Dynamical Teichmüller Spaces

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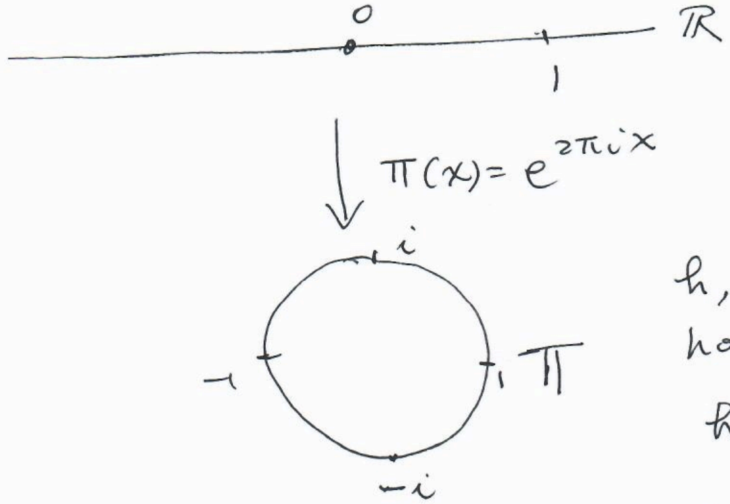
$\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$, the unit circle in $\mathbb{C}$

$\mathbb{R}$ is the real line



H, an increasing function, $H(x+1) = H(x) + 1$

$H(0) = 0$

$\downarrow$

h, an orientation-preserving homeomorphism of $\mathbb{T}$

$h(1) = 1$

$\pi \circ H = h \circ \pi$, lifting

We call h or H a circle homeomorphism.

Let
$$\varepsilon_h(t) = \sup_{x \in \mathbb{R}} \left| \log \left| \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \right| \right|, \quad t > 0,$$
be the quasisymmetric distortion function for h.

Then h is quasisymmetric if $\sup_{t>0} \varepsilon_h(t) = M < \infty$.

Furthermore, h is symmetric if $\varepsilon_h(t) \to 0^+$ as $t \to 0^+$.

If h is a C^1-diffeomorphism, then h is symmetric and we have the modulus of continuity
$$\omega_h(t) = \sup_{\substack{|x-y| \le t \\ x, y \in \mathbb{R}}} |\log H'(x) - \log H'(y)|.$$

In this case $\varepsilon_h(t) \le C\, \omega_h(t)$ for some constant $C>0$. We say h is $C^{1+\alpha}$ if $\omega_h(t) \le C t^{\alpha}$ for some $0<\alpha\le 1$ and some constant $C>0$.

$$Q = \{ h \mid h \text{ is } \cancel{\text{symmetric}} \text{ quasisymmetric} \}$$

$$S = \{ h \mid h \text{ is symmetric} \}$$

$$C^{1+H} = \{ h \mid h \text{ is } C^{1+\alpha} \text{ for some } 0<\alpha\le 1 \}$$

- universal Teichmüller space $TQ = Q/M$
- universal symmetric Teichmüller space $TS = S/M$
- universal smooth Teichmüller space $\mathcal{T}\mathcal{C}^{1+H} = C^{1+H}/M$

where M = the space of all Möbius transformation preserving the unit disk in $\mathbb{C}$.

- $TQ = \{ h \in Q \mid h(1)=1,\ h(i)=i,\ h(-1)=-1 \}$
- $TS = \{ h \in S \mid h(1)=1,\ h(i)=i,\ h(-1)=-1 \}$.
- $\mathcal{T}\mathcal{C}^{1+H} = \{ h \in C^{1+H} \mid h(1)=1,\ h(i)=i,\ h(-1)=-1 \}$.

K-metric means the Kobayashi metric

T-metric means the Teichmüller metric

1) On TQ, K-metric = T-metric

A simple proof is to use holomorphic motions.

2) On TS, K-metric = T-metric

3) On $T\mathcal{C}^{+H}$, K-metric = T-metric

expect a holomorphic motion proof.

We will talk 2) and 3) in the next talk.

Note that all of them are complex Banach manifold through Bers embedding.

A more important Teichmüller space relating to dynamics is $AT = Q/S$, the asymptotical conformal universal Teichmüller space.

