SYMMETRIC RIGIDITY FOR CIRCLE ENDOMORPHISMS HAVING BOUNDED GEOMETRY

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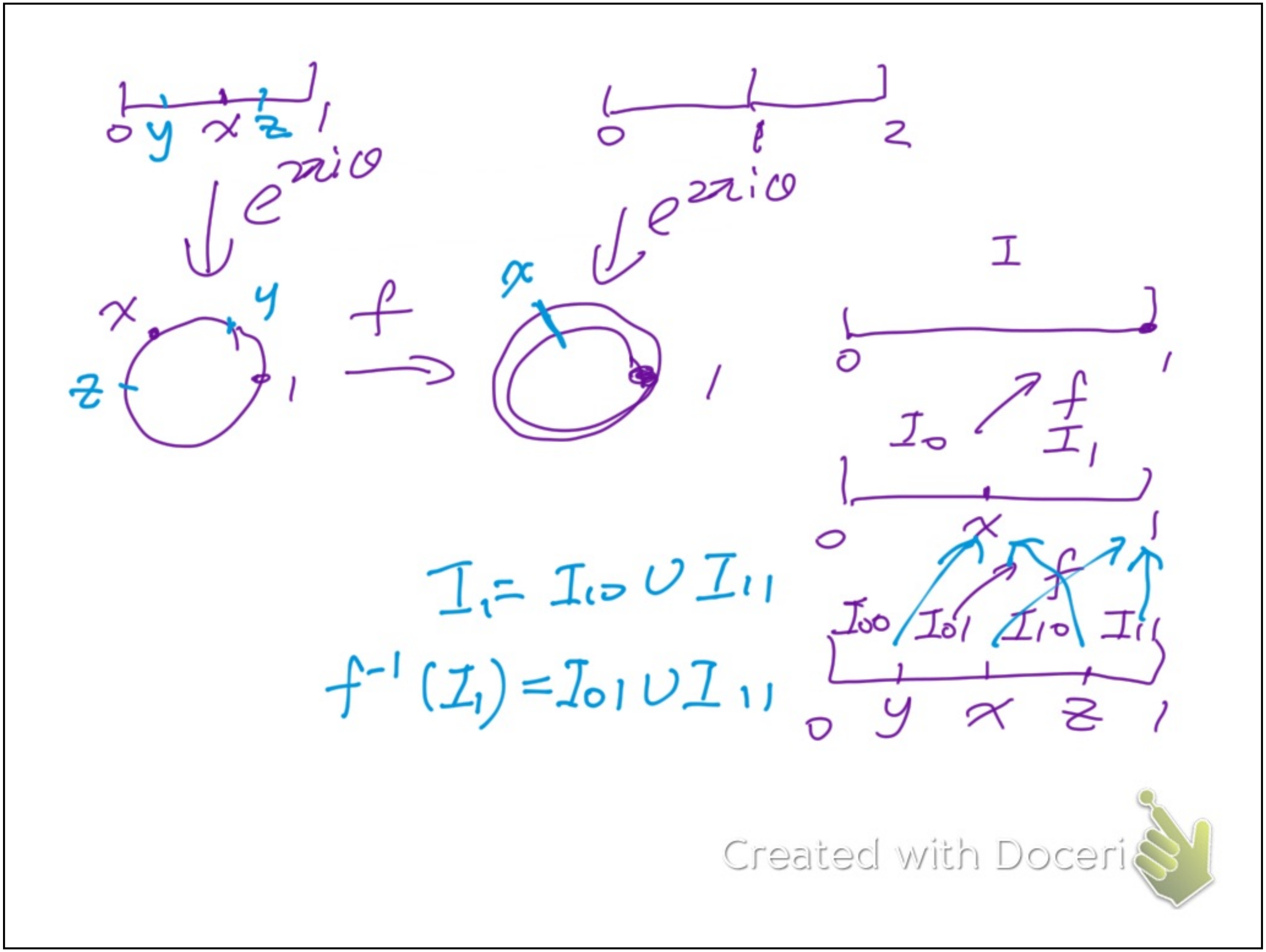
Theorem 1 (Main Theorem). *Suppose f and g are two circle endomorphisms of the same degree $d \geq 2$ having bounded geometry such that $f(1) = g(1) = 1$ and suppose f and g both preserve the Lebesgue measure on the unit circle. Let h be the conjugacy from f to g with $h(1) = 1$. That is, $h \circ f = g \circ h$. If h is a symmetric homeomorphism, then h must be the identity.*


