w_{μ_n} be the normalized quasiconformal homeomorphism with Beltrami coefficient μ_n. Consider $w_{\nu_n} = w_\mu \circ w_{\mu_n}^{-1}$ where

$$\nu_n \circ w_{\mu_n} = \frac{\mu - \mu_n}{1 - \overline{\mu}_n \mu} \cdot \frac{\overline{(w_{\mu_n})_z}}{(w_{\mu_n})_z}.$$

Then $k_n = \|\nu_n\|_\infty \to 0$ as $n \to \infty$. Let $K_n = (1 + k_n)/(1 - k_n) \to 1$ as $n \to \infty$. For any four points z_1, z_2, z_3, and z_4 in T positively orientated,

$$Cr(w_\mu(z_1), w_\mu(z_2), w_\mu(z_3), w_\mu(z_4)) \le \lambda(K_n) \cdot Cr(w_{\mu_n}(z_1), w_{\mu_n}(z_2), w_{\mu_n}(z_3), w_{\mu_n}(z_4)).$$

Since w_{μ_n} is conformal on the annulus A_n, the Koebe distortion theorem implies that for any $z, w \in T$ with $|z - w| \le 1/n^2$,

$$\frac{|w'_{\mu_n}(z)|}{|w'_{\mu_n}(w)|} \le \frac{1 + n|z - w|}{(1 - n|z - w|)^3} \le \frac{1 + \frac{1}{n}}{(1 - \frac{1}{n})^3}.$$

Consider $h = w_\mu|T$ and H is its lift to $\mathbb{R}$. Take $z_1 = \mathrm{e}^{2\pi\mathrm{i}(x+t)}$, $z_4 = \mathrm{e}^{2\pi\mathrm{i}x}$, $z_3 = \mathrm{e}^{2\pi\mathrm{i}(x-t)}$, and $z_2 = \mathrm{e}^{2\pi\mathrm{i}(x+\pi)}$ for $x \in [0,1]$ and $|1 - \mathrm{e}^{2\pi\mathrm{i}t}| < 1/n^2$, we get

$$\frac{H(x) - H(x-t)}{H(x+t) - H(x)} \leq \widetilde{\lambda}\left(\frac{1}{n}\right)\lambda(K_n)\frac{1+\frac{1}{n}}{(1-\frac{1}{n})^3} = \mathrm{e}^{\varepsilon(\frac{1}{n})},$$

where we modify $\widetilde{\lambda}(t)$ in (3.6) such that it is still a bounded positive function with $\widetilde{\lambda}(t) \to 1$ as $t \to 0$ and $\varepsilon(t) \to 0$, as $t \to 0$, is a bounded positive function. Similarly, we have that

$$\frac{H(x) - H(x-t)}{H(x+t) - H(x)} \leq \mathrm{e}^{\varepsilon(1/n)}.$$

This implies that $h = w_\mu|T$ is a symmetric homeomorphism. Conclude from our calculation from the previous two paragraphs, we get that

$$\mathcal{P}(\mu) = [w_\mu|T] : \mathcal{BM}^0(\Delta) \to \mathcal{TS} \tag{3.13}$$

is an onto map. We will prove in the next section that

$$\mathcal{P}(\mu) = [w_\mu|T] : \mathcal{BM}^\alpha(\Delta) \to \mathcal{TC}^{1+\alpha} \tag{3.14}$$

is an onto map and only when ω is a Hölder modulus of continuity, we have this property.

4 Bers' Embedding on Circle Diffeomorphisms

The process from $\mu \in \mathcal{BM}(\Delta)$ to $\mathcal{P}(\mu) = [w_\mu|T]$ is not holomorphic because the reflection with respect to T is not. The holomorphic process is so called Bers' embedding.

Given $\mu \in \mathcal{BM}(\Delta)$, extend it on Δ_∞ by assigning value 0. We still use μ to denote it. Then it is a Beltrami coefficient in $\mathcal{BM}(\mathbb{C})$. By Theorem 3.1, we have the normalized quasiconformal homeomorphism w^μ with Beltrami coefficient μ. Then w^μ depends on $\mu \in \mathcal{BM}(\Delta)$ holomorphically.

Since w^μ on Δ_∞ is analytic and $w'(z) \neq 0$, we have the derivative

$$Dw = \log w',$$

which is a 0-form; the non-linearity

$$Nw = (\log w')' dz = \frac{w''}{w'}dz,$$

which is a 1-form; and the Schwarzian derivative

$$Sw = \left((Nw)' - \frac{1}{2}(Nw)^2\right)dz^2 = \left(\frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2\right)dz^2 = \phi(z)dz^2,$$

which is a 2-form. All three forms satisfy the chain rule:

$$D(w_2 \circ w_1) = Dw_1 + w_1^*(Dw_2),$$
$$N(w_2 \circ w_1) = Nw_1 + w_1^*(Nw_2),$$
$$S(w_2 \circ w_1) = Sw_1 + w_1^*(Sw_2).$$

Moreover, we have that $Dw = 0$ if and only if $w = \text{const}$, a constant function; $Nw = 0$ if and only if $w = az + b$, a linear function; and $Sw = 0$ if and only if $w = (az+b)/(cz+d)$, a Möbius transformation. Since ϕ is holomorphic on Δ_∞ and since the Schwarzian derivative is invariant under post composition of Möbius transformation, we are enable to embed the universal Teichmüller space into the space $\mathcal{QD}$ of all holomorphic quadratic differentials on Δ_∞,

$$\pi : \tau = [\mu] \in \mathcal{TQ} \hookrightarrow Sw^\mu \in \mathcal{QD}. \tag{4.1}$$