Conjecture: On AT, K-metric = T-metric

Note: each point in AT is an equivalent class $\{h \circ h_1 \mid h_1 \in S\}$

We are interested in two subspaces called dynamical Teichmüller spaces of AT.

consider $q(z)=z^2 : \mathbb{T} \to \mathbb{T}$, $Q(x)=2x$

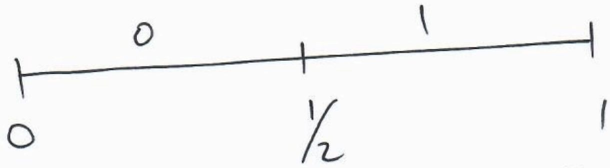
we identify $\mathbb{T}$ with $[0,1]/0\sim 1$, then

$q(x)=2x \pmod 1$. (In general, we can consider

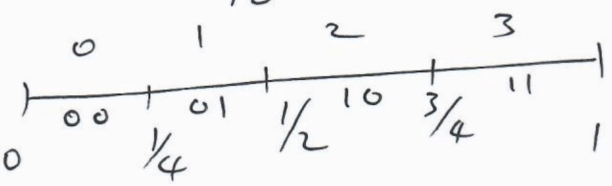
$q_d(z)=z^d$ for $d\geq 2$).

$q(0)=0$ is a fixed point

$q^{-1}(0)=\{0, \frac{1}{2}\}$



$q^{-2}(0)=\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$



$0=0\cdot 2+0$, $1=0\cdot 2+1$, $2=1\cdot 2+0$, $3=1\cdot 2+1$.

In general $q^{-n}(0)$ cuts $[0,1]$ into 2^n-equal sized intervals $I_{n,k}$, $k=0,1,\dots,2^n-1$

$$k=i_0 2^{n-1}+i_1 2^{n-2}+\cdots+2i_{n-2}+i_{n-1}$$

$$\Sigma_n^+=\{w_n=i_0 i_1\cdots i_{n-1} \mid i_k=0,1\}=\prod_0^{n-1}\{0,1\}$$

discrete Topology

$$\Sigma_n^+ \to \Sigma^+ = \{ w = i_0 i_1 \cdots i_{n-1} \cdots \mid i_k = 0, 1 \}$$

$$= \prod_0^{\infty} \{0, 1\}$$

$\uparrow$ discrete topology

Σ^+ with the product topology is a compact metric space. The standard metric (Lebesgue metric) is $d_L^+(w, w') = \sum_{k=1}^{\infty} \frac{|i_{k-1} - i'_{k-1}|}{2^k}$.

Then one can check $I_{n,k} = I_{w_n}$

and $I_{n,k} = I_{n+1, 2k} \cup I_{n+1, 2k+1} = I_{w_n 0} \cup I_{w_n 1}$.

$\forall\, w = i_0 i_1 \cdots i_{n-1} \cdots$, $w_n = i_0 i_1 \cdots i_{n-1}$

$$\cdots \subset I_{w_n} \subset I_{w_{n-1}} \subset \cdots \subset I_{w_1} \subset [0,1]$$

$\Rightarrow$ $\{x_w\} = \bigcap_{n=1}^{\infty} I_{w_n}$ is a single point set

$$x_w = i_0 + \frac{1}{2} i_1 + \cdots + \frac{i_{n-1}}{2^{n-1}} + \cdots \in [0,1]$$

$\pi_0 : \Sigma^+ \to [0,1]$, $\pi_0(w) = x_w$ is a onto and 1-1 except for a countable subset

$\{\omega_n 0\cdots 0\cdots, \omega_n 11\cdots\}$. On this countable subset.

$$\pi_0(\omega_n 0\cdots 0\cdots) = \pi_0(\omega_n 1\cdots 1\cdots)$$

Moreover, $\quad \pi_0 \circ \sigma^+ = g \circ \pi_0$

where $\sigma^+ : \Sigma^+ \to \Sigma^+$, $\sigma(\omega) = i_1 i_2 \cdots$, if $\omega = i_0 i_1 i_2 \cdots$,

is the ~~the~~ left shift.

$$\begin{tikzcd}
\Sigma^+ \arrow[r, "\sigma^+"] \arrow[d, "\pi_0"] & \Sigma^+ \arrow[d, "\pi_0"] \\
{[0,1]} \arrow[r, "g"] & {[0,1]}
\end{tikzcd}$$