0 y x z 1
$e^{2\pi i\theta}$
0 1 2
$e^{2\pi i\theta}$
x y z 1
f
x 1
I
0 1
I_0 f I_1
0 x 1
I_{00} I_{01} f I_{10} I_{11}
0 y x z 1
$I_1 = I_{10} \cup I_{11}$
$f^{-1}(I_1) = I_{01} \cup I_{11}$
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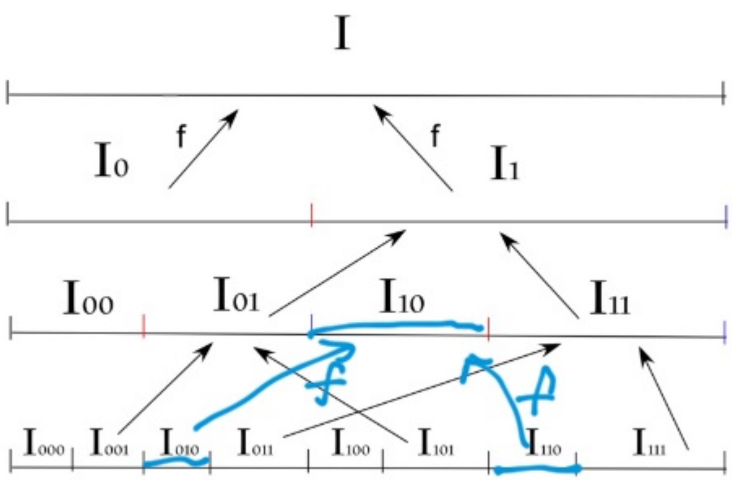


Markov partitions of the unit interval for a degree 2 covering map f(z).

w_n = length n word of 0's and 1's

$I_{w_n} = I_{w_n 0} \cup I_{w_n 1}$

$f^{-1}(I_{w_n}) = I_{0w_n} \cup I_{1w_n}$

Bounded Geometry

$\frac{|I_{w_n}|}{|I_{w_n 0}|} \le C$ and $\frac{|I_{w_n}|}{|I_{w_n 1}|} \le C$ $\forall w_n$