This definition is independent of the choice of representation μ in the equivalent class τ because if $\mathcal{P}(\mu) = \mathcal{P}(\nu)$, then $w_\mu|T = w_\nu|T$. Both $w^\mu \circ w_\mu^{-1}$ and $w^\nu \circ w_\nu^{-1}$ are holomorphic on Δ. Since $w_\mu|T = w_\nu|T$, we can extend the holomorphic map

$$(w^\mu \circ w_\mu^{-1}) \circ (w^\nu \circ (w_\nu)^{-1})^{-1} = w^\mu \circ (w_\mu^{-1} \circ w_\nu) \circ (w^\nu)^{-1}$$

on $w^\nu(\Delta)$ to $w^\nu(\Delta_\infty)$ by $w^\mu \circ (w^\nu)^{-1}$ homomorphically and to $w^\nu(T)$ continuously. Since $w^\nu(T)$ is a quasicircle, we get a holomorphic map on the Riemann sphere, which is a Möbius transformation. Since it fixes -1, 1, and i, this Möbius transformation must be the identity. This implies that $w^\mu = w^\nu$ and thus $Sw^\mu = Sw^\nu$ on Δ_∞. Conversely, if $Sw^\mu = Sw^\nu$ on Δ_∞, then $S(w^\mu \circ (w^\nu)^{-1}) = 0$, which implies that $w^\mu \circ (w^\nu)^{-1}$ is a Möbius transformation and thus the identity since it fixes -1, 1, i. So $w^\mu = w^\nu$ on $\Delta_\infty \cup T$. Since $w^\mu \circ w_\mu^{-1}$ and $w^\nu \circ w_\nu^{-1}$ are both conformal on Δ with the same image and both fix -1, 1, and i, they are the same, that is, $w^\mu \circ w_\mu^{-1} = w^\nu \circ w_\nu^{-1}$ on $\Delta \cup T$. This implies that $w_\mu|T = w_\nu|T$ and $\mathcal{P}(\mu) = \mathcal{P}(\nu)$.

Let

$$\rho_\infty(z)|dz| = \frac{|dz|}{|z|^2 - 1}$$

be the hyperbolic metric on Δ_∞. For a quadratic holomorphic differential $q = \phi dz^2$ in $\mathcal{QD}$, define its norm

$$\|q\| = \sup_{z \in \Delta_\infty} |\phi(z)\rho_\infty^{-2}(z)|.$$

Since w^μ is conformal on Δ_∞, we have that $\|Sw^\mu\| \leq 6$ (refer to [20]). This says that the embedding π in (4.1) embeds the universal Teichmüller space $\mathcal{TQ}$ into a bounded set in the complex Banach space $\mathcal{QD}$.

Given a modulus of continuity $\omega(t)$. Consider the Teichmüller space $\mathcal{TC}^{1+\omega}$. For any $\tau = [\mu] \in \mathcal{TC}^{1+\omega}$, (3.9) implies that we have a representation $\mu \in \tau$ such that $|\mu(z)| \leq C\omega(1-|z|)$ for all $z \in \Delta$ for some constant $C > 0$.

Given any point $z \in \Delta_\infty$, let $z^* \in \Delta$ be the reflection of z with respect to the unit circle. Let $r = 1 - |z^*|$. For any $0 < \beta < 1$, let

$$A_{r,\beta} = \{z \in \mathbb{C} \mid 1 - r^\beta < |z| < 1\}.$$

Define

$$\mu_{r,\beta}(\xi) = \begin{cases} \dfrac{\mu(\xi)}{C\omega(r^\beta)}, & \xi \in A_{r,\beta}; \\ 0, & \xi \in \mathbb{C}. \end{cases} \tag{4.2}$$

Since $\omega(\cdot)$ is an increasing function, we have that $\|\mu_{r,\beta}\|_\infty \leq 1$.

Let $w^{c\mu_{r,\beta}}$ be the normalized solution of the Beltrami equation (3.4) for $c \in \Delta$. We still use $Sw^{c\mu_{r,\beta}}$ to denote the coefficient of the holomorphic quadratic differential $Sw^{c\mu_{r,\beta}}$. The measurable Riemann Mapping Theorem implies that

$$f(c) = \rho_\infty^{-2}(z) Sw^{c\mu_{r,\beta}}(z) : \Delta \to \mathbb{C}$$

is an analytic function with upper bound 6. The Schwarz lemma implies that

$$|f(c)| = |\rho_\infty^{-2}(z) S w^{c\mu_{r,\beta}}(z)| \le 6|c|, \quad c \in \Delta$$

since $f(0) = 0$. In particular, when $c_0 = C\omega(r^\beta)$, we have that

$$|\rho_\infty^{-2}(z) S w^{c_0\mu_{r,\beta}}(z)| \le 6C\omega((1 - |z^*|)^\beta).$$

Note that $c_0\mu_{r,\beta}(\xi) = \mu(\xi)$ on $A_{r,\beta}$ and 0 on $\mathbb{C} \setminus A_{r,\beta}$.

Now consider the map $w^\nu = w^\mu \circ (w^{c_0\mu_{r,\beta}})^{-1}$. By the composition formula, we have

$$\nu = \frac{\mu - c_0\mu_{r,\beta}}{1 - \overline{c_0\mu_{r,\beta}}\mu}\theta, \quad \text{where } |\theta| = 1. \tag{4.3}$$

This implies that $\nu(\xi) = 0$ for all $\xi \in A_{r,\beta} \cup (\mathbb{C} \setminus \Delta)$. Thus w^ν is conformal on $\xi \in D_{r,\beta} = A_{r,\beta} \cup (\widehat{\mathbb{C}} \setminus \Delta)$. The hyperbolic metric on the disk $D_{r,\beta}$ is

$$\rho_{r,\beta}(\xi)|d\xi| = \frac{(1 - r^\beta)|d\xi|}{|\xi|^2 - (1 - r^\beta)^2}.$$

Since w^ν is a conformal map on $D_{r,\beta}$, we have that

$$|\rho_{r,\beta}^{-2}(\xi) S w^\nu(\xi)| \le 6, \quad \forall \xi \in D_{r,\beta}.$$

Therefore, we have that for any $\xi \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$,

$$|\rho_\infty^{-2}(\xi) S w^\nu(\xi)| = \frac{(1 - r^\beta)^2(|\xi|^2 - 1)^2}{(|\xi|^2 - (1 - r^\beta)^2)^2} |\rho_{r,\beta}^{-2}(\xi) S w^\nu(\xi)| \le 6\frac{(1 - r^\beta)^2(|\xi|^2 - 1)^2}{(|\xi|^2 - (1 - r^\beta)^2)^2}.$$

When $\xi = z$, we have that $|z| = 1/(1 - r)$,

$$|\rho_\infty^{-2}(z) S w^\nu(z)| \le C_1 r^{2(1-\beta)},$$

where $C_1 > 0$ is a constant.