On $[0,1]$, we have the Lebesgue metric,

On Σ, $\quad d_L$ is the Lebesgue metric

$$\sigma : (\Sigma, d_L^+) \circlearrowleft \xrightarrow{\text{change metric}} \sigma : (\Sigma, d_h^+)$$

$$\begin{tikzcd}
 \arrow[d, "\pi_0"] & & \arrow[d, "\pi_0"] \\
 \arrow[rr, "h"] & &
\end{tikzcd}$$

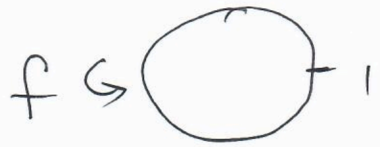
$$d_h^+(\omega, \omega') = |h(x_\omega) - h(x_{\omega'})|$$

$f = h \circ g \circ h^{-1} : \mathbb{T} \to \mathbb{T}$ is a circle endomorphism of degree 2, $f(\phi) = 1$.

F, lift of f, a homeomorphism of $\mathbb{R}$

$F(x+1) = F(x) + 2$

$$\downarrow \pi$$



Consider $$\varepsilon_f(t) = \sup_{\substack{x \in \mathbb{R} \\ n \in \mathbb{N}}} \left| \log \left| \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \right| \right|$$

Then we say f is uniformly quasisymmetric (uqs) if $$\sup_{t>0} \varepsilon_f(t) = M < \infty.$$

~~More~~ Furthermore, We say f is uniformly symmetric (us) if

$$\varepsilon_f(t) \longrightarrow 0^+ \text{ as } t \to 0^+$$

We say f preserves the Lebesgue measure (plm) if

$|f^{-1}(A)| = |A|$ for all Borel subset A of $\mathbb{T}$

The first dynamical Teichmüller space is

$$TUS = \{ h \in Q \mid f \text{ is us} \} / S$$

<u>Theorem (J)</u>: For any $\tau = [h] = \{ h \circ h_1 \mid h_1 \in S \}$, we have one $\hat{h} = h \circ h_1$ such that $\hat{f} = \hat{h} \circ g \circ \hat{h}^{-1}$ plm.

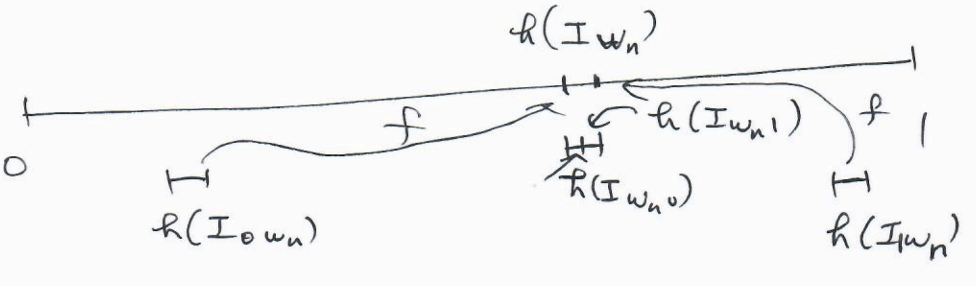
$$\Rightarrow TUS = \{ h \in Q \mid f \text{ is us and plm} \} / S$$

Furthermore, we like to study our second dynamical Teichmüller space

$$TUQ = \{ h \in Q \mid f \text{ is uqs and plm} \} / S$$

$$\subsetneqq \{ h \in Q \mid f \text{ is uqs} \} / S .$$

$$f \text{ plm} \iff h(I_{w_n}) = |h(I_{0w_n})| + |h(I_{1w_n})|$$



$$h(I_{w_n}) = \sum_{w_{n+1} \in (\sigma^{-1})(w_n)} | h(I_{w_{n+1}}) |$$

Theorem (Adamski-Hu-J-Wang)

Suppose both $f_1 = h_1 \circ g \circ h_1^{-1}$ and $f_2 = h_2 \circ g \circ h_2^{-1}$ have bounded geometric and plm. Suppose $h \in S$ such that $f_1 = h \circ f_2 \circ h^{-1}$. Then $h = id$.