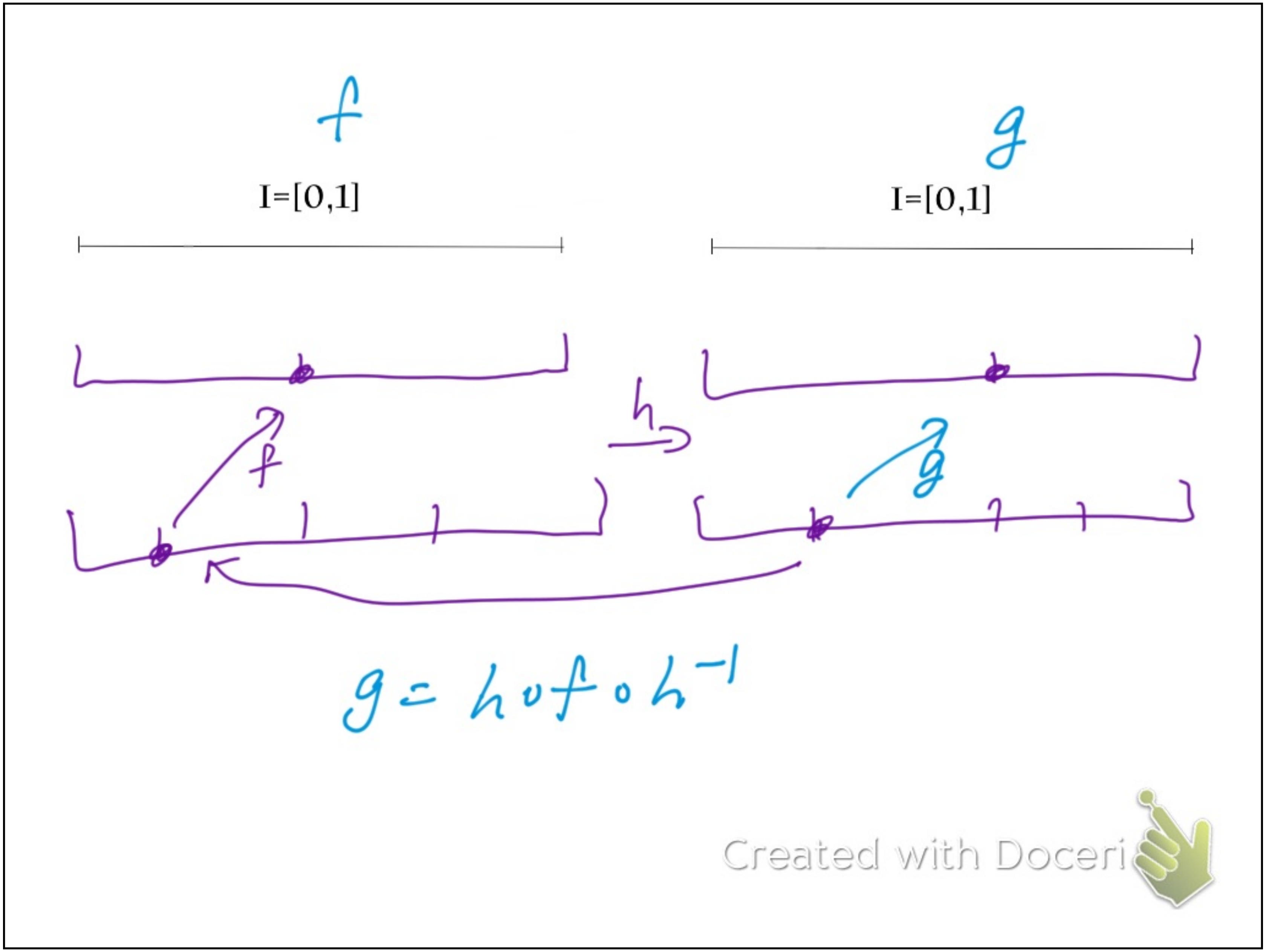
$|I_{w_n 0}| \ge \frac{1}{C}|I_{w_n}|$

Leb. inv.

$|I_{w_n}| = |I_{0w_n}| + |I_{1w_n}|$

By construction

$|I_{w_n}| = |I_{w_n 0}| + |I_{w_n 1}|$


f
g
I=[0,1]
I=[0,1]
h
f
g
g = h∘f∘h^-1
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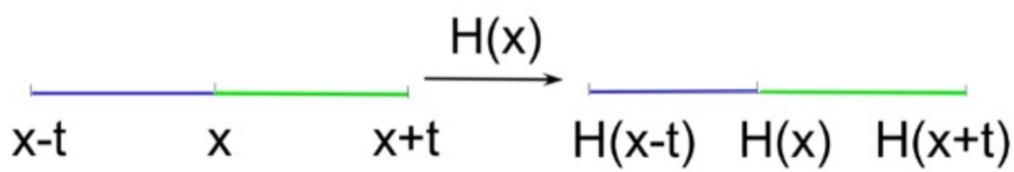
Definition 1. *A circle homeomorphism h is called quasisymmetric (refer to [2]) if there exists a constant $M \geq 1$ such that*

$$\frac{1}{M} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq M \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

It is called symmetric (refer to [3]) if there exists a positive bounded function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and