Since $w^\nu = w^\mu \circ (w^{c_0\mu_{r,\beta}})^{-1}$, we have that

$$\rho_\infty^{-2}(z) S w^\nu(z) = \rho_\infty^{-2}(z) S w^\mu(z) - \rho_\infty^{-2}(z) S w^{c_0\mu_{r,\beta}}(z).$$

Thus we have that

$$\begin{aligned} |\rho_\infty^{-2}(z) S w^\mu(z)| &\le |\rho_\infty^{-2}(z) S w^\nu(z)| + |\rho_\infty^{-2}(z) S w^{c_0\mu_{r,\beta}}(z)| \\ &\le C_1(1 - |z^*|)^{2(1-\beta)} + 6C\omega((1 - |z^*|)^\beta) \\ &\le C_2\left(\left(1 - \frac{1}{|z|}\right)^{2(1-\beta)} + \omega\left(\left(1 - \frac{1}{|z|}\right)^\beta\right)\right), \end{aligned} \tag{4.4}$$

where $C_2 > 0$ is a constant.

Replacing $\omega(t)$ by $\varepsilon(t)$ for each $[\mu] \in \mathcal{TS}$ in the above calculation, Inequality (4.4) becomes

$$|\rho_\infty^{-2}(z) S w^\mu(z)| \le C_2\left(\left(1 - \frac{1}{|z|}\right)^{2(1-\beta)} + \varepsilon\left(\left(1 - \frac{1}{|z|}\right)^\beta\right)\right), \tag{4.5}$$

Define

$$\mathcal{QD}^0 = \left\{ q = \phi dz^2 \in \mathcal{QD} \,\Big|\, \sup_{1<|z|\le r} |\rho_\infty^{-2}(z) q(z)| \to 0, \ r \to 1 \right\}. \tag{4.6}$$

Then we have an embedding

$$\pi : [\mu] \in \mathcal{TS} \hookrightarrow S w^\mu \in \mathcal{QD}^0. \tag{4.7}$$

Combining with (3.13), we have a chain

$$\mathcal{BM}^0(\Delta) \xrightarrow{\mathcal{P}} \mathcal{TS} \overset{\pi}{\hookrightarrow} \mathcal{QD}^0, \tag{4.8}$$

where $\mathcal{P}$ is onto and π is an embedding. This fact combining with a similar argument to that in the last two paragraphs in this section, we have

Theorem 4.1 ([13]) *One can give the Bers complex manifold structure on $\mathcal{TS}$.*

We continue on the smooth case and will prove a similar result in the smooth case. We already knew that if $\tau \in \mathcal{TC}^{1+\omega}$, then there are a representation $\mu \in \tau$ and a constant $C > 0$ such that

$$|\mu(z)| \le C\omega(1-|z|), \quad \text{for a.e. } z \in \Delta. \tag{4.9}$$

We need to prove the converse.

Given $\mu \in \mathcal{BM}^\omega(\Delta)$, we still consider $w^{c\mu_{r,\beta}}$ with $\mu_{r,\beta}$ defined in (4.2). Then $g(z) = w^{c\mu_{r,\beta}}(z)$ is an analytic function defined on Δ_∞ with $g(\infty) = \infty$. Consider $f(\xi) = 1/g(1/z)$. Then it is an analytic function defined on Δ with $f(0) = 0$ and

$$f'(\xi) = g'(z)\frac{z^2}{g(z)^2}. \tag{4.10}$$

From the Measurable Riemann Mapping Theorem (Theorem 3.1), we know $g(z) = z + b_0 + b_1/z + \cdots$, so $f'(0) = 1$. From the area theorem for univalent functions (see [20]), we have that

$$\left|(1-|\xi|^2)\frac{f''(\xi)}{f'(\xi)} - 2\overline{\xi}\right| \le 4, \quad \forall \xi \in \Delta.$$

Since $f(\Delta)$ is a quasi-disk, from the Carathéodory theorem (see [24]), f or g can be extended as a homeomorphism on $\overline{\Delta}$ or $\overline{\Delta}_\infty$, respectively. Then (4.10) implies that g' and f' have the same modulus of continuity on T, which is the common boundary of both Δ and Δ_∞. What we have seen is that to study the modulus of continuity of the derivative of $w^{c\mu_{r,\beta}}(z)$ on T, we can use the model of Δ_∞ with the coordinate $\xi = 1/z$. Under this model, we have that

$$\left|\left(1-\frac{1}{|z|}\right)Nw^{c\mu_{r,\beta}}(z)\right| \le 6.$$

Following this, we have that for any fixed $z \in \Delta_\infty$, $k(c) = (1-1/|z|)Nw^{c\mu_{r,\beta}}(z)$ is an analytic function of $c \in \Delta$ with upper bound 6 and $k(0) = 0$. The Schwarz lemma implies that

$$\left|\left(1-\frac{1}{|z|}\right)Nw^{c\mu_{r,\beta}}(z)\right| \le 4|c|.$$

In particular, when $c_0 = C\omega(r^\beta)$,

$$\left|\left(1-\frac{1}{|z|}\right)Nw^{c_0\mu_{r,\beta}}(z)\right| \le 4\omega\left(\left(1-\frac{1}{|z|}\right)^\beta\right).$$

Remember that $c_0\mu_{r,\beta}(\xi) = \mu(\xi)$ on $A_{r,\beta}$ and 0 on $\mathbb{C} \setminus A_{r,\beta}$.