$\Rightarrow$ $TUS = \{ h \in Q \mid f$ is us and plm $\}$

$TUQ = \{ h \in Q \mid f$ is uqs and plm $\}$

Since f is uqs (or us) $\Rightarrow$ f has bounded geometry

It makes the following conjecture is easier to study:

Conjecture: On TUS or TUQ, K-metric = T-metric.

In general $f = h \circ g \circ h^{-1}$, $h \in TUS$ or TUQ, is not differentiable, even may be totally singular. However, it has the dual derivative as we explain below.

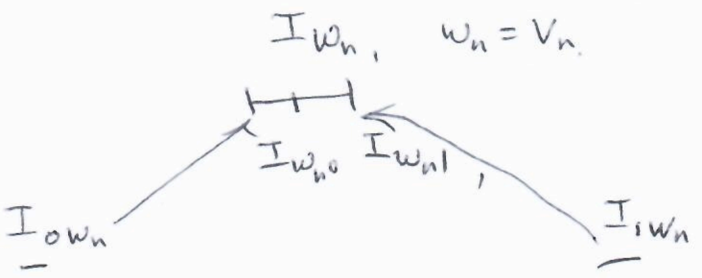
Dual symbolic representation:

$$\Sigma_n^- = \{ v_n = j_{n-1} \cdots j_1 j_0 \} = \{0,1\}^{\prod_0^{n-1}}$$

with product topology

$$v_n = j_{n-1} j_{n-2} \cdots j_1 j_0 = i_0 i_1 \cdots i_{n-2} i_{n-1} = w_n .$$

$$\Sigma_n^- \longrightarrow \Sigma^- = \{ v = \cdots j_{n-1} \cdots j_1 j_0 \} = \{0,1\}^{\prod_0^{\infty}}$$



$0 w_n = 0 v_n$

$1 w_n = 1 v_n$

In other words, $I_{w_n 0}$, $I_{w_n 1}$ are n-closed in Σ^+

$I_{0w_n} = I_{0v_n}$, $I_{1w_n} = I_{1v_n}$ are n-closed in Σ^-.

$\sigma^-(v) = \cdots j_{n-1} \cdots j_1$, if $v = \cdots j_{n-1} \cdots j_1 j_0$, is called the right shift.

If $h \in TUQ$, then h induces a metric on Σ^-

$$d_h^-(v, v') = |h(I_{w_n})| \left(= |h(I_{0w_n})| + |h(I_{1w_n})|\right)$$

if $v_n = v_n'$ but $v_{n+1} \neq v_{n+1}'$

Theorem (Hu-J-Wang), $\forall h \in TUQ$,

$\sigma^{-} : (\Sigma^{-}, d_h^{-}) \circlearrowleft$ is a differentiable a.e and the derivative $\dfrac{d_h^{-}\sigma^{-}(v)}{d_h^{-}v}$ is a L^1-function on (Σ^{-}, d_h^{-})

— We proved this theorem by using Martingale theory.

Theorem (J). $\forall h \in TUS$,

$\sigma^{-} : (\Sigma^{-}, d_h^{-}) \circlearrowleft$ is a C^1-map, i.e., its derivative $\dfrac{d_h^{-}\sigma^{-}(v)}{d_h^{-}v}$ is a continuous function on Σ^{-}.