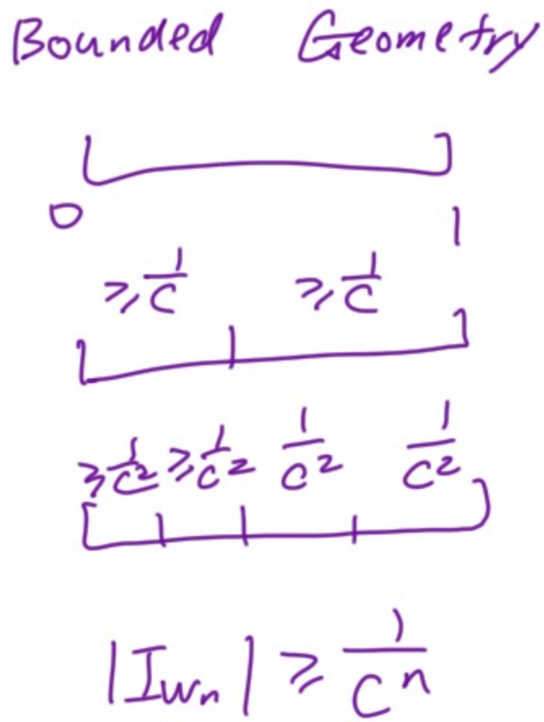
$$\frac{1}{1+\epsilon(t)} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq 1 + \epsilon(t) \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

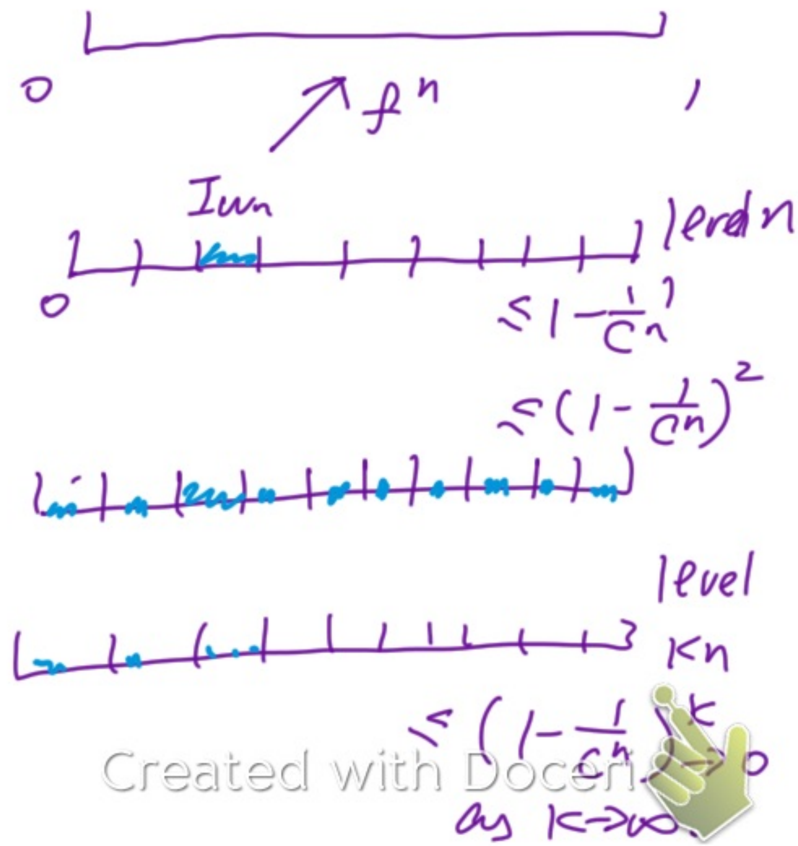


Lemma 1. *Suppose f is a circle endomorphism having bounded geometry. Let $\{\eta_n\}_{n=1}^{\infty}$ be the corresponding sequence of partitions. Suppose $I_{w_n} \in \eta_n$ is a fixed partition interval for some $n \geq 1$. Then*

$$\lim_{k\to\infty} \sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^k \neq w_n} |I_{w_n^1 \cdots w_n^k}| = 0,$$

where the ω_n^i are all words of length n.





$$1 \leq \Phi = \sup_{I \subseteq [0,1]} \frac{|h(I)|}{|I|} \leq \infty$$

$$X = \{x \in [0,1] \mid \exists I_k^x = [a_k, b_k], \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x, \lim_{k\to\infty} \frac{|h(I_k^x)|}{|I_k^x|} = \Phi\}$$

$=a$ $=b$

Lemma 2. *Suppose f and g are both circle endomorphisms having bounded geometry. Then X is a non-empty subset of T.*