Since $w^\nu = w^\mu \circ (w^{c_0\mu_{r,\beta}})^{-1}$ is conformal on $D_{r,\beta} = A_{r,\beta} \cup (\widehat{\mathbb{C}} \setminus \Delta)$. Similar to what we have done for the Schwarzian derivative, we have that

$$\left|\left(1-\frac{1}{|z|}\right)Nw^\mu(z)\right| \le C_3\left(\left(1-\frac{1}{|z|}\right)^{1-\beta} + \omega\left(\left(1-\frac{1}{|z|}\right)^\beta\right)\right).$$

Let $z = re^{2\pi i\theta}$, then we have that

$$|Dw^{\mu}(z) - Dw^{\mu}(e^{2\pi i\theta})| \le C_4\bigg(\Big(1-\frac{1}{|z|}\Big)^{1-\beta} + \int_1^r \frac{\omega((1-\frac{1}{s})^{\beta})}{1-\frac{1}{s}}ds\bigg).$$

(Note that C_4 may depend on β.) Define

$$\widehat{\omega}(t) = t^{1-\beta} + \int_0^t \frac{\omega(s^{\beta})}{s}ds. \tag{4.11}$$

If $\widetilde{\omega}(t) \to 0$ as $t \to 0$ ($\Longleftrightarrow \widetilde{\omega}(1) < \infty$), then $Dw^{\mu}|T$ exists and is a $\widetilde{\omega}$-continuous function.

On the other hand, we have an explicit formula for $\dot{w}^{\mu_{r,\beta}} = dw^{c\mu_{r,\beta}}/dc|_{c=0}$,

$$\dot{w}^{\mu_{r,\beta}}(z) = \frac{1}{2\pi i}\int\int_{\mathbb{C}} \mu_{r,\beta}(\xi)\frac{z(z-1)}{\xi(\xi-1)(\xi-z)}d\xi \wedge d\overline{\xi}.$$

If $0 < c \le |\mu_{r,\beta}(\xi)| \le C$ for all $\xi \in A_{r,\beta}$, then

$$k'(0) = N\dot{w}^{\mu_{r,\beta}}(z) \neq 0.$$

The Keobe 1/4-Theorem implies that there is a constant $C_5 > 0$ such that

$$\bigg|\Big(1-\frac{1}{|z|}\Big)Nw^{c_0\mu_{r,\beta}}(z)\bigg| \ge C_5\omega\bigg(\Big(1-\frac{1}{|z|}\Big)^{\beta}\bigg) \ge C_5\omega\Big(1-\frac{1}{|z|}\Big).$$

So we have a constant $C_6 > 0$ such that

$$|Dw^{\mu}(z) - Dw^{\mu}(e^{2\pi i\theta})| \ge C_6\bigg(\Big(1-\frac{1}{|z|}\Big)^{1-\beta} + \int_1^r \frac{\omega(1-\frac{1}{s})}{1-\frac{1}{s}}ds\bigg). \tag{4.12}$$

Define

$$\widetilde{\omega}(t) = \int_0^t \frac{\omega(s)}{s}ds. \tag{4.13}$$

Thus, $Dw^{\mu}|T$ is exact $\widehat{\omega}$-continuous. Therefore, if we want $\widetilde{\omega}(t)$ is in the same class as $\omega(t)$, that is,

$$\widetilde{\omega}(t) = C\omega(t), \tag{4.14}$$

then $\omega(t)/t = C\omega'(t)$, which implies that $\omega(t) \approx t^{\alpha}$ for some $0 < \alpha \le 1$. In this case, $\widehat{\omega}(t) \approx t^{\beta\alpha}$ (when β closes to 1).

Concluding from our calculations we just did, we define

$$\mathcal{TC}^{1+H} = \bigcup_{0<\alpha\le 1} \mathcal{TC}^{1+\alpha} \tag{4.15}$$

as the Teichmüller space of all C^1 circle diffeomorphisms whose derivative is Hölder continuous functions. For $q = \phi dz^2 \in \mathcal{QD}$ and $0 < \alpha \le 1$, define

$$\|q\|_{\alpha} = \sup_{z\in\Delta_{\infty}} \bigg|\Big(1-\frac{1}{|z|}\Big)^{-\alpha}\rho_{\infty}^{-2}(z)q(z)\bigg|.$$

We define a subspace

$$\mathcal{QD}^H = \{q = \phi dz^2 \in \mathcal{QD} \mid \|q\|_{\alpha} < \infty \text{ for some } 0 < \alpha \le 1\}. \tag{4.16}$$

It is a complex Banach vector space. From our above calculation, we have an embedding

$$\pi : [\mu] \in \mathcal{TC}^{1+H} \hookrightarrow Sw^{\mu} \in \mathcal{QD}^H. \tag{4.17}$$

For the circle homeomorphism $w_\mu|T$, we can weld it by two conformal maps w^μ defined on Δ_∞ and w^{μ^*} defined on Δ. Here we define $\mu^*(z) = \mu(z^*)^*$ on Δ_∞ and 0 on Δ. Similarly, we have that $Dw^{\mu^*}|T$ exists and is an $\widetilde{\omega}$-continuous function. Thus $w_\mu|T = (w^{\mu^*})^{-1} \circ w^\mu$ is a $C^{1+\widetilde{\omega}}$-diffeomorphism of T. In particular, when $\omega(t) = t^\alpha$ for some $0 < \alpha \le 1$, then $\widetilde{\omega}(t) = t^\beta$ for some $0 < \beta \le 1$. Let

$$\mathcal{BM}^H(\Delta) = \bigcup_{0<\alpha\le 1} \mathcal{BM}^\alpha(\Delta)$$

Thus we a chain

$$\mathcal{BM}^H(\Delta) \xrightarrow{\mathcal{P}} \mathcal{TC}^{1+H} \overset{\pi}{\hookrightarrow} \mathcal{QD}^H. \tag{4.18}$$

Now we are ready to give the Bers complex manifold structure on $\mathcal{TC}^{1+H}$. We will still work on it for arbitrary modulus of continuous ω, and eventually see, why it only works for the Hölder case.