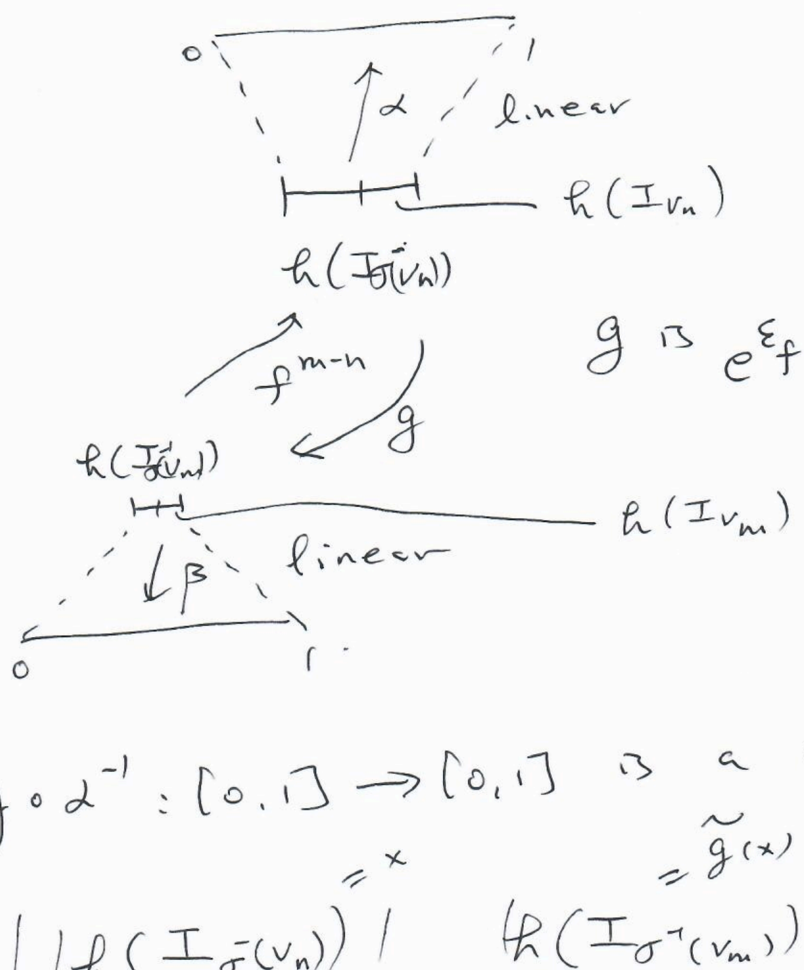
The proof uses the following lemma.

Lemma Suppose $h : [0,1] \to [0,1]$, $h(0) = 0$, $h(1) = 1$, is a M-quasisymmetric homeomorphism. Then $|h(x) - x| \le M - 1$, $\forall x \in [0,1]$.

Moreover, $M-1$ is the best estimation.

· From lemma, we see

$v_n = w_n$



g is $e^{\varepsilon_f(h(I_{v_n}))} - qs$

$\tilde{g} = \beta \circ g \circ \alpha^{-1} : [0,1] \to [0,1]$ is a $e^{\varepsilon_f(h(I_{v_n}))} - qs$

$$\Rightarrow \left| \underbrace{\frac{|h(I_{\sigma^{-1}(v_n)})|}{|h(I_{v_n})|}}_{=x} - \underbrace{\frac{|h(I_{\sigma^{-1}(v_m)})|}{|h(I_{v_m})|}}_{=\tilde{g}(x)} \right|$$

$$= |g(x) - x| \le e^{\varepsilon_f(\cancel{h}(I_{v_n}))} - 1 \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \left\{ \frac{|h(I_{\sigma^-(v_n)})|}{|h(I_{v_n})|} \right\}_{n=1}^{\infty} \text{ is a Cauchy}$$

sequence $\Rightarrow$

$$\frac{d_h^- \sigma^-(v)}{d_h^- v} = \lim_{n\to\infty} \frac{|h(I_{\sigma^-(v_n)})|}{|h(I_{v_n})|} \quad \text{exists}$$

and is a continuous function on Σ^-.

(*) We have two characterizations for all $\frac{d_h^* \sigma^-(v)}{d_h^- v}$ for $h \in TUS$

(**) We still want to know some characterization of $\frac{d_h^- \sigma^-(v)}{d_h^- v}$ for $h \in TUQ$.

Another smooth dynamical Teichmüller space

$$TC^{1+H}\mathcal{E} = \{ h \in Q \mid f = h\circ g\circ h^{-1} \text{ is } C^{1+\alpha} \text{ for some } 0<\alpha\le 1 \} / S$$

my solution to Katok conjecture in 1-dim

$$\{ h \in Q \mid f = h\circ g\circ h^{-1} \text{ is } C^{1+\alpha} \text{ expanding} \}/S$$

Ruelle-Perron-Fröbius Thm

$\{ h \in Q \mid f \text{ is } C^{1+\alpha}$ ~~expanding~~ and plm $\}$

For $h \in TC^{1+H}\mathcal{E}$, f has both the derivative and the dual derivative and both are Hölder continuous.

Theorem (J). $TUS = \overline{TC^{1+H}\mathcal{E}}$,

i.e, $TC^{1+H}\mathcal{E}$ is not a complete space under T-metric, and the completsion of $TC^{1+H}\mathcal{E}$ is TUS under T-metric.