① $\Phi = \infty$ $\lim_{k\to\infty} \frac{|h(I_k^x)| < 1}{|I_k^x| \to 0} = \infty$ $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x$ ✓

② $\Phi < \infty$ and $a = b = x \in X$ ✓

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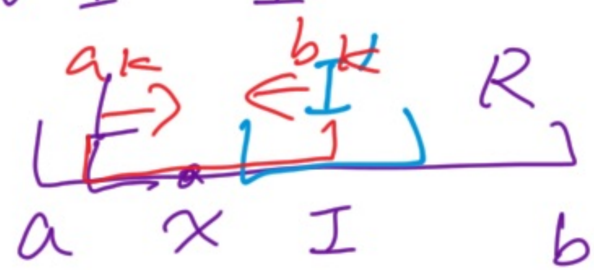
3. $\bar{\Phi} < \infty$ and $a < b$.

$I = [a, b]$

$$\bar{\Phi} = \lim_{k\to\infty} \frac{|h(I_k)|}{|I_k|} = \lim_{k\to\infty} \frac{h(b_k) - h(a_k)}{b_k - a_k} = \frac{|h(I)|}{|I|}$$

($h(b_k) \to h(b)$, $h(a_k) \to h(a)$, $b_k \to b$, $a_k \to a$)

$\forall\, I' \subset I$



$$\bar{\Phi} = \frac{|h(L)| + |h(I')| + |h(R)|}{|L| + |I'| + |R|}$$

$\le \bar{\Phi}$ $\quad \le \bar{\Phi}$ $\quad \le \bar{\Phi}$

$$\frac{|h(I')|}{|I'|} = \bar{\Phi} \Rightarrow I \subset X^{\sim} \quad \checkmark$$

Lemma 3. *Suppose f and g are both circle endomorphisms having bounded geometry and both preserve the Lebesgue measure m. Then X is dense in $[0, 1]$. That is, $\overline{X} = [0, 1]$.*

Proof. We will prove that for any $n \geq 1$ and for any partition interval $I_{w_n} \in \eta_n$, $I_{w_n} \cap X \neq \emptyset$. It will then follow from inequality (3) that $\overline{X} = [0, 1]$. We prove it by contradiction.

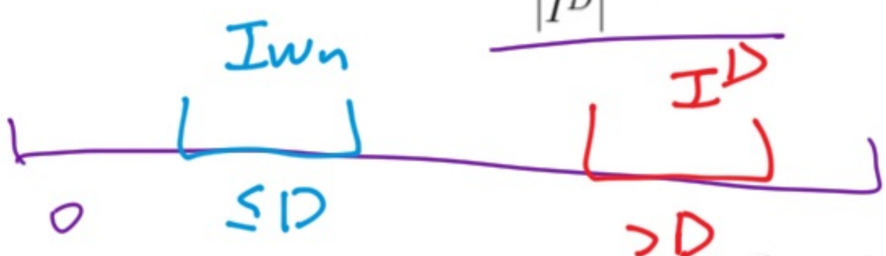
Assume we have a partition interval I_{w_n} such that $I_{w_n} \cap X = \emptyset$. Then we can find a number $D < \Phi$ such that