Let $\mathcal{QD}^{1+\omega}(2)$ be the open ball of $\mathcal{QD}^{1+\omega}$ of radius 2. Following Ahlfors and Weill [3], for any $q = \phi dz^2$ in this ball, we define

$$\mu(z) = -\frac{1}{2}\phi\Big(\frac{1}{\bar z}\Big)\Big(\frac{z}{\bar z}\Big)^2\Big(1-\Big|\frac{1}{z}\Big|^2\Big)^2 \quad \text{for } z\in\Delta; \quad 0 \text{ for } z \in \Delta_\infty.$$

Let w^μ be the normalized solution of the Beltrami equation (3.4) with the Beltrami coefficient μ.

Take two independent solutions w_1 and w_2 of the ordinary differential equation

$$w'' = -\frac{\phi}{2}w \quad \text{on } \Delta_\infty.$$

Then we have that

$$w^\mu(z) = \frac{w_1(z)}{w_2(z)} \quad \text{for } z\in\Delta_\infty; \quad \frac{w_1(\frac{1}{\bar z}) + (z-\frac{1}{\bar z})w_1'(\frac{1}{\bar z})}{w_2(\frac{1}{\bar z}) + (z-\frac{1}{\bar z})w_2'(\frac{1}{\bar z})} \quad \text{for } z \in \Delta$$

and $Sw^\mu = \phi$ on Δ_∞. Since $\mu \in \mathcal{BM}^\omega(\Delta)$, we have a holomorphic embedding

$$\iota(q) = \mu(q) : \mathcal{QD}^{1+\omega}(2) \to \mathcal{BM}^\omega(\Delta).$$

As we have already seen that $\mathcal{P}(\mu) = [w_\mu|T] \in \mathcal{TC}^{1+\widetilde{\omega}}(\Delta)$, so in order that the following chain (4.19) stays in the same class we need $\omega(t) = t^\alpha$ for some $0 < \alpha \le 1$ and then $\widetilde{\omega}(t) = t^{\widetilde{\alpha}}$ for some $0 < \widetilde{\alpha} \le 1$. In this case, we have a homeomorphism

$$\mathcal{P}\circ\iota : \mathcal{QD}^H(2) \to \mathcal{U} = h_0(\mathcal{QD}^H(2)) \subset \mathcal{TC}^{1+H}. \tag{4.19}$$

Let h_0 be the inverse. Then $(h_0, \mathcal{U})$ is a local chart at the basepoint $\mathcal{P}(0)$.

Given any point $\tau = \mathcal{P}(\mu)$ in $\mathcal{TC}^{1+H}$ for $\mu \in \mathcal{BM}^H(\Delta)$. Consider w^μ and Sw^μ. For any $\mathcal{P}(\nu) \in \mathcal{U}$ for $\nu \in \mathcal{BM}^H(\Delta)$, consider the map $w^\delta = w^\mu \circ w^\nu$, then $\delta \in \mathcal{BM}^H(\Delta)$. It defines a homeomorphism $\mathcal{P}(\nu) \to \mathcal{P}(\delta)$ mapping $\mathcal{U}$ to $\mathcal{U}_\tau$ homeomorphically. Let β_τ be the inverse of this homeomorphism and

$$h_\tau = h_0 \circ \beta_\tau : \mathcal{U}_\tau \to \mathcal{QD}^H(2).$$

Then $(h_\tau, \mathcal{U}_\tau)$ is a chart at τ. Thus we have a cover of charts on $\mathcal{TC}^{1+H}(\Delta)$,

$$\mathcal{C} = \{(h_\tau, \mathcal{U}_\tau)\}_{\tau\in\mathcal{TC}^{1+H}},$$

such that the transition map for any two charts is

$$h_{\tau,\tau'} = h_\tau \circ h_{\tau'}^{-1} : h_{\tau'}(\mathcal{QD}^H(2)) \to h_\tau(\mathcal{QD}^H(2))$$

whenever $\mathcal{U}_{\tau'} \cap \mathcal{U}_\tau \neq \emptyset$.

Let $\tau = \mathcal{P}(\mu)$ and $\tau' = \mathcal{P}(\mu')$ for any $\mu, \mu' \in \mathcal{BM}^H(\Delta)$ and let $\mathcal{P} \circ \iota(q) = [\mu(q)]$ for $q \in \mathcal{QD}^H$. From our above calculation, one can see the transition map is

$$h_{\tau,\tau'}(q) = q + (w^{\mu(q)})^*(S((w^\mu)^{-1} \circ w^{\mu'})).$$

From the Measurable Riemann Mapping Theorem (Theorem 3.1), it is a holomorphic map. Thus the cover of charts $\mathcal{C}$ gives us the Bers type complex manifold structure model on the complex Banach space $\mathcal{QD}^H$. Under this Bers type complex manifold structure, the smooth Teichmüller space $\mathcal{TC}^{1+H}$ becomes a sub-complex manifold of the universal Teichmüller space $\mathcal{TQ}$ and a sub-complex manifold of the universal aymptotically conformal Teichmüller space $\mathcal{TS}$. We have already proved the following theorem.

Theorem 4.2 *We can give the Bers complex manifold structure on the smooth Teichmüller space $\mathcal{TC}^{1+H}$. And it is the largest space in the space of all C^1 circle diffeomorphisms on which we can assign the Bers complex manifold structure.*

5 Teichmüller's Metric and Kobayashi's Metric

In the previous section, we have shown that the smooth Teichmüller space $\mathcal{TC}^{1+H}$ can be given the Bers complex manifold structure which is compatible with the Bers complex manifold structure of the universal Teichmüller structure $\mathcal{TQ}$ and with the Bers complex manifold structure of the universal asymptotically conformal Teichmüller structure $\mathcal{TQ}$. As a subspace, $\mathcal{TC}^{1+H}$ has an induced Teichmüller's metric $d_T(\cdot,\cdot)$ on it. As a complex manifold, it has Kobayashi's metric on it. In this section, we will prove they are the same.

First let us give a brief introduction of Kobayashi's metric. Consider the unit disk Δ as a hyperbolic Riemann surface, its hyperbolic metric

$$d_\Delta(z,w) = \tanh^{-1}(z,w) = \frac{1}{2}\log\frac{1+\frac{|z-w|}{|1-\overline{z}w|}}{1-\frac{|z-w|}{|1-\overline{z}w|}}, \quad z,w \in \Delta. \tag{5.1}$$

is Kobayashi's metric on it. Consider a connected complex manifold M modeled by a complex Banach space, let $\mathcal{H}(\Delta, M)$ be the space of all holomorphic maps from Δ into M. Kobayashi's pseudo-metric $d_K(\cdot,\cdot)$ on M is defined to be the largest pseudo metric on M such that

$$d_K(f(z), f(w)) \le \rho_\Delta(z,w), \quad \forall z,w \in \Delta \quad \text{and} \quad \forall f \in \mathcal{H}(\Delta, M). \tag{5.2}$$

Another way to describe Kobayashi's metric on M is as follows. Given $p, q \in M$, consider the subspace $\mathcal{H}_{p,q}$ consisting of all $f \in \mathcal{H}(\Delta, M)$ such that $f(0) = p$ and $f(s) = q$ for some $s \in \Delta$. Let $r = \inf_{f\in\mathcal{H}_{p,q}} |s|$ and

$$d_1(p,q) = \frac{1}{2}\log\frac{1+r}{1-r}. \tag{5.3}$$

Note that if $\mathcal{H}_{p,q} = \emptyset$, then $d_1(p,q) = \infty$. Now consider the space

$$\mathcal{C}_n = \{p_0 = p, p_1, \ldots, p_{n-1}, p_n = q\}$$

of n-chains connecting $p, q \in M$ and define

$$d_n(p,q) = \inf_{\mathcal{C}_n} \sum_{i=1}^n d_1(p_{i-1}, p_i). \tag{5.4}$$

It is clear that $d_{n+1}(p,q) \le d_n(p,q)$ for all $n > 0$ and $p, q \in M$. Kobayashi's pseudo-metric is

$$d_K(p,q) = \lim_{n\to\infty} d_n(p,q), \quad p, q \in M. \tag{5.5}$$

By using this description of Kobayashi's metric, one can show that for the complex plane $\mathbb{C}$, its Kobayashi's pseudo metric is 0 and that for the hyperbolic disk Δ, its Kobayashi's pseudo metric is the hyperbolic metric. Furthermore, using this description, one can see that a holomorphic map contracts Kobayashi's metrics. Precisely, let M and M' be two complex manifolds, let $F : M \to M'$ be a holomorphic map, then

$$d'_K(F(p), F(q)) \le d_K(p,q), \quad \forall p, q \in M,$$

where $d_K(\cdot,\cdot)$ and $d_K(\cdot,\cdot)$ are Kobayashi's metrics on M and M', respectively.

It is well-known among experts that when $M = \mathcal{TQ}$ the universal Teichmüller space, Kobayashi's metric and Teichmüller's metric coincide, that is, $d_K(\cdot,\cdot) = d_T(\cdot,\cdot)$ (see [7, 8, 11, 25]). It is also known that when $M = \mathcal{TS}$ the universal asymptotically conformal Teichmüler space, Kobayashi's metric and Teichmüller's metric coincide, that is, $d_K(\cdot,\cdot) = d_T(\cdot,\cdot)$ (see [6, 14]). We now prove that

Theorem 5.1 *For $M = \mathcal{TC}^{1+H}$, Kobayashi's metric and Teichmüller's metric coincide.*

Let us use $d_{T,H}(\cdot,\cdot)$ and $d_{K,H}(\cdot,\cdot)$ to denote Techmüller's metric and Kobayashi's metric on $\mathcal{TC}^{1+H}$, respectively. Then for any two points $\tau, \eta \in \mathcal{TC}^{1+H}$,

$$d_{T,H}(\tau,\eta) = \inf\{d_{\mathcal{BM}}(\mu,\nu) \mid \mu,\nu \in \mathcal{BM}(\Delta), \mathcal{P}(\mu) = \tau, \mathcal{P}(\nu) = \eta\},$$

where

$$d_{\mathcal{BM}}(\mu,\nu) = \tanh^{-1}\left\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\right\|_\infty = \frac{1}{2}\log\left(\frac{1+\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty}{1-\|\frac{\mu-\nu}{1-\overline{\mu}\nu}\|_\infty}\right).$$

Furthermore, we have

$$d_{T,H}(\tau,\eta) = \frac{1}{2}\inf_{\mu\in\tau,\nu\in\eta}\log K(w_\mu \circ w_\nu^{-1}) = \frac{1}{2}\inf_{\mu\in\tau,\nu\in\eta}\log K(w^\mu \circ (w^\nu)^{-1}).$$

Let $d_K(\cdot,\cdot)$ and $d_T(\cdot,\cdot)$ be Teichmüller's metric and Kobayashi's metric on the universal Teichmüller space $\mathcal{TQ}$. From the definition, we have that

$$d_{K,H}(\tau,\eta) \ge d_K(\tau,\eta) = d_T(\tau,\eta) = d_{T,H}(\tau,\eta)$$

for any $\tau, \eta \in \mathcal{TC}^{1+H}$. Thus we have the easy part in the proof of Theorem 5.1,

Lemma 5.2 *For any two points $\tau, \eta \in \mathcal{TC}^{1+H}$,*

$$d_{K,H}(\tau,\eta) \ge d_{T,H}(\tau,\eta). \tag{5.6}$$

The hard part in the proof of Theorem 5.1 is

Lemma 5.3 *For any two points $\tau, \eta \in \mathcal{TC}^{1+H}$,*

$$d_{K,H}(\tau,\eta) \le d_{T,H}(\tau,\eta). \tag{5.7}$$

We will divide the proof of Lemma 5.3 into trhee subsections. For any $\nu \in \eta \cap \mathcal{TC}^{1+H}$, we can define a right translation (biholomorphicmap) $[w_\mu|T] \to [w_\mu \circ (w_\nu)^{-1}|T]$ on $\mathcal{TC}^{1+H}$ which moves η to $\mathcal{P}(0) = [0]$ and preserves both Teichmüller's metric and Kobayashi's metric. Thus to prove (5.7) for any points $\tau, \eta \in \mathcal{TC}^{1+H}$, we only need to prove

$$d_{K,H}([0], \tau) \le d_{T,H}([0], \tau). \tag{5.8}$$

Before to prove this inequality, we review some properties in Teichmüller theory without proofs. The reader who is interested in them may refer to [12, 19, 26].