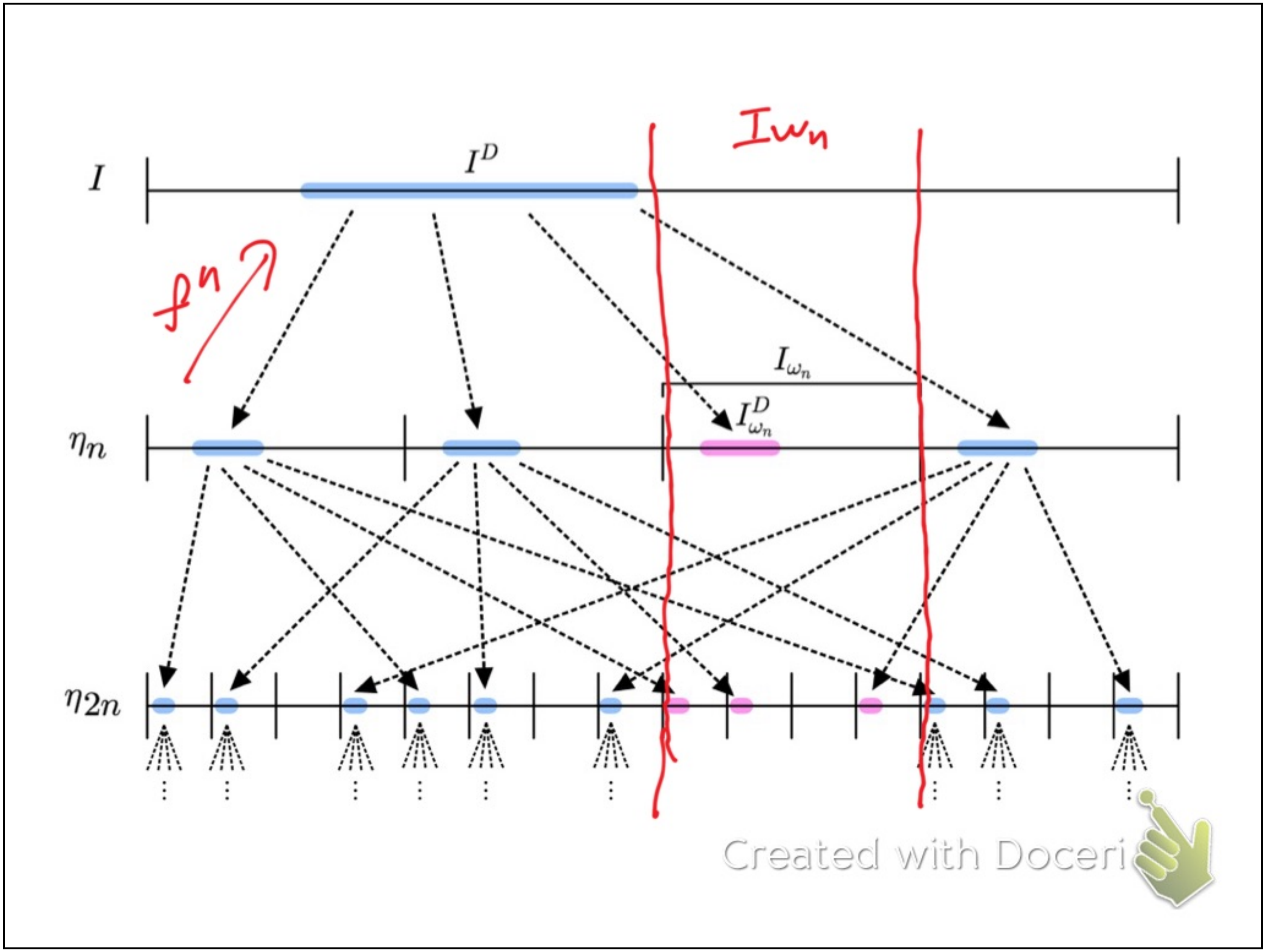
$$\frac{|h(I)|}{|I|} \leq D \tag{7}$$

for all $I \subset I_{w_n}$.