5.1 Extremal Point

Suppose ϕ is a holomorphic function on Δ. Let

$$\|\phi\| = \int_\Delta |\phi(z)| dxdy, \quad z = x + \mathrm{i}y.$$

Given a point $\tau = [\mu] \in \mathcal{TQ}$, let

$$k_0 = \inf_{\mu \in \tau} \|\mu\|_\infty.$$

From the normal family theory in quasiconformal theory, we have a $\mu_0 \in \tau$ such that $\|\mu_0\|_\infty = k_0$. We call μ_0 an extremal point in τ.

A sequence $\{\varphi_n\}$ of holomorphic functions is called a *Hamilton sequence* for μ_0 if $\|\phi_n\| = 1$ and $\lim_{n\to\infty} \sup \int_\Delta \mu_0 \varphi_n dxdy = \|\mu_0\|_\infty$.

Theorem 5.4 (Hamilton–Krushkal Theorem) *Given any point $\tau = [\mu] \in \mathcal{TQ}$, if $\mu_0 \in \tau$ is an extremal point, then μ_0 has a Hamilton sequence $\{\phi_n\}$.*

5.2 Frame Point

Given a point $\tau = [\mu] \in \mathcal{TQ}$, an element $\mu_1 \in \tau$ is called a frame point if there is a compact set $D \subset \Delta$ such that

$$\|\mu_1 | (\Delta \setminus D)\|_\infty < k_0.$$

Lemma 2.9 says that if $\tau \neq [0] \in \mathcal{TC}^{1+H}$, then it always has a frame point.

Theorem 5.5 (Strebel's Frame Mapping Theorem) *For ant $\tau \neq [0] \in \mathcal{TQ}$, if it has a frame point, then it has a unique extremal point μ_0 in the Teichmüller form,*

$$\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0},$$

for a holomorphic function φ_0 with $\|\phi_0\| = 1$. Moreover, for any $\nu \in \tau$,

$$K_0 = \frac{1+k_0}{1-k_0} \le \int_\Delta \frac{|1 + \nu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\nu|^2} |\varphi_0| dxdy.$$

5.3 Holomorphic Functions

Suppose $\{\varphi_n\}$ is a sequence of holomorphic functions with $\|\phi_n\| = 1$. Suppose $D \subset \Delta$ is a compact subset. We claim that $\{\varphi_n\}$ is uniformly bounded on D. We prove the claim by contradiction. Suppose not, then there exists a sequence of points $\{z_n\} \subset D$ and a subsequence of $\{\varphi_n\}$, still denoted by $\{\varphi_n\}$, such that $|\varphi_n(z_n)| \ge n$. Since D is compact, $\{z_n\}$ has an accumulation point $z_0 \in D$. Then there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that z_n converges to z_0. Choose a small $r > 0$ such that the closed disk $D_r(z_0) = \{|z - z_0| \le r\} \subset \Delta$. Then $z_n \in D_{r/4}(z_0)$ when n is large enough, say $n > N$.

For any $n > N$, one can apply the Cauchy integral formula for $\varphi_n(z_n)$ to obtain

$$n \leq |\varphi_n(z_n)| \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} \frac{|\varphi_n(z)|}{|z-z_n|} r' d\theta$$

for each $\frac{r}{2} \leq r' \leq r$. And then

$$n \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| \frac{4}{r} r d\theta = \frac{2}{\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta.$$

Multiplying the previous inequality by r' and integrating both sides in radial direction from $\frac{r}{2}$ to r, we obtain

$$\frac{3}{8} n r^2 = n \int_{\frac{r}{2}}^{r} r' dr' \leq \frac{2}{\pi} \int_{\frac{r}{2}}^{r} r' \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta dr' \leq \frac{2}{\pi} \|\varphi_n\| = \frac{2}{\pi}.$$

Hence $\frac{3}{8} n r^2 \leq \frac{2}{\pi}$ for any $n > N$. This is a contradiction when n is large enough. We proved the claim.

Applying the Cauchy integral formula for derivatives $\{\phi_n'\}$, one can see it is also uniformly bounded on D and thus $\{\phi_n\}$ is a uniformly bounded equi-continuous family. The Ascoli–Arzela Theorem implies $\{\phi_n\}$ has a convergent subsequence, still denoted as $\{\phi_n\}$, on D. Taking an increasing sequence of compact sets $\{D_m\}$ such that $\Delta = \bigcup_m D_m$, we get a convergent subsequence of $\{\phi_n\}$, still denoted as $\{\phi_n\}$, on Δ. Suppose ϕ_0 is its limiting function. By Fatou's Lemma, $\|\phi_0\| \leq 1$.

5.4 The Proof of Lemma 5.3

For any $\tau \in \mathcal{TC}^{1+H}$, take $\mu \in \tau$ in Lemma 2.9. Let $k = \|\mu\|_\infty$. Let

$$\Delta_n = \left\{ z \in \Delta \;\middle|\; |z| < r_n = 1 - \frac{1}{n} \right\} \quad \text{and} \quad A_n = \Delta \setminus \Delta_n.$$

Let $l_n = \|\mu | A_n\|_\infty$. Lemma 2.9 implies that $l_n < k_0$ for n large enough, say $n > N$. So μ is a frame point in τ. This implies that τ has a unique extremal point μ_0 in the Teichmüller form $\mu_0 = k_0 |\phi_0| / \phi_0$ for some holomorphic function ϕ_0 with $\|\phi_0\| = 1$. Moreover, $0 < k_0 < k$.

Let $f_n(z) = w_\mu(r_n z)$. It maps Δ to a quasi-disk $D_n = f_n(\Delta)$. Let $g_n : D_n \to \Delta$ be the Riemann mapping. Then $h_n = g_n \circ f_n$ is a quasiconformal self-homeomorphism of Δ and $\tau_n = [h_n | T]$ is in $\mathcal{TQ}$. From Lemma 2.9, for N large enough, every point τ_n has a frame point for $n > N$. Thus for every $n > N$, τ_n has a unique extremal point $\mu_{n,0}$ in the Teichmüller form,

$$\mu_{n,0} = k_{n,0} \frac{|\phi_{n,0}|}{\phi_{n,0}}$$

with a holomorphic function $\phi_{n,0}$ with $\|\phi_{n,0}\| = 1$. By our definition, one can see that $k_{n,0} \geq k_0$ for all $n > N$.

Now we define $F_n(z) = g_n^{-1} \circ w_{\mu_{n,0}}(z/r_n)$ for $z \in \Delta_n$ and $F_n(z) = w_\mu(z)$ for $z \in A_n$. It agrees on the circle $\partial \Delta_n$. Thus it is a quasiconformal self-homeomorphism of Δ. The Beltrami coefficient ν_n of F_n is $\mu_{n,0}(z/r_n)$ on Δ_n and μ on A_n. Thus $\nu_n \in \tau \in \mathcal{TC}^{1+\alpha}$. And $\|\nu_n\|_\infty > k_0$. We have a holomorphic map

$$p(c) = \left[c \frac{\nu_n}{\|\nu_n\|_\infty} \right] : \Delta \to \mathcal{TC}^{1+\alpha}$$

such that $p(0) = [0]$ and $p(\|\nu_n\|_\infty) = \tau$. This implies that

$$d_{K,\alpha}([0], \tau) \leq d_1([0], \tau) \leq \frac{1}{2} \log \frac{1 + \|\nu_n\|_\infty}{1 - \|\nu_n\|_\infty}.$$

Our final step is to prove $\|\nu_n\|_\infty \to k_0$ as $n \to \infty$.

From Subsection 5.3, there exists a subsequence of $\{\varphi_{n,0}\}$, still denoted by $\{\varphi_{n,0}\}$, converging uniformly to a holomorphic function $\widehat{\varphi}$ on any compact subset $D \subset \Delta$. Furthermore, $\|\widehat{\phi}\| \leq 1$. We claim that $\|\widehat{\phi}\| > 0$. We prove the claim by contradiction.

Suppose $\|\widehat{\varphi}\| = 0$. Then $\{\varphi_{n,0}\}$ has a subsequence, we still denote by $\{\phi_{n,0}\}$, converging uniformly to zero on any compact subset $D \subset \Delta$. For any $\epsilon > 0$, we first choose a compact subset $D \subset \Delta$ such that

$$\|\mu|(\Delta \setminus D)\|_\infty < \epsilon.$$

There exists $N_1 > N$ such that

$$\int_D |\varphi_{n,0}(z)| dxdy \leq \epsilon$$

and such that $D \subset \Delta_n$ for all $n > N_1$.

From Subsection 5.2,

$$K_{n,0} = \frac{1 + k_{n,0}}{1 - k_{n,0}} \leq \int_\Delta \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy.$$

This says

$$K_{n,0} \leq \int_{\Delta \setminus D} \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy + \int_D \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy.$$

Then, for $K = (1 + k)/(1 - k)$,

$$K_{n,0} \leq \int_{\Delta \setminus D} \frac{1 + \epsilon}{1 - \epsilon} |\varphi_{n,0}| dxdy + K \int\int_D |\varphi_{n,0}| dxdy,$$

and hence

$$K_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} \int_\Delta |\varphi_{n,0}| dxdy + \left(K - \frac{1 + \epsilon}{1 - \epsilon}\right) \int\int_D |\varphi_{n,0}| dxdy.$$

Therefore

$$1 < k_0 < k_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} + \left(K - \frac{1 + \epsilon}{1 - \epsilon}\right) \cdot \epsilon.$$

This is a contradiction when ϵ is sufficient small. Therefore $\|\widehat{\varphi}\| > 0$.

Now let $\widehat{\mu} = \widehat{k} \frac{|\widehat{\varphi}|}{\widehat{\varphi}}$, where $\widehat{k} = \lim_{n \to \infty} k_{n,0}$ (by taking a limit of a convergent subsequence if it is necessary). Then $\mu_{n,0} \to \widehat{\mu}$ a.e. on Δ. By the convergence theorem (see [20, Theorem 4.6]) of families of quasiconformal maps, we obtain

$$\lim_{n \to \infty} w_{\mu_{n,0}}|T = w_{\mu_{n,0}}|T = w_\mu|T = w_{\widehat{\mu}}|T.$$

By the uniqueness of the extremal point in τ, $\widehat{k} = k_0$. Thus $k_{n,0} \to k_0$ as $n \to \infty$ for a subsequence of n's. We completed the proof of Lemma 5.3. Both Lemmas 5.2 and 5.3 give a proof of Theorem 5.1.

Remark 5.6 (Added Remark) We can only give the Bers complex Banach manifold structure on the union space $\mathcal{TC}^{1+H}$ of $\mathcal{TC}^{1+\alpha}$ over all $0 < \alpha \leq 1$ due to the estimation before (4.11), which is the largest space in the space of all C^1 orientation-preserving circle diffeomorphisms on which we can do so (see Theorem 4.2). Moreover, we proved that Kobayashi's metric and proved that Teichmüller's metric are the same on $\mathcal{TC}^{1+H}$ with the Bers complex manifold structure (Theorem 5.1). For a smaller space, we noticed recently that in [21], Matsuzaki has a similar study and claimed that the Teichmüller space $\mathcal{TC}^{1+\alpha}$ for any fixed $0 < \alpha \leq 1$ can be given the Bers complex Banach manifold structure. If it is the case, then our proof of Theorem 5.1 works in this case too.

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