Since $X \neq \emptyset$, we have an interval $I^D \subseteq [0, 1]$ such that

$$\frac{|h(I^D)|}{|I^D|} > D. \tag{8}$$




I
I^D
I_{ω_n}
f^n
I_{ω_n}
$I^D_{\omega_n}$
η_n
η_{2n}
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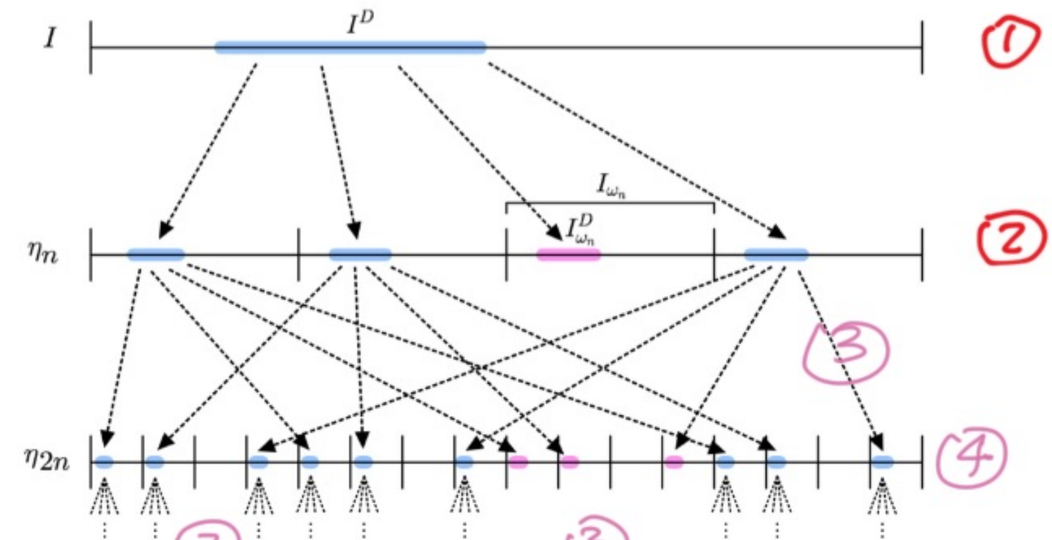


$$|I^D| = |I^D_{\omega_n}| + \sum_{\omega_n^1\neq\omega_n} |I^D_{\omega_n^1}| = |I^D_{\omega_n}| + \sum_{\omega_n^1\neq\omega_n} |f^{-n}(I^D_{\omega_n^1})|$$

$$= |I^D_{\omega_n}| + \sum_{\omega_n^1\neq\omega_n}\left(|I^D_{\omega_n\omega_n^1}| + \sum_{\omega_n^2\neq\omega_n}|I^D_{\omega_n^2\omega_n^1}|\right)$$

$$= |I^D_{\omega_n}| + \sum_{\omega_n^1\neq\omega_n}|I^D_{\omega_n\omega_n^1}| + \sum_{\omega_n^2\neq\omega_n}\sum_{\omega_n^1\neq\omega_n}|I^D_{\omega_n^2\omega_n^1}| = \dots$$

$$= |I^D_{w_n}| + \sum_{l=1}^{k-1}\sum_{w_n^l\neq w_n}\cdots\sum_{w_n^1\neq w_n}|I^D_{w_n w_n^l\dots w_n^1}| + \sum_{w_n^k\neq w_n}\cdots\sum_{w_n^1\neq w_n}|I^D_{w_n^k\dots w_n^1}|$$

rewrite

$h(I^D_{w_n}) + h\sum \cdots + \sum\cdots\sum$

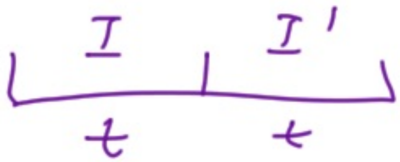
$\leq D$ $\quad \leq D \quad \to 0$

$$\frac{h(I^D)}{|I^D|} \leq D$$

Proof of Theorem 1. We will prove that $\Phi = 1$. Equivalently, we will prove that $\Phi > 1$ cannot happen, regardless of $\Phi < \infty$ or $\Phi = \infty$.

We proceed with a proof by contradiction. Assume $\Phi > 1$ (possibly ∞). Then we have two numbers $1 < D_1 < D_2 < \Phi$.

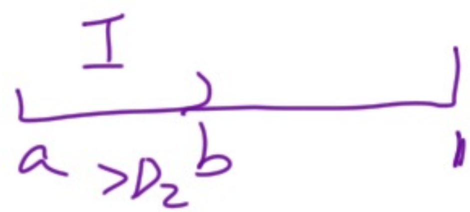
h symm. $\exists t_0$ s.t $\forall I, I'$ both length $t < t_0$



$$\frac{D_1}{D_2} < \frac{|h(I)|}{|h(I')|} < \frac{D_2}{D_1}$$

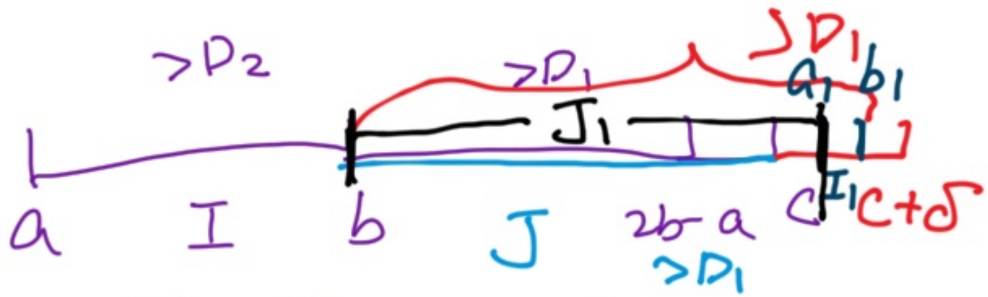
$\bar{X}$ is dense: $\exists I = [a,b]$ $|I| < t_0$ s.t.

$$\frac{|h(I)|}{|I|} > D_2$$



We are going to prove

$$\frac{|h[a,1]|}{|[a,1]|} > D_1$$



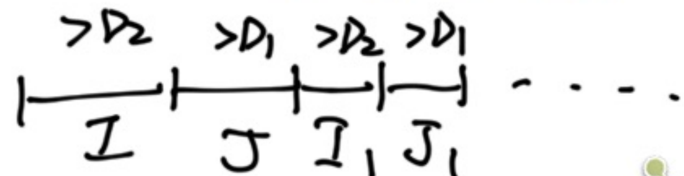
$J = [b, c], \quad c \geq 2b-a$

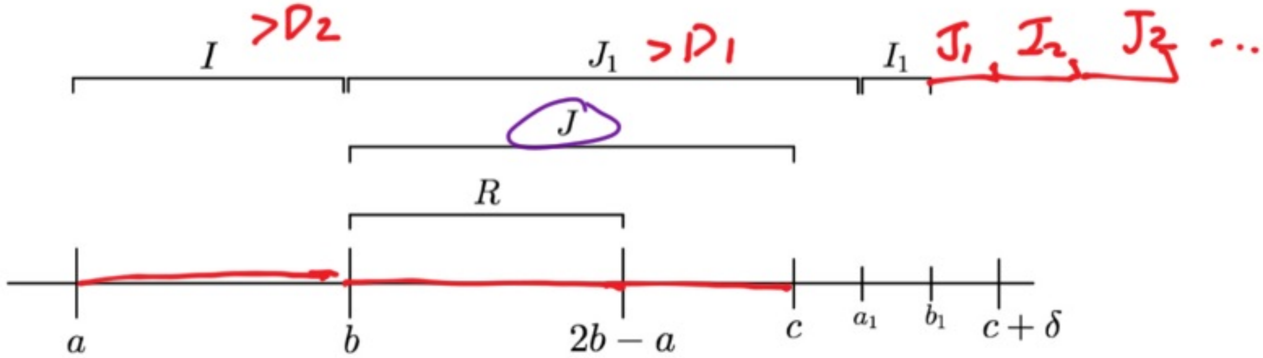
$$\frac{|h(J)|}{|J|} > D_1$$

$\exists \delta$ s.t. $\forall x \in (c, c+\delta)$ $\dfrac{|h([b,x])|}{|[b,x]|} > D_1$

$\bar{X}$ is dense $\Rightarrow I_1 = [a_1, b_1] \subset [c, c+\delta]$

$$\frac{|h(I_1)|}{|I_1|} > D_2$$





$I_k = [a_k, b_k]$ $\quad \exists\, b_\infty = \lim_{k\to\infty} b_k \le 1$

$$\frac{|h([b, b_\infty])|}{|[b, b_\infty]|} > D_1$$

$\beta = \sup\{\text{all possible } b_\infty\}$

If $\beta \neq 1$ $\quad \frac{|h([b,\beta])|}{|[b,\beta]|} > D_1$ $\quad [b,\beta] = J$.

$\Rightarrow \beta = 1$ $\quad\quad \Rightarrow\Leftarrow$ β is sup

$$\frac{|h([a,1])|}{|[a,1]|} > D,$$

$$\frac{|h([0,b])|}{|[0,b]|} > D, \qquad \Rightarrow \in$$

$\underline{\Phi} = 